Energy Positivity from Holographic Volume Susceptibility

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Energy conditions from geometry and causality

From bulk causality to boundary energy conditions:

- Average Null Energy Condition
  - Holographically, from Gao-Wald
- Strong Subadditivity of Entanglement Entropy
  - From extremality of HRT
  - Also implies boundary energy conditions
- Quantum Null Energy Condition
  - Holographically, from nesting of entanglement wedges near boundary

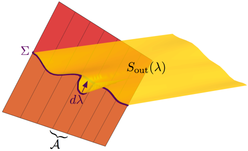
Bulk geometry, causality $\rightarrow$ energy positivity relations.

Diagonal and off-diagonal variations

Quantum Null Energy Condition

$$\langle T_{kk}(p)\rangle \geq \frac{\hbar}{2\pi\sqrt{h}} S''_{\text{out}}$$

$S_{\text{out}} = S_{\text{out}}[X^a]$ is a functional of the coordinates X^a of the entangling surface.



What does S''_{out} mean?

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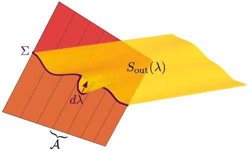

What does S''_{out} mean? The second (variational) derivative of S_{out} is a matrix; use a local basis for variations.

- Off-diagonal variations: $\sup(\delta_1 X^a) \cap \sup(\delta_2 X^a) = \emptyset$.
- Diagonal variations: $\delta_1 X^a, \delta_2 X^a \propto \delta(X^a - y^a)$ (at same pt).
- S''_{out} means a diagonal variation (in a null direction).

Off-diagonal variations of entanglement entropy

Off-diagonal variations (*entanglement density* or *entanglement susceptibility*) are also interesting:

Entanglement Susceptibility

$$S''_{\text{off-diagonal}}(y_1, y_2) := s^a s^b \left. \frac{\delta^2 S}{\delta X^a(y_1)\delta X^b(y_2)} \right|_{y_1 \neq y_2}$$

Why? Every off-diagonal matrix element is non-positive, because of strong-subadditivity.

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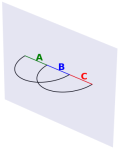
This can be enough to prove energy conditions in some perturbative cases (involving shockwaves) [Khandker, Kundu, Li].

The famous holographic argument for SSA

[Headrick, Takayanagi]



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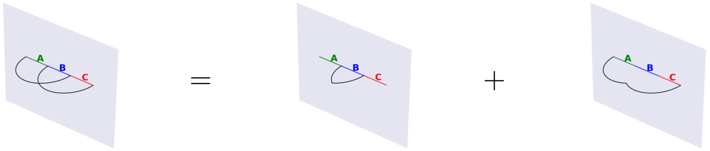

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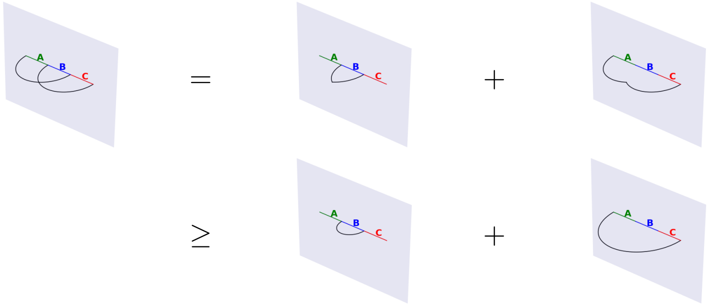

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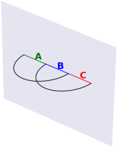

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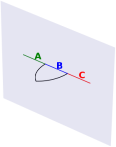

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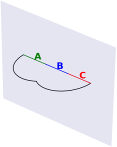

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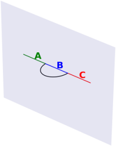

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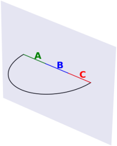

$$S(AB) + S(BC) \geq S(B) + S(ABC)$$

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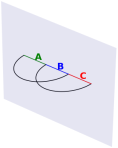

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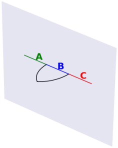

