New results on the entanglement entropy of singular regions in CFTs

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It from Qubit
Simons Collaboration on
Quantum Fields, Gravity and Information

Instituto
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June 22, 2019

Talk based on

arXiv:1904.11495

with **Horacio Casini** and **William Witczak-Krempa**

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arXiv:1904.11495
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+ some mentions to previous work

Phys.Rev. B96 (2017) no.3, 035117
with **Lauren Sierens**, **Rajiv Singh**, **Rob Myers**, **Roger Melko**

Phys.Rev. B93 (2016) 045131
with **William Witczak-Krempa**

JHEP 1512 (2015) 168
JHEP 1508 (2015) 068
with **Rob Myers**

Phys.Rev.Lett. 115 (2015) 021602
with **Rob Myers**, **William Witczak-Krempa**

1. EE of singular regions in CFTs: known facts and conjectures
2. EE of singular regions in CFTs: New results
   - Vertex-induced universal terms
   - Wedge entanglement vs corner entanglement
   - Singular regions and EE divergences

1. EE of singular regions in CFTs: known facts and conjectures

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Given smooth spatial entangling region V with characteristic length scale H,

$$S_n^{(d)} = b_{d-2}\frac{H^{d-2}}{\delta^{d-2}} + b_{d-4}\frac{H^{d-4}}{\delta^{d-4}} + \cdots + \begin{cases} b_1 \frac{H}{\delta} + (-1)^{\frac{d-1}{2}} s_n^{\text{univ}}\,, & (\text{odd } d)\,, \\ b_2 \frac{H^2}{\delta^2} + (-1)^{\frac{d-2}{2}} s_n^{\text{univ}} \log\left(\frac{H}{\delta}\right) + b_0\,, & (\text{even } d)\,. \end{cases}$$

where δ, UV regulator.

Universal terms in $d = 3, 4$

Even d: $s_n^{\text{univ}} \Leftrightarrow$ logarithmic term, linear combination of local integrals on $\Sigma \equiv \partial V$ weighted by theory-dependent "charges".

Odd d: $s_n^{\text{univ}} \Leftrightarrow$ constant term, no longer controlled by local integral on $\Sigma \equiv \partial V$. Less robust than logarithmic terms $\Rightarrow$ May use Mutual Information as a regulator. [Casini; Casini, Huerta, Myers, Yale]

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- $d = 4$, $\Sigma \Leftarrow$ smooth surface [Solodukhin; Fursaev]

$$s_n^{\text{univ}} = -\frac{1}{2\pi}\left[f_a(n)\int_\Sigma \mathcal{R} + f_b(n)\int_\Sigma k^2 - f_c(n)\int_\Sigma W\right]\log\left(\frac{H}{\delta}\right)$$

where $f_a(1) = a$, $f_b(1) = f_c(1) = c$ trace-anomaly coefficients. Geometry and theory dependences factorize term by term.

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- $d = 3$, $\Sigma \Leftarrow$ smooth curve

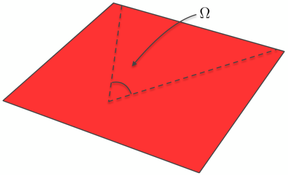
$$S_n^{(3)} = b_1 \frac{H}{\delta} - s_n^{\text{univ}}$$

e.g., $\Sigma = \mathbb{S}^1$, then s_1^{univ} = free energy of CFT on $\mathbb{S}^3$ [Casini, Huerta, Myers; Dowker], non-local quantity. Geometry and theory dependences entangled.

Corner entanglement in $d = 3$

Situation changes when geometric singularities present on Σ. Consider corner of opening angle Ω on a time slice of a $d = 3$ CFT,

$$S_{\rm EE}^{\rm corner} = b_1 \frac{H}{\delta} - a_n^{(3)}(\Omega) \log\left(\frac{H}{\delta}\right) + b_0$$



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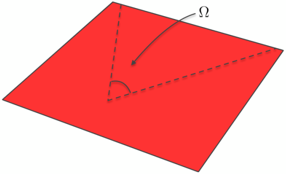

Logarithmic universal term arises, controlled by $a_n^{(3)}(\Omega)$. Vast literature, free fields, lattice models, holography, etc. [Many people]

Angular and theory dependences do not disentangle (*e.g.*, simple result for holographic theories [Drukker, Gross, Ooguri; Hirata, Takayanagi] vs horrendous expressions for free fields [Casini, Huerta]).

