Holographic Entropy Arrangement

Veronika Hubeny

UC DAVIS
UNIVERSITY OF CALIFORNIA

Physics Department & center for Quantum Mathematics and Physics

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Quantum Information and String Theory
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It from Qubit
Simons Collaboration on Quantum Fields, Gravity and Information

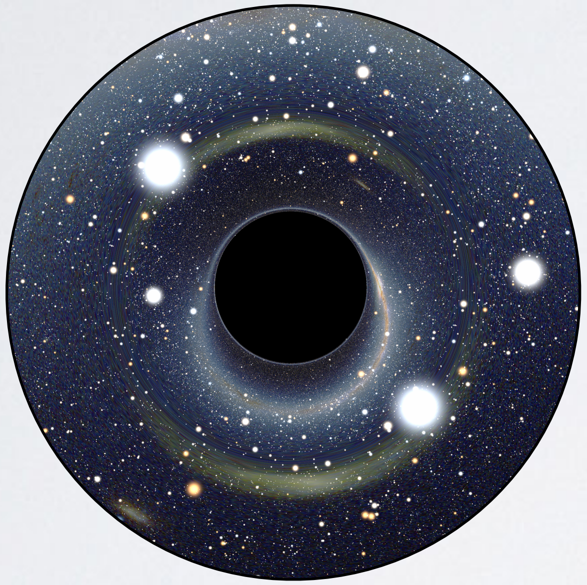
[based mainly on 1808.07871, 1812.08133 w/ Mukund Rangamani & Max Rota
+ w.i.p. w/ Sergio Hernandez-Cuenca, Mukund Rangamani, Max Rota
+ in part on 1905.06985 w/ Temple He & Matt Headrick]

AdS/CFT after 21 years

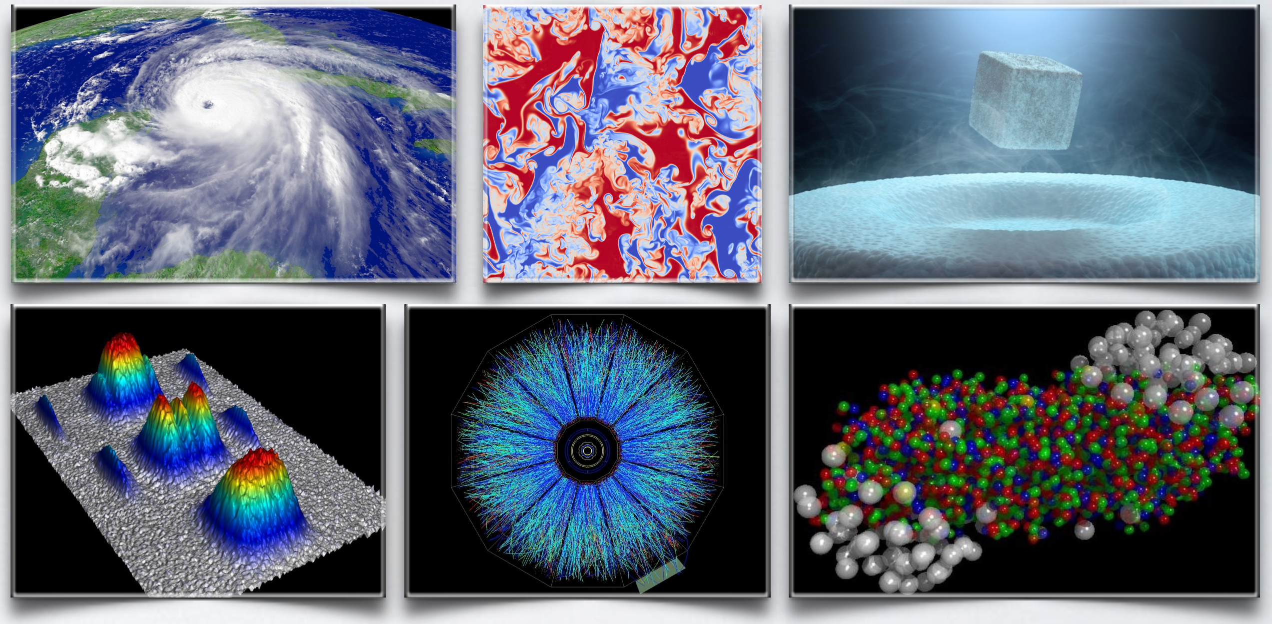
String theory (gravity) $\Longleftrightarrow$ field theory (no gravity)

"in bulk" = higher dimensions — *"on boundary" = lower dimensions*

describes gravitating systems, e.g. black holes



describes experimentally accessible systems



Invaluable tool to:

- Study strongly interacting field theory (hard, but describes many systems) by working with higher-dimensional gravity on AdS (easy).
- Study quantum gravity in AdS (hard, but needed to understand spacetime) by using the field theory (easy for certain things)

Pre-requisite:

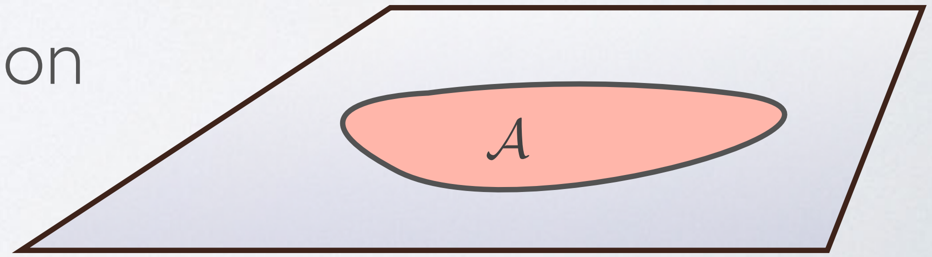
We need to understand the AdS/CFT dictionary…

- How does bulk spacetime emerge from the CFT?
  - Which CFT quantities give the bulk metric?
  - What determines bulk dynamics (Einstein's eq.)?
  - How does one recover a local bulk operator from CFT quantities?
- What part of bulk can we recover from a restricted CFT info?
  - What bulk region does a CFT state (at a given instant in time) encode?
  - What bulk region does a spatial subregion of CFT state encode?
- (How) does the CFT "see" inside a black hole?
  - Does it unitarily describe black hole formation & evaporation process?
  - How does it resolve curvature singularities?