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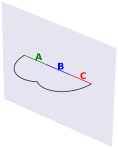

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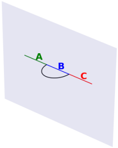

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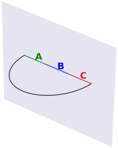


$$S(AB) + S(BC) \geq S(B) + S(ABC)$$

$$0 \geq S(ABC) - S(AB) - S(BC) + S(B)$$

$$A, C \text{ tiny perturbations of } B \implies s^a s^b \left. \frac{\delta^2 S}{\delta X^a(y_1)\delta X^b(y_2)} \right|_{y_1 \neq y_2} \leq 0$$

Maximal volume slices

Maximal volume slices are also believed to play an important role in holography, e.g. CV conjecture:

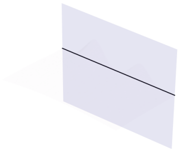
$$\mathcal{C} \sim \frac{V_{\max}}{G_N \ell}$$

Some similar properties to entanglement entropy:

- Leading divergences are local, geometrical
- Obeys nesting
- Strong **super**additivity relation

[Carmi, Myers, Rath; Carmi; Couch, Eccles, Jacobson, Nguyen]

Holographic Volume Superadditivity



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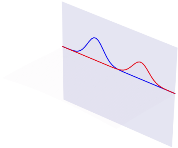

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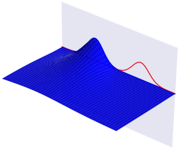

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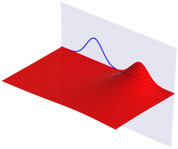

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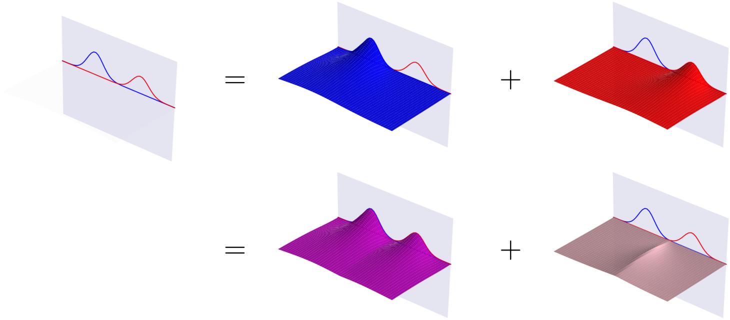

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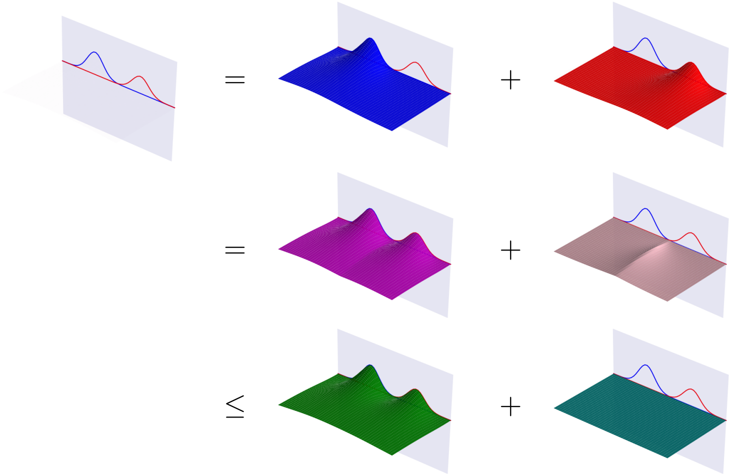