Corner entanglement in $d = 3$

Still, remarkable amount of universality observed [PB, Myers, Witczak-Krempa]

$$a_1^{(3)}(\Omega) = \sigma\,(\Omega - \pi)^2 + \dots\,, \quad \sigma = \frac{\pi^2}{24} C_T \tag{1}$$

Conjectured to hold $\forall$ CFTs in $d = 3$.

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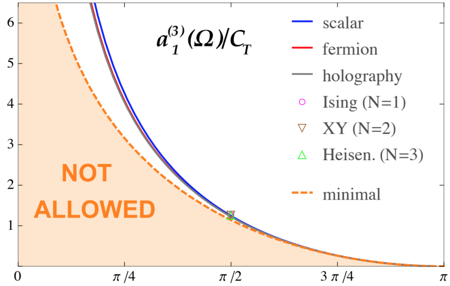

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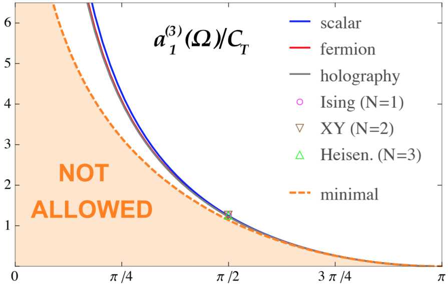

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Universal lower bound $\Leftrightarrow$ $a_1^{(3)}(\Omega) \geq \frac{\pi^2 C_T}{3} \log[1/\sin(\Omega/2)]$ [PB, Witczak-Krempa]

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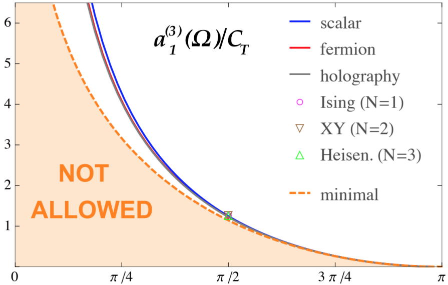

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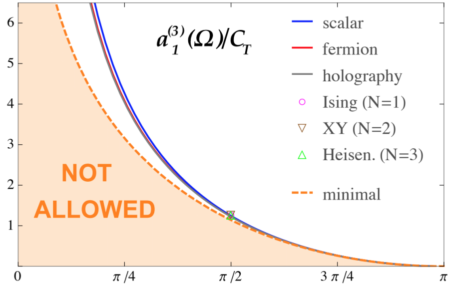

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Rényi entropy generalization is trickier...

Cone entanglement in $d = 4$

Fundamentally different from corner, theory dependence completely disentangled from angular dependence (which is the same for all CFTs)

[Klebanov, Nishioka, Pufu, Safdi]

$$S_n^{(4)\,\text{cone}} = b_2 \frac{H^2}{\delta^2} - a_n^{(4)}(\Omega) \log^2\left(\frac{H}{\delta}\right) + b_0 \log\left(\frac{H}{\delta}\right) + c_0$$

$$a_n^{(4)}(\Omega) = \frac{1}{4} f_b(n) \frac{\cos^2\Omega}{\sin\Omega} \quad \forall \text{ CFTs}$$

Other singular regions in $d = 4$

- Polyhedral corner of opening angles $\theta_1, \theta_2, \ldots, \theta_j$

$$S_n^{(4)\,\text{polyh.}} = b_2 \frac{H^2}{\delta^2} - w_1 \frac{H}{\delta} + v_n(\theta_1, \theta_2, \cdots, \theta_j) \log\left(\frac{L}{\delta}\right) + \mathcal{O}(\delta^0)$$

log instead of $\log^2$ universal term. $v_n(\theta_1, \theta_2, \cdots, \theta_j)$ conjectured to be controlled by some linear combination of $f_a(n)$, $f_b(n)$. [Sierens, PB, Singh, Myers, Melko]