Recent hints / expectations: entanglement plays a crucial role…

Entanglement entropy

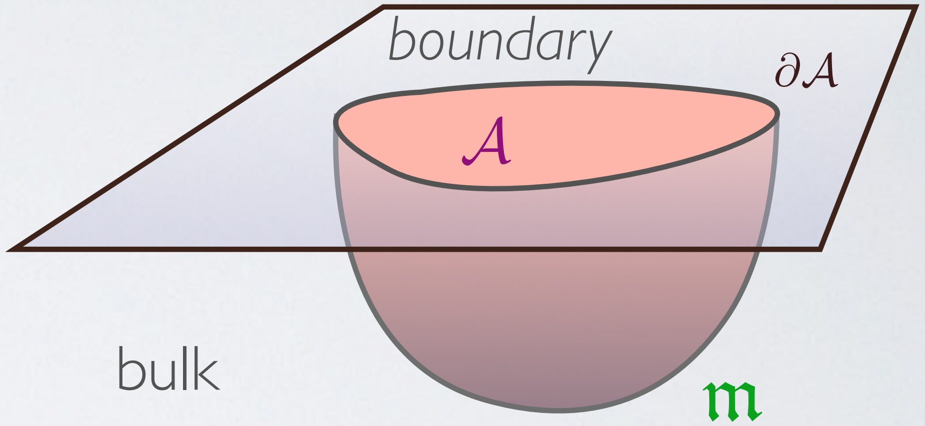
- Entanglement
  - Most non-classical manifestation of QM
  - Quantum resource for performing tasks which can't be performed using classical resources
  - Plays increasingly central role in Quantum Information, Q. many body systems, QFT & even QG!
- Entanglement entropy
  - Natural measure of entanglement of subsystem $\mathcal{A}$ w/ $\mathcal{H} = \mathcal{H}_{\mathcal{A}} \otimes \mathcal{H}_{\bar{\mathcal{A}}}$
  - In full state ρ , the reduced density matrix for $\mathcal{A}$ is $\rho_{\mathcal{A}} = \mathrm{Tr}_{\bar{\mathcal{A}}}\, \rho$
  - Then EE $S_{\mathcal{A}}$ corresponds to the measure of mixedness of $\rho_{\mathcal{A}}$, i.e. EE = the von Neumann entropy $S_{\mathcal{A}} = -\mathrm{Tr}\rho_{\mathcal{A}} \log \rho_{\mathcal{A}}$
- e.g. in a local QFT:
  - Choose subsystem $\mathcal{A}$ to be a local region



Holographic Entanglement Entropy

Proposal [RT=Ryu & Takayanagi, '06] for *static* configurations:

In the bulk, entanglement entropy $S_{\mathcal{A}}$ for a boundary region $\mathcal{A}$ is captured by the area of a minimal co-dimension-2 bulk surface $\mathfrak{m}$ *at constant t* anchored on entangling surface $\partial\mathcal{A}$ & homologous to $\mathcal{A}$

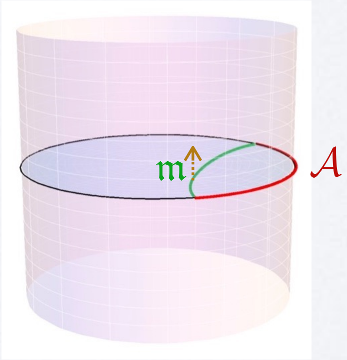


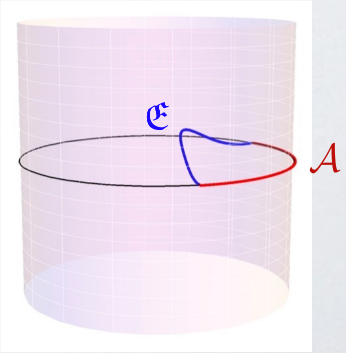
$$S_{\mathcal{A}} = \min_{\partial\mathfrak{m}=\partial\mathcal{A}} \frac{\text{Area}(\mathfrak{m})}{4\,G_N}$$

In *time-dependent* situations, RT prescription needs to be covariantized:

[HRT = VH, Rangamani, Takayanagi '07]

minimal surface $\mathfrak{m}$ at constant time → extremal surface $\mathfrak{E}$ in the full bulk





This gives a well-defined quantity in any (arbitrarily time-dependent asymptotically AdS) spacetime.

Entanglement relations

- Universal:
  - Sub-additivity (SA) $S(A) + S(B) \geq S(AB)$
  - Araki-Lieb (AL) $S(A) + S(AB) \geq S(B)$
  - Strong sub-additivity (SSA) $S(AB) + S(BC) \geq S(B) + S(ABC)$
  - Weak monotonicity (WM) $S(AB) + S(BC) \geq S(A) + S(C)$

- True in holography:
  - Monogamy of mutual information (MMI)

$$S(AB) + S(BC) + S(CA) \geq S(A) + S(B) + S(C) + S(ABC)$$

  - 5-party cyclic inequality (C5)

$$\begin{aligned} &S(ABC) + S(BCD) + S(CDE) + S(DEA) + S(EAB) \\ &\qquad \geq S(AB) + S(BC) + S(CD) + S(DE) + S(EA) + S(ABCDE) \end{aligned}$$

  - + four further 5-party relations [Bao, Nezami, Ooguri, Stoica, Sully, Walter '15; Hernandez-Cuenca '19]
  - k-region cyclic inequality (Ck) for k=odd is obvious...