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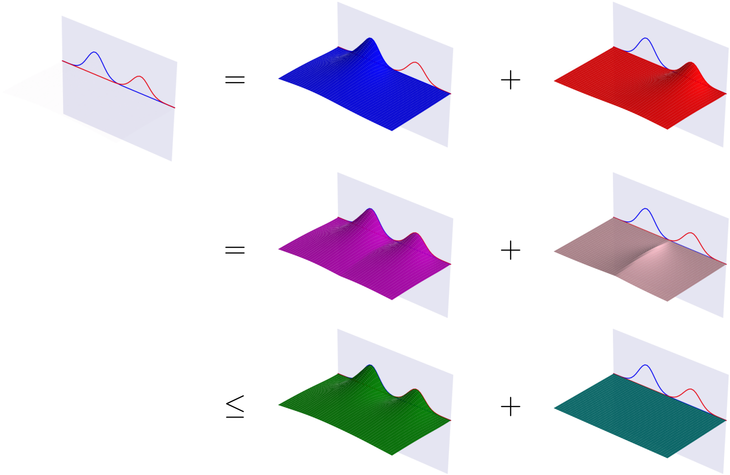

$$V_1 + V_2 \leq V_{1+2} + V_0$$

$$\text{tiny perturbations} \implies t^a t^b \left. \frac{\delta^2 V_{\max}}{\delta X^a(y_1)\delta X^b(y_2)} \right|_{y_1 \neq y_2} \geq 0$$

Off-diagonal variations of volume

Perform a similar variation of boundary conditions for maximal volume surfaces.

Holographic Volume Susceptibility

The non-local contribution to the volume at one point, due to variations at another point, as we did for entropy.

$$V''_{\text{off-diagonal}}(y_1, y_2) := t^a t^b \left. \frac{\delta^2 V_{\max}}{\delta X^a_\partial(y_1) \delta X^b_\partial(y_2)} \right|_{y_1 \neq y_2}$$

Now the maximal volume susceptibility is **positive**, by strong **super**additivity.

Example: Perturbations around vacuum

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In Fefferman-Graham coordinates,

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- Let the maximal volume surface σ be given by $t = T(\vec{y}, z)$.
- Assume time reflection symmetry: $T = 0$ is our initial maximal volume surface.
- Perturb the boundary conditions $\delta T(\vec{y}, z = 0) \equiv \delta t(\vec{y})$, and expand volume functional around 0:

$$\delta V[\delta t] = \int_{\sigma[\delta t]} \delta\sqrt{H} = \int_{\sigma[\delta t]} (\cdots)\, \delta T + \int_{\sigma[\delta t]} (\cdots)\, \delta T \delta T + \dots$$

- Impose e.o.m.: δT drops out, $\delta T \delta T$ term localizes to boundary (Hamilton-Jacobi).

Perturbations around vacuum

Result: Susceptibility in perturbed space

$$\frac{\delta^2 V}{\delta t(y_1)\delta t(y_2)} = \frac{1}{2}\left(1 - \frac{8\pi G_N}{d}\rho(y_1) z_0^d\right) g(y_1, z|y_2)$$
$$+ \frac{1}{4}\int d^{d-1}y\, dz\, z^{2-d}\left(\delta^{ab}h_{tt} + 2\delta^{ac}\delta^{be}h_{ce}\right)\times$$
$$\times \partial_a G(y, z|y_1, z_0)\, \partial_b G(y, z|y_2, z_0)$$

($g =$ bulk-bd'y prop; $G =$ bulk-bulk; $\rho = \langle T_{tt}\rangle$, $z_0 =$ cutoff.)

First term: diagonal (δ in the limit $z_0 \to 0$).
Second term: off-diagonal, $\implies \geq 0$.
Integrate second term over y_1 to obtain

$$0 \leq \int d^{d-1}y_1 \left(\frac{\delta^2 V}{\delta t(y_1)\delta t(y_2)}\right)_{\text{off-diag}} = z_0^d \frac{8\pi G_N}{d}\rho(y_2) + (\text{subleading})$$

How did we get $\langle T_{tt} \rangle \geq 0$?

- Famously, in all QFTs, $\exists$ states that violate $\langle T_{tt} \rangle \geq 0$.
  - Reflections off moving mirrors
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- Casimir effect?
  - Solution: conformal anomaly is subtracted from boundary stress tensor in the Fefferman-Graham expansion
  - This bound concerns additional excitations above the anomalous (Casimir) energy.

Summary

Holographic volume susceptibility

- A new quantity to study in field theory and holography.
- Strong superadditivity $\implies$ Positivity of susceptibility
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