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- Infinite wedge of opening angle Ω

$$S_n^{(4)\,\text{wedge}} = b_2 \frac{H^2}{\delta^2} - f_n(\Omega) \frac{H}{\delta} + \mathcal{O}(\delta^0)$$

$f_n(\Omega)$ non-universal overall factor, but based on holographic and free scalar calculations, $\partial_\Omega \left(f_n(\Omega)/a_n^{(3)}(\Omega) \right) \overset{(?)}{=} 0$

[Klebanov, Nishioka, Pufu, Safdi]

2. EE of singular regions in CFTs: New results

Vertex-induced universal terms

- Setup: free scalar in d-dim. Rényi entropy from Euclidean partition function on $\mathbb{R}^d$ for a field which picks up a phase when entangling region V is crossed. [Casini, Huerta]

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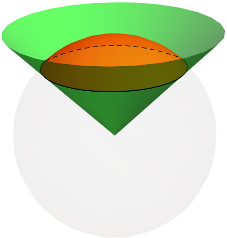
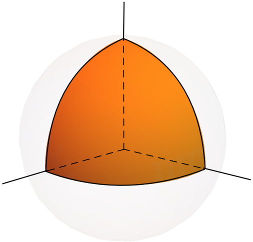
$$S_n|_{\log^2} = \frac{-f_b(n)}{8\pi} \log^2 \delta \int_\gamma k^2$$

where $\gamma \Leftrightarrow$ boundary of area on the surface of $\mathbb{S}^2$ resulting from $\mathbb{S}^3 \cap V$.

Vertex-induced universal terms





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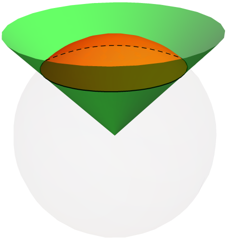

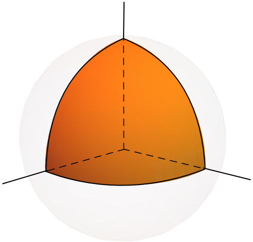

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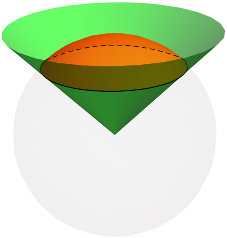

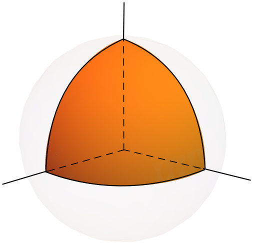

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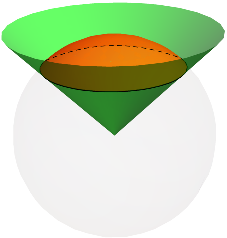

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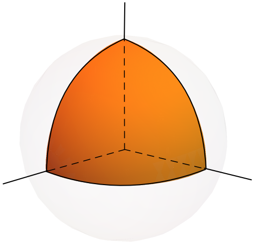

For polyhedral corners, γ are always great circles $\Rightarrow$ $k = 0$, no $\log^2$ term.

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Polyhedral corners

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$$S_n^{(4)\,\text{polyh.}} = b_2 \frac{H^2}{\delta^2} - w_1 \frac{H}{\delta} + v_n(\theta_1, \theta_2, \cdots, \theta_j) \log\left(\frac{L}{\delta}\right) + \mathcal{O}(\delta^0)$$

Universal function $v_n(\theta_1, \theta_2, \cdots, \theta_j)$ does not arise from log term controlled by Solodukhin's local-integrals formula. It arises however from non-local constant piece. Its evaluation for free fields requires full calculation of spectral function on sphere with a cut —fully analogous to corner in $d = 3$, very different from cone.