QI interpretation

- Universal:
  - Sub-additivity (SA) $S(A)+S(B)\geq S(AB)$

    ⇨ Mutual Information $I(A:B)\equiv S(A)+S(B)-S(AB)\geq 0$
  - Strong sub-additivity (SSA) $S(AB)+S(BC)\geq S(B)+S(ABC)$

    ⇨ Conditional mutual information $I(A:C|B)\equiv I(A:BC)-I(A:B)\geq 0$
- True in holography:
  - Monogamy of mutual information (MMI)

$$S(AB)+S(BC)+S(CA)\geq S(A)+S(B)+S(C)+S(ABC)$$

⇨ Tripartite information $I3(A:B:C)\equiv I(A:B)+I(A:C)-I(A:BC)\leq 0$

→ gives interesting structure information on nature of entanglement in holography cf. [Hayden, Headrick, Maloney]

Our Goal

1) Obtain a full set of information quantities $\{Q(A:B:C:\cdots)\}$
for arbitrary number N of parties ("colors")

- Want information quantities which:
  - arise as linear combinations of entanglement entropies of subsystems
  - can vanish for some configurations in geometric states in holographic CFTs (we'll call these *faithful*)
  - are independent of other faithful IQs (we'll call these *primitive*)
- These are characterized by ***holographic entropy arrangement***
  - = set of all primitive information quantities Q
  - consists of hyperplanes in *entropy space*

NB: distinct from Mutual Information Arrangement mentioned by Rota

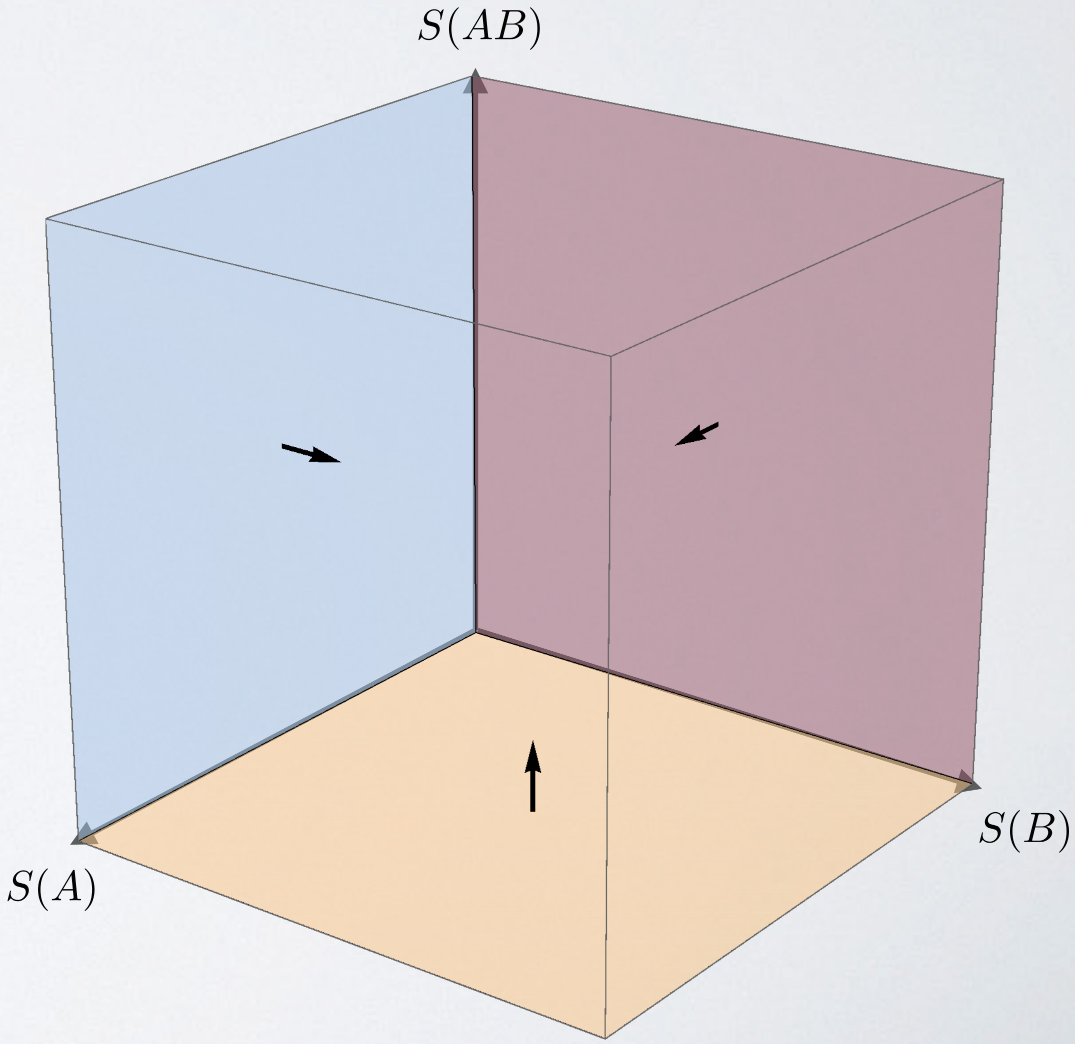
2) Obtain a full set of *universal holographic inequalities*

- These generate the ***holographic entropy polyhedron***
  - obtain a candidate list of inequalities by using a *sieve* on arrangement
  - prove directly?

Entropy space for 2 parties

- Define all entanglement entropies
  - Consider partitioning of Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_{\overline{AB}}$
  - Independent EEs → entropy vector $\vec{S} = \{S(A), S(B), S(AB)\}$
  - Lives in entropy space $\mathbb{R}^3$

- Entanglement Relations
  - Positivity of EEs $S(X) \geq 0$



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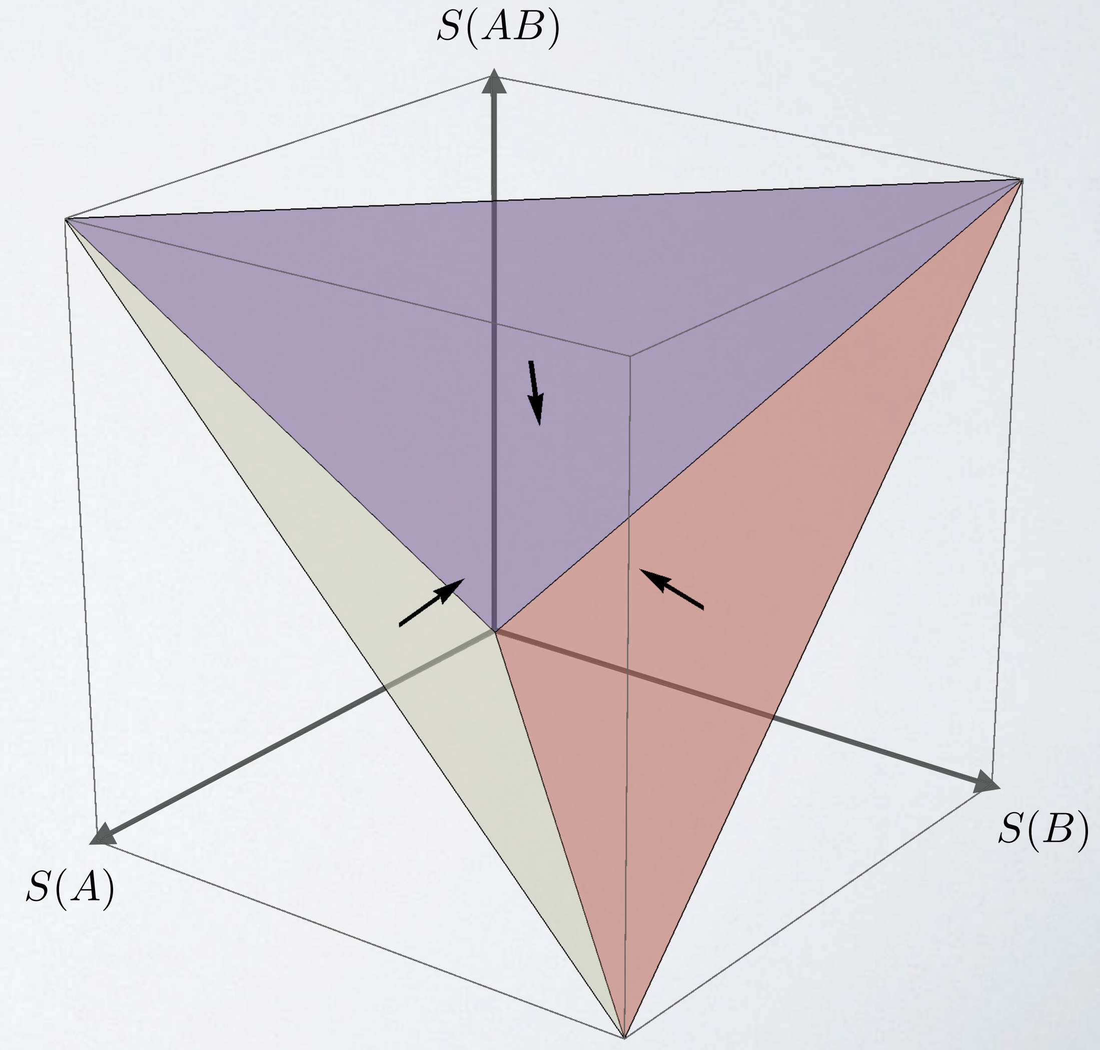
  - Positivity of EEs $S(X) \geq 0$
  - SA $S(A) + S(B) \geq S(AB)$
  - AL_1 $S(A) + S(AB) \geq S(B)$
  - AL_2 $S(B) + S(AB) \geq S(A)$
  - positivity of EE is redundant…
  - $\text{SA}+\text{AL}_1+\text{AL}_2$ form entropy cone

    = holographic entropy polyhedron



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  - Positivity of EEs $S(X) \geq 0$
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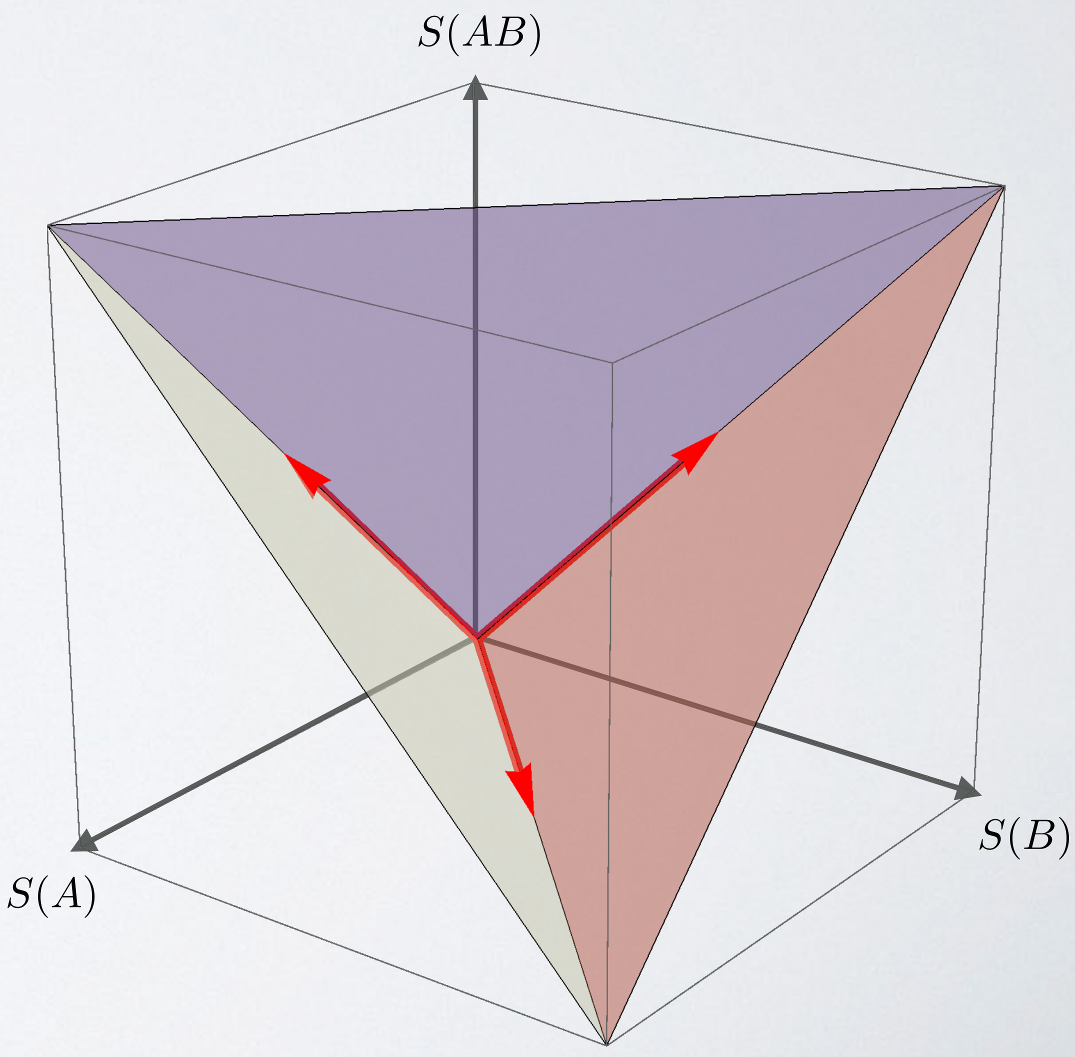
  - AL$_2$ $S(B) + S(AB) \geq S(A)$
  - positivity of EE is redundant…
  - SA+AL$_1$+AL$_2$ form entropy cone
  - specified by 'extreme rays'