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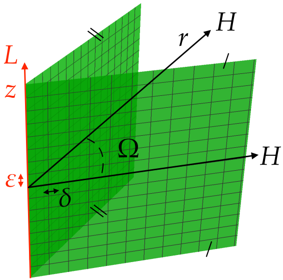
Result:

$$f(\Omega) = a(\Omega)\left[1 + \log \alpha_{\rm UV}\right] \alpha_{\rm IR}$$

where $\alpha_{\rm UV} = \epsilon/\delta$, $\alpha_{\rm IR} = L/H$ are ratios of UV and IR regulators along the two different directions.



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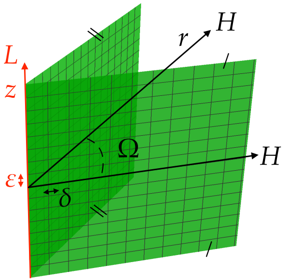
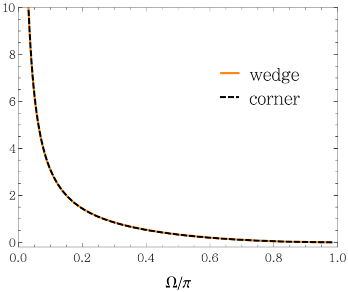


Same angular dependence. Overall factor of $f(\Omega)$ ill-defined, polluted by ambiguous choices of regulators. Ok with [Klebanov, Nishioka, Pufu, Safdi]

Wedge EE vs Corner EE

Einstein gravity bulk ⇒ Ryu-Takayanagi prescription for EE



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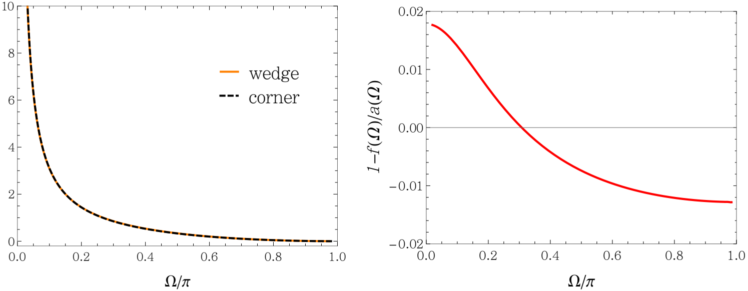

Remarkably close... But different

$$a(\Omega) \overset{\Omega\to 0}{=} \frac{\kappa}{\Omega} + \dots\,, \quad a(\Omega) \overset{\Omega\to \pi}{=} \sigma \cdot (\Omega-\pi)^2 + \dots\,, \quad \frac{\kappa}{\sigma} = 4\Gamma\left(\tfrac{3}{4}\right)^4 \simeq 9.0198$$

$$f(\Omega) \overset{\Omega\to 0}{=} \frac{\tilde{\kappa}}{\Omega} + \dots\,, \quad f(\Omega) \overset{\Omega\to \pi}{=} \tilde{\sigma} \cdot (\Omega-\pi)^2 + \dots\,, \quad \frac{\tilde{\kappa}}{\tilde{\sigma}} = \frac{2^{2/3}256\sqrt{\pi}\,\Gamma\left(\tfrac{5}{6}\right)}{3\Gamma\left(\tfrac{1}{6}\right)^2} \simeq 8.7469$$

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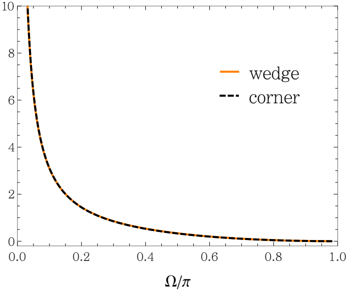

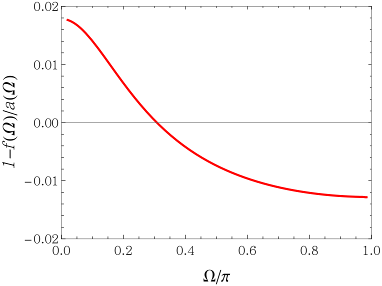

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Wedge EE $\neq$ corner EE in general.