Entropy space for 3 parties

- Partition Hilbert space

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C \otimes \mathcal{H}_{\overline{ABC}}$$

- Entropy space is $\mathbb{R}^7$:
- Entropy vector:

$$\vec{S} = \{S(A), S(B), S(C), S(AB), S(AC), S(BC), S(ABC)\}$$

- General form of *information quantity* (= entanglement entropy relation)

$$Q(\vec{S}) = q_A\, S(A) + q_B\, S(B) + q_C\, S(C) + q_{AB}\, S(AB) + q_{AC}\, S(AC) + q_{BC}\, S(BC) + q_{ABC}\, S(ABC)$$

rational coefficients

Entropy space for 3 parties

- Partition Hilbert space

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C \otimes \mathcal{H}_{\overline{ABC}}$$

- Entropy space is $\mathbb{R}^7$:
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$$Q(\vec{S}) = q_A\, S(A) + q_B\, S(B) + q_C\, S(C) + q_{AB}\, S(AB) + q_{AC}\, S(AC) + q_{BC}\, S(BC) + q_{ABC}\, S(ABC)$$

- Entropy relations (equalities) are specified by *hyperplanes* in entropy space:

$$Q(\vec{S}) = 0$$

Entropy space for N parties

- Partition Hilbert space into N factors
- Entropy space is $\mathbb{R}^D$ with $D = 2^N - 1$
- Entropy vector $\vec{S} = \{S(X)\}$ where X is any collection of parties
- General form of *information quantity*

$$Q(\vec{S}) = \sum_X q_X \, S(X)$$

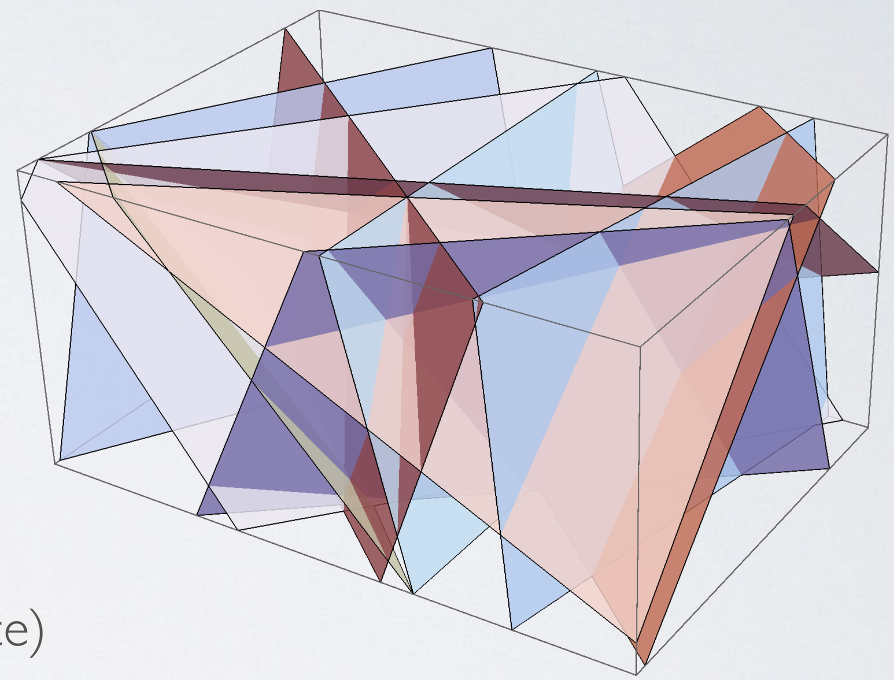
(D terms)

- Entropy relations specified by *hyperplanes* in entropy space:

$$Q(\vec{S}) = 0$$

Set of information quantities

- Mathematical framework to study information quantities describing interesting EE relations



= *arrangement of hyperplanes*

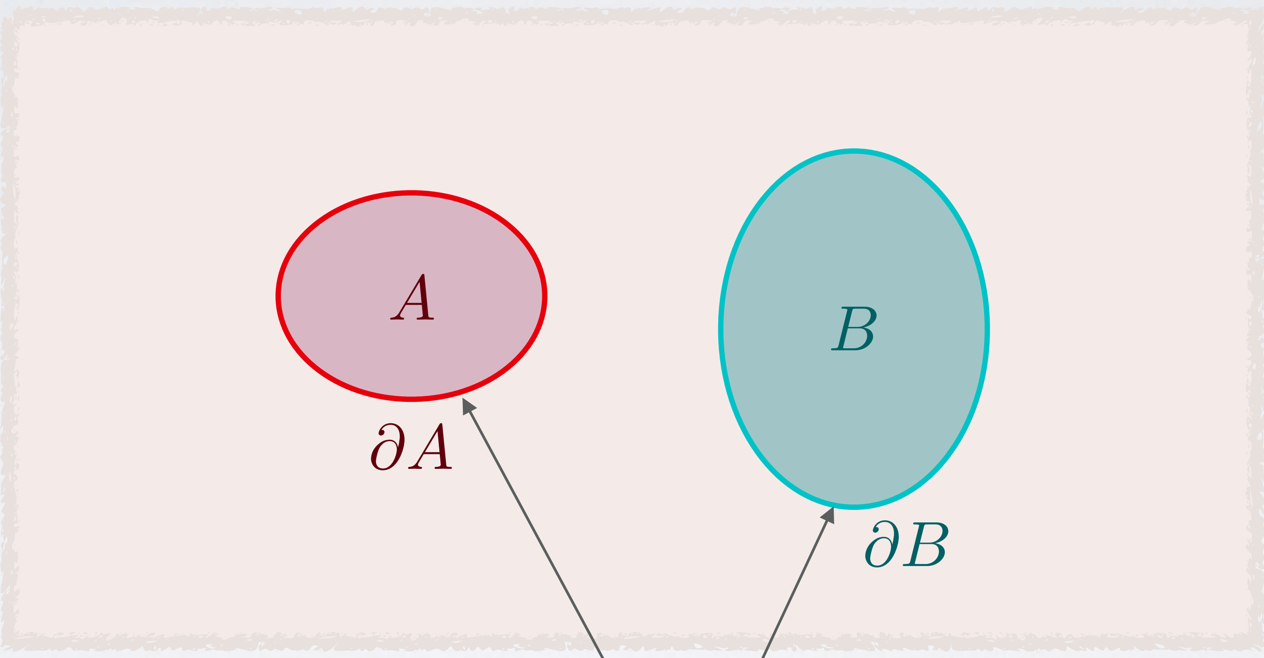
- In our case the arrangement is:
  - finite (→ consists of finite # of hyperplanes)
  - essential (→ {normal vectors} span the full entropy space)
  - central (→ {hyperplanes} intersect only at the origin)
  - symmetric (under permutations of colors & purifier)

- To obtain holographic entropy polyhedron, we additionally need to specify hyperplane orientation (where possible)

cf. holographic entropy cone of [Bao, Nezami, Ooguri, Stoica, Sully, Walter '15]

Entanglement in QFT

- Natural 'decomposition' of Hilbert space = spatial regions



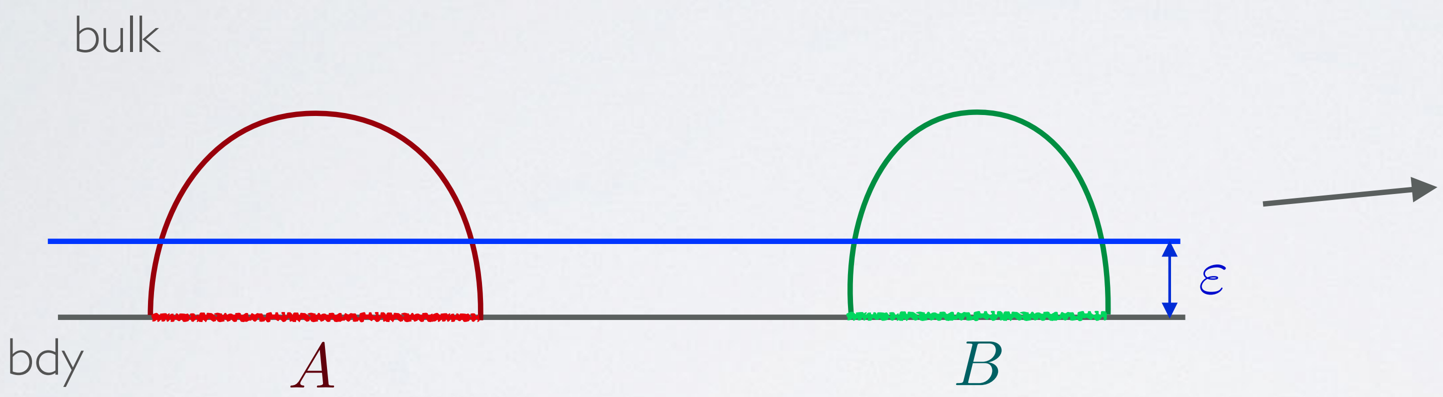
  - bounded by entangling surfaces (later denoted by $\partial A \equiv$ a and $\partial B \equiv$ b)

- Entanglement entropy has a UV divergence
  - $\sim$ area of entangling surface

Position in entropy space

- Two options to ‘localize’ a configuration in entropy space:

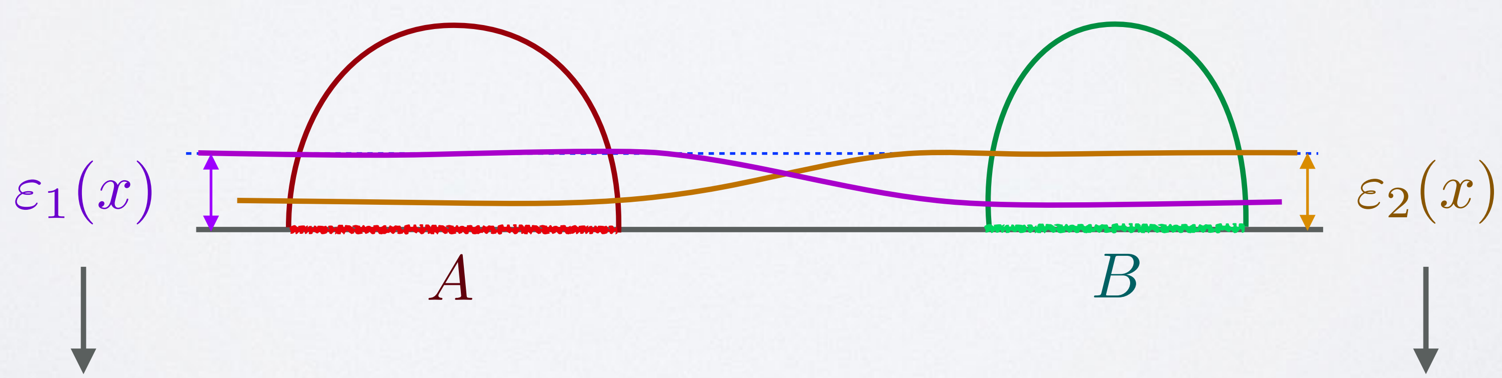
1) Introduce a UV regulator:



finite area ‘above’ ε

- But position (& even direction) in entropy space is cutoff-dependent:

cf:



$S(B)$ increases at const. $S(A)$

$S(A)$ increases at const. $S(B)$

Position in entropy space

- Two options to 'localize' a configuration in entropy space:

2) Consider multi-boundary wormholes:

e.g.:

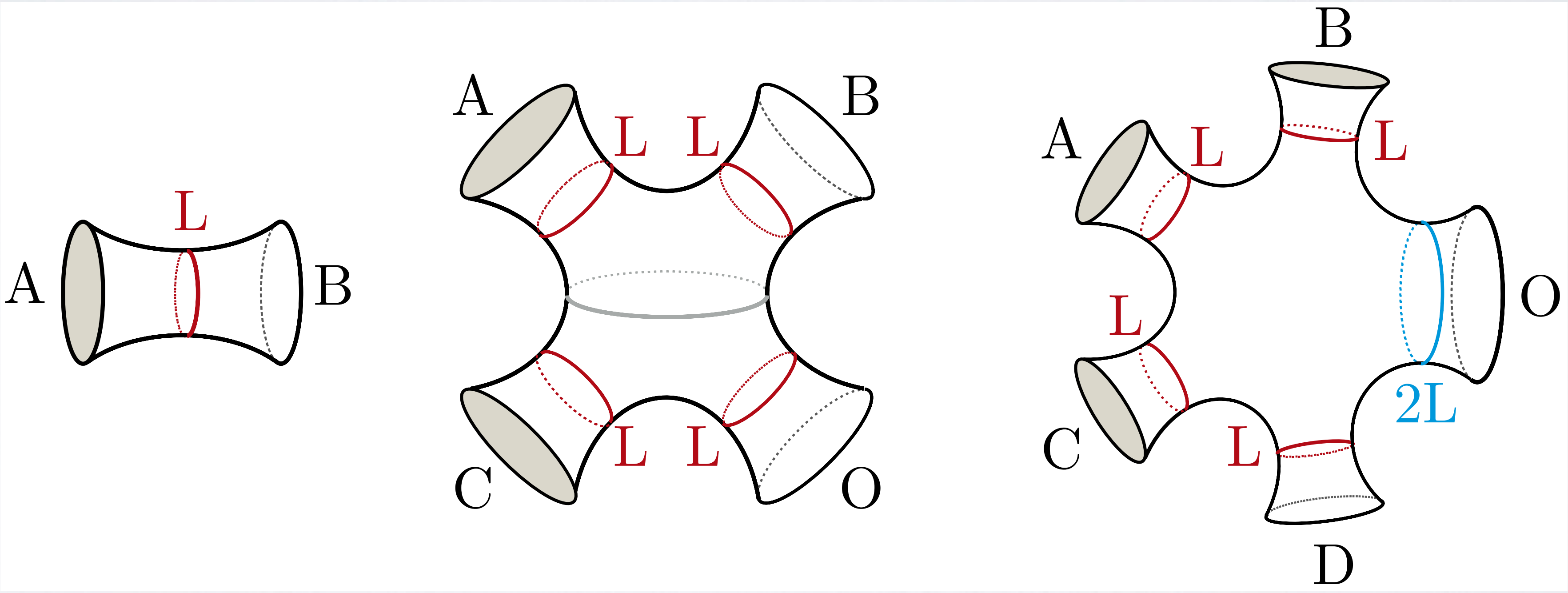


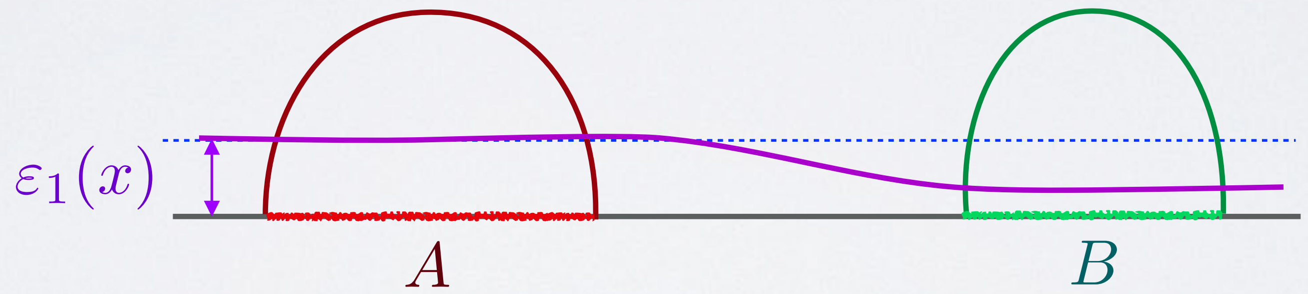
Fig. from [Bao, Nezami, Ooguri, Stoica, Sully, Walter]

Each region covers one entire bdy (so ∄ entangling surfs)

- But requires multiple CFTs…

Proto-entropy

- However, certain combinations of EEs (information quantities) are UV-finite
  - e.g. for disjoint regions, any "balanced" IQ is UV-finite
  - Ex.: saturation of SA: $S(A) + S(B) = S(AB)$



same parts of surfaces appear on both sides of the equality
$\Rightarrow$ cancel out independently of the cutoff

$\Rightarrow$ under varying cutoff, vectors $\vec{S}_{\varepsilon(x)}$ span lower-dimensional subspace of entropy space.

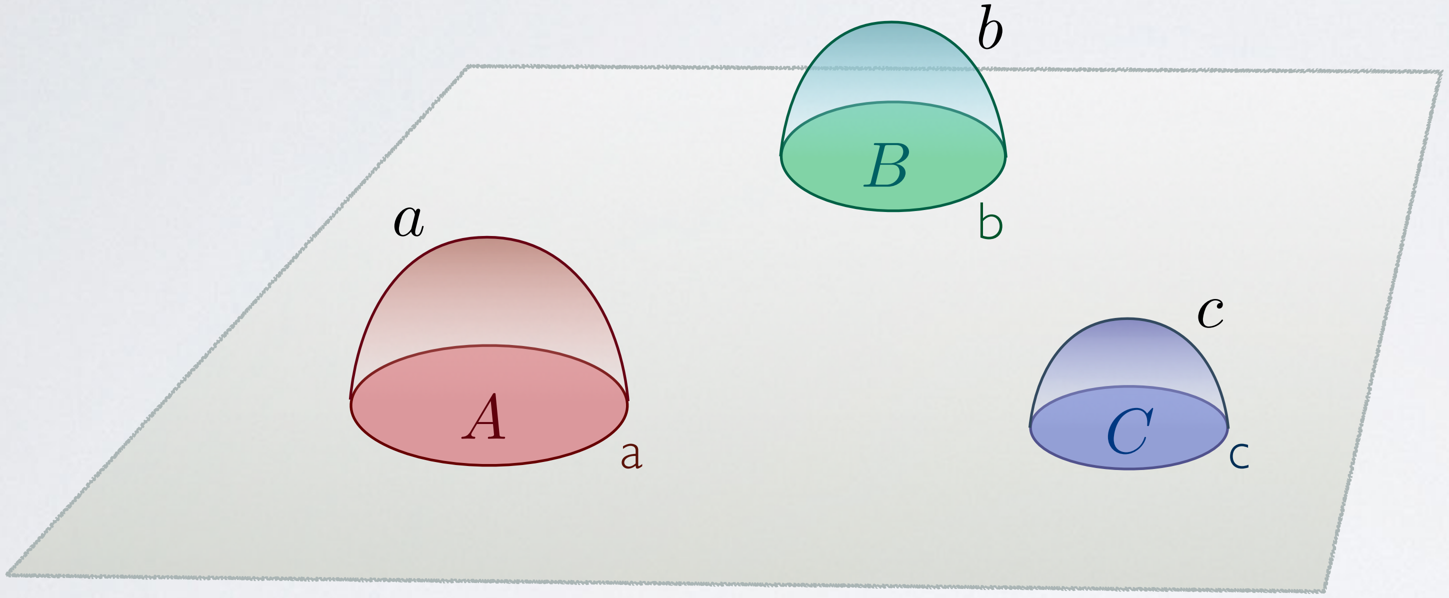
- Suggests hyperplanes are the natural / fundamental constructs
  - Think of QI relations as combinations of surfaces (= ***proto-entropy***), rather than their areas…