Intermezzo: EMI model

Computing EE in QFTs is difficult in general...
Usual (semi)-analytic handles

- Highly symmetric regions, *e.g.*, (hyper)spheres
- Highly symmetric theories, *e.g.*, $d = 2$ CFTs
- Free fields
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Defining property suggested by its name: $I(A,B) + I(A,C) = I(A, B \cup C) \Rightarrow$ Strongly constrains EE and MI expressions.

$$S^{\rm EMI} = \kappa \int_{\partial A} d^{d-2}\mathbf{r}_1 \int_{\partial A} d^{d-2}\mathbf{r}_2 \, \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{r}_1 - \mathbf{r}_2|^{2(d-2)}}$$

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[Casini, Fosco, Huerta; Swingle]
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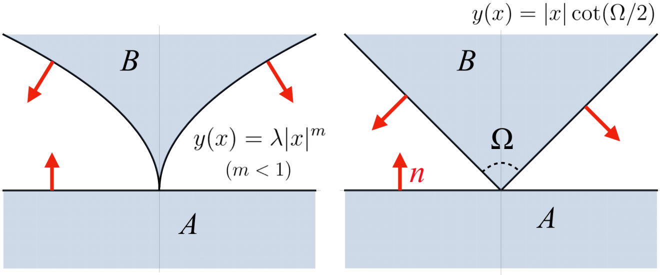
$$S^{\rm EMI} = \kappa \int_{\partial A} d^{d-2}\mathbf{r}_1 \int_{\partial A} d^{d-2}\mathbf{r}_2 \, \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{r}_1 - \mathbf{r}_2|^{2(d-2)}}$$

Free fermion in $d = 2$ only theory known to satisfy extensivity property. Still EMI expressions capture generic features of EE and MI in general dimensions. Computationally, even simpler than Ryu-Takayanagi formula.

Finite MI for touching regions



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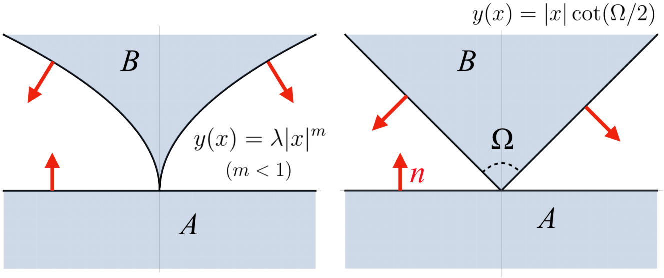
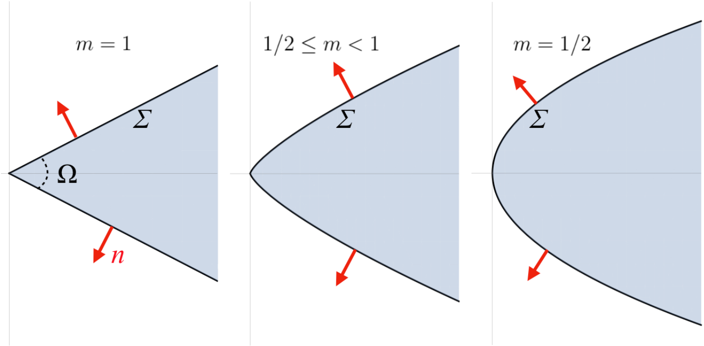


$$I(A,B) = 4\pi\kappa \tan(\Omega/2)\log(L/\delta) + \mathcal{O}(\delta^0) \quad \text{for straight corner}$$

$$I(A,B) = \frac{4\pi\kappa}{(1-m)}\left[\frac{L^{1-m}}{\lambda} - \frac{1}{\lambda^{\frac{1}{m}}\delta^{1-\frac{1}{m}}}\right] \quad \text{for curved corner}$$

Divergent for $m \geq 1$ but finite for $m < 1 \Rightarrow$ two regions touching at a point through a sufficiently sharp corner have non-divergent MI.

New EE singularities (or lack thereof)



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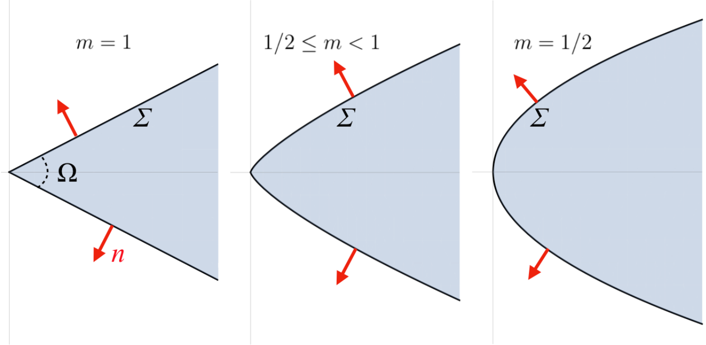

$$S_{\rm EE}^{(m=1)}=\frac{4\kappa H}{\delta}-a(\Omega)\log\left(H/\delta\right)+\mathcal{O}(\delta^0)\,,\quad a(\Omega)=2\kappa\left[1+(\pi-\Omega)\cot\Omega\right]$$

$$S_{\rm EE}=\frac{4\kappa H}{\delta}+\mathcal{O}(\delta^0)\,,\quad 1/2\leq m<1\,.$$

For $1/2 \leq m < 1$, curvature divergence at the tip, still no additional EE divergence $\Rightarrow$ Corners less sharp than straight corner do not modify the EE structure of divergences.

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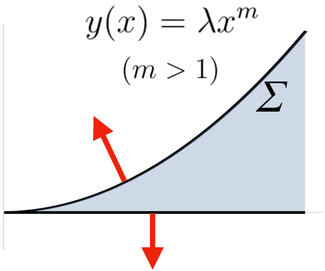

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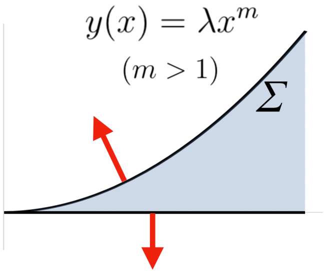

$$S_{\rm EE} = \frac{4\kappa H}{\delta} - \frac{2\kappa\pi c_m}{\lambda^{\frac{1}{m}}\delta^{1-\frac{1}{m}}} + \mathcal{O}(\delta^0)\,.$$

New non-universal divergence (same as for MI). The sharper de corner, the closer to the area-law one, without ever reaching it.

Things to remember

- Straight lines emanating from vertices produce logarithmic enhancement of entanglement divergences with respect to smooth regions.
  - $d = 3$ corner: $\log \delta \leftarrow$ from constant term; $a_n^{(3)}(\Omega)$ non-local nature.
  - $d = 4$ cone: $\log^2 \delta$ term $\leftarrow$ from $\log \delta$ term; $a_n^{(4)}(\Omega)$ local, controlled by $f_b(n)$, angular dependence fully determined.
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- Wedge entanglement $\neq$ corner entanglement in general.
- MI of regions touching through sufficiently sharp needles does not diverge.
- Corners less sharp than straight corners do not modify structure of divergences of EE. If sharper than straight corners $\Rightarrow$ new non-universal divergences approaching area-law.

- Actual computation of $v(\theta_1, \theta_2, \theta_3)$ for free fields requires full evaluation of spectral function on $\mathbb{S}^3$ with boundary conditions on two-dimensional spherical polyhedron. Challenging, perhaps not particularly illuminating...

A few questions for the future

- Actual computation of $v(\theta_1, \theta_2, \theta_3)$ for free fields requires full evaluation of spectral function on $\mathbb{S}^3$ with boundary conditions on two-dimensional spherical polyhedron. Challenging, perhaps not particularly illuminating...
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- Is there an upper bound for the corner function: $\mathfrak{a}(\Omega) \geq a^{(3)}(\Omega)$ $\forall$ CFTs?
- Does the EMI model correspond to any real CFT in $d \geq 3$? If not, what properties does the EMI fail to satisfy? Are there less restrictive conditions one can impose on the mutual information leading to interesting models?