Strategy

- Work in space of bulk extremal (HRT) surfaces
  - Consider proto-entropy rather than entropy itself
- This recasts the geometric problem into an algebraic one
  - Shapes & areas of surfaces irrelevant, only their 'existence' (i.e. being invoked by HRT) matters
  - (NB: complementary viewpoint to bit thread picture)
  - A-priori still hard problem: scan over all configurations in all geometric states for all holographic CFTs
- Simplification
  - Suffices to consider vacuum state of CFT on $\mathbb{R}^{2,1}$
  - Suffices to consider equivalence classes of configurations (generating the same information quantities)

Building up hyperplanes for N=3

- Consider simplest configuration w/ 3 uncorrelated regions
  - 3 entangling surfaces: a,b,c
  - 3 bulk surfaces, called correspondingly a, b, c



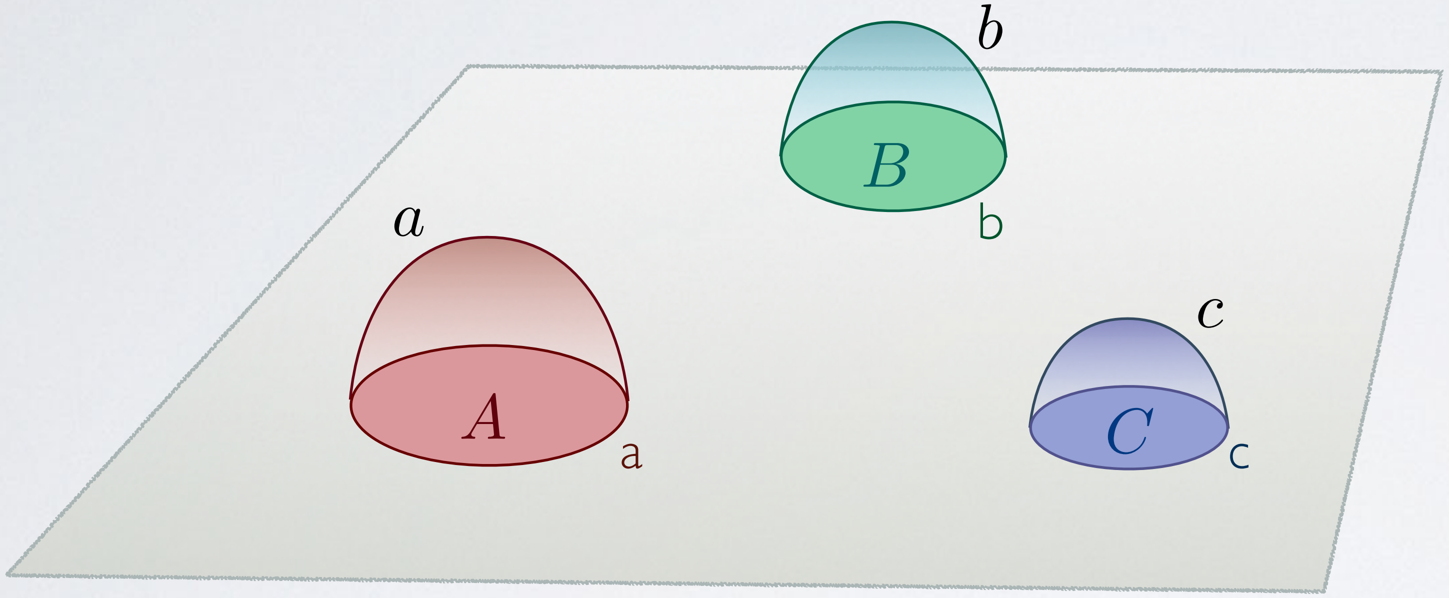
  - Construct entropy vector

S(.)	A	B	C	AB	AC	BC	ABC
a	1						
b							
c							

$$S(A) = \frac{1}{4G_N}\text{Area}[a]$$

Building up hyperplanes for N=3

- Consider simplest configuration w/ 3 uncorrelated regions
  - 3 entangling surfaces: a,b,c
  - 3 bulk surfaces, called correspondingly a, b, c



- Construct entropy vector & read off corresponding q relations:

S(.)	A	B	C	AB	AC	BC	ABC
a	1			1	1		1
b		1		1		1	1
c			1		1	1	1

$$\rightarrow \underbrace{q_A + q_{AB} + q_{AC} + q_{ABC}}_{\text{all terms involving A } := \alpha} = 0$$

Building up hyperplanes for N=3

Why ? Recall:

$$Q(\vec{S}) = q_A\, S(A) + q_B\, S(B) + q_C\, S(C) + q_{AB}\, S(AB) + q_{AC}\, S(AC) + q_{BC}\, S(BC) + q_{ABC}\, S(ABC)$$

$$= q_A\, a + q_B\, b + q_C\, c + q_{AB}\,(a+b) + q_{AC}\,(a+c) + q_{BC}\,(b+c) + q_{ABC}\,(a+b+c)$$

$$= a\,\underbrace{(q_A + q_{AB} + q_{AC} + q_{ABC})} + b\,\underbrace{(q_B + q_{AB} + q_{BC} + q_{ABC})} + c\,\underbrace{(q_C + q_{AC} + q_{BC} + q_{ABC})}$$

$$= 0$$

must $= 0$ individually
since we can vary $a, b,$ and c independently