Thank you

Vertex-induced universal terms

For a free scalar (analogously for free fermion), Rényi entropy computable as [Casini, Huerta]

$$S_n(V) = \frac{1}{1-n}\log(\mathrm{Tr}\,\rho_V^n) = \frac{1}{1-n}\sum_{k=0}^{n-1}\log Z[e^{2\pi i\frac{k}{n}}]$$

where $Z[e^{2\pi i a}]$ Euclidean partition function on $\mathbb{R}^d$ for a field which picks up a phase $e^{2\pi i a}$ when entangling region V is crossed.

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where $Z[e^{2\pi i a}]$ Euclidean partition function on $\mathbb{R}^d$ for a field which picks up a phase $e^{2\pi i a}$ when entangling region V is crossed. One can exploit relation with Green function:

$$\partial_{m^2}\log Z[e^{2\pi i a}] = -\frac{1}{2}\int_{\mathbb{R}^d} d^d\vec{r}\, G_a(\vec{r},\vec{r})\,,$$

where

$$(-\nabla^2_{\vec{r}_1} + m^2)G_a(\vec{r}_1,\vec{r}_2) = \delta(\vec{r}_1-\vec{r}_2)\,,$$
$$\lim_{\epsilon\to 0^+} G_a(\vec{r}_1+\epsilon\vec{\eta},\vec{r}_2) = e^{2\pi i a}\lim_{\epsilon\to 0^+} G_a(\vec{r}_1-\epsilon\vec{\eta},\vec{r}_2)\,,\quad \vec{r}_1\in V\,,$$

and $\vec{\eta}$ is orthogonal to V.

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For regions emanating from vertices in $d=4$, cut on $\mathbb{S}^3 \Leftrightarrow$ certain area on the surface of a $\mathbb{S}^2$.

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Spectral function can be related to analogous Rényi entropy in $(d-1)$-dim. de Sitter space (as long as boundary conditions match).

$$\partial_{m^2} \log Z \left[\mathrm{dS}_{(d-1)}\right] = \frac{1}{2} \operatorname{Tr} \left[\frac{1}{\nabla^2_{\mathbb{S}^{d-1}} - d(d-1)g(d) - m^2} \right] .$$

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High-mass expansion (valid for $m \gg L_{\mathrm{dS}} \equiv 1$, UV cutoff δ hidden in $c_{n,i}$)

$$S_n^{\mathrm{dS}_{(d-1)}} = c_{n,(d-3)} m^{d-3} + \cdots + c_{n,0} + \boxed{\frac{c_{n,-1}}{m}} + \cdots$$

Various possible combinations of m, δ and local integrals over entangling surface. All trace of m in divergent terms involving δ must disappear as $m \to 0 \Rightarrow$ terms involving negative powers of m combined with δ forbidden.

Vertex-induced universal terms

Only possible form of $c_{n,-1}$ in $d = 4$:

$$c_{n,-1} = \alpha_n \int_\gamma k^2 \, ,$$

where $\gamma \Leftrightarrow$ boundary of the entangling region in $\mathrm{dS}_3 \Leftrightarrow$ boundary of area on the surface of $\mathbb{S}^2$ resulting from $\mathbb{S}^3 \cap V$.

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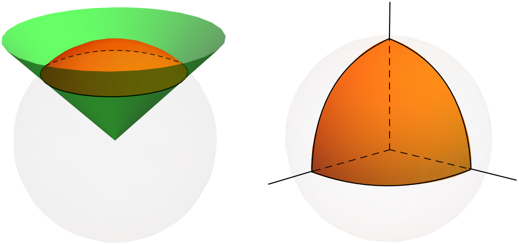

Entangling regions in $dS_3 \Leftrightarrow$ orange surfaces.
$\gamma \Leftrightarrow$ black curved segments bounding them.

Vertex-induced universal terms

Local nature of $c_{n,-1}$ prevents it from feeling the background geometry curvature $\Rightarrow$ we can fix α_n *e.g.*, using cylinder Rényi entropy in flat space,

$$S_n^{(\mathrm{dS}_3)}|_{m^{-1}} = \frac{f_b(n)}{8m}\int_\gamma k^2$$

Combined with

$$S_n|_{\log} = -\frac{\log(\delta/L)}{\pi}\int_0^\infty dm^2 m\,\frac{\partial S_n^{\mathrm{dS}_{(d-1)}}}{\partial m^2}\,,$$

we finally get

$$\boxed{S_n|_{\log^2} = \frac{-f_b(n)}{8\pi}\log^2\delta\int_\gamma k^2}$$

Wedge EE vs corner EE

When entangling region takes the form $C \times \mathbb{R}_L$, dimensional reduction possible for free fields. d-dim. field $\Leftrightarrow$ ∞ $(d-1)$-dim. independent fields of mass $M_k^2 = m^2 + (2\pi k/L)^2$. [Casini, Huerta]

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$$f(\Omega) = a(\Omega)\,[1 + \log \alpha_{\rm UV}]\,\alpha_{\rm IR}$$

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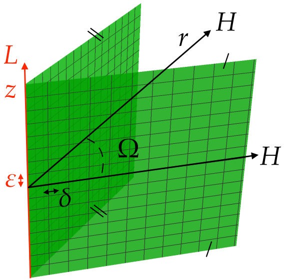

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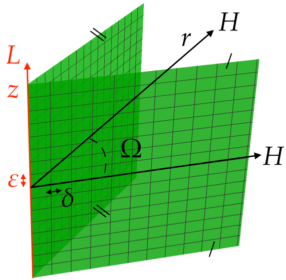

Angular dependence agrees. Overall factor of $f(\Omega)$ ill-defined, polluted by ambiguous choices of regulators. Agreement with [Klebanov, Nishioka, Pufu, Safdi]

Even dimensions

$s_n^{\text{univ}} \Leftrightarrow$ logarithmic term, linear combination of local integrals on $\Sigma \equiv \partial V$ weighted by theory-dependent "charges".

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- $d = 4$, $\Sigma \Leftarrow$ smooth surface [Solodukhin; Fursaev]

$$s_n^{\text{univ}} = -\frac{1}{2\pi}\left[f_a(n) \int_\Sigma \mathcal{R} + f_b(n) \int_\Sigma k^2 - f_c(n) \int_\Sigma W \right] \log\left(\frac{H}{\delta}\right)$$

where $f_a(1) = a$, $f_b(1) = f_c(1) = c$ trace-anomaly coefficients. Geometry and theory dependences factorize term by term. Rényi-index dependence changes from theory to theory.

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- $d = 6, 8, \ldots \Leftarrow$ similar story (more independent integrals and charges). [see *e.g.*, Safdi; Miao]

Odd dimensions

$s_n^{\text{univ}} \Leftrightarrow$ constant term, no longer controlled by local integral on $\Sigma \equiv \partial V$. Less robust than logarithmic terms, *e.g.*, one cannot distinguish H from $H + a\delta$, which pollutes $s_n^{\text{univ}} \Rightarrow$ Use Mutual Information as regulator [Casini; Casini, Huerta, Myers, Yale]

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$$S_n^{(3)} = b_1 \frac{H}{\delta} - s_n^{\text{univ}}$$

e.g., $\Sigma = \mathbb{S}^1$, then s_1^{univ} = free energy $\mathcal{F}$ of CFT on $\mathbb{S}^3$ [Casini, Huerta, Myers; Dowker], non-local quantity.

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- $d = 5, 7, \ldots \Leftarrow$ similar story for s_n^{univ}, analogous connection between $s_1^{\text{univ}}(\mathbb{S}^{d-2})$ and $\mathcal{F}(\mathbb{S}^d)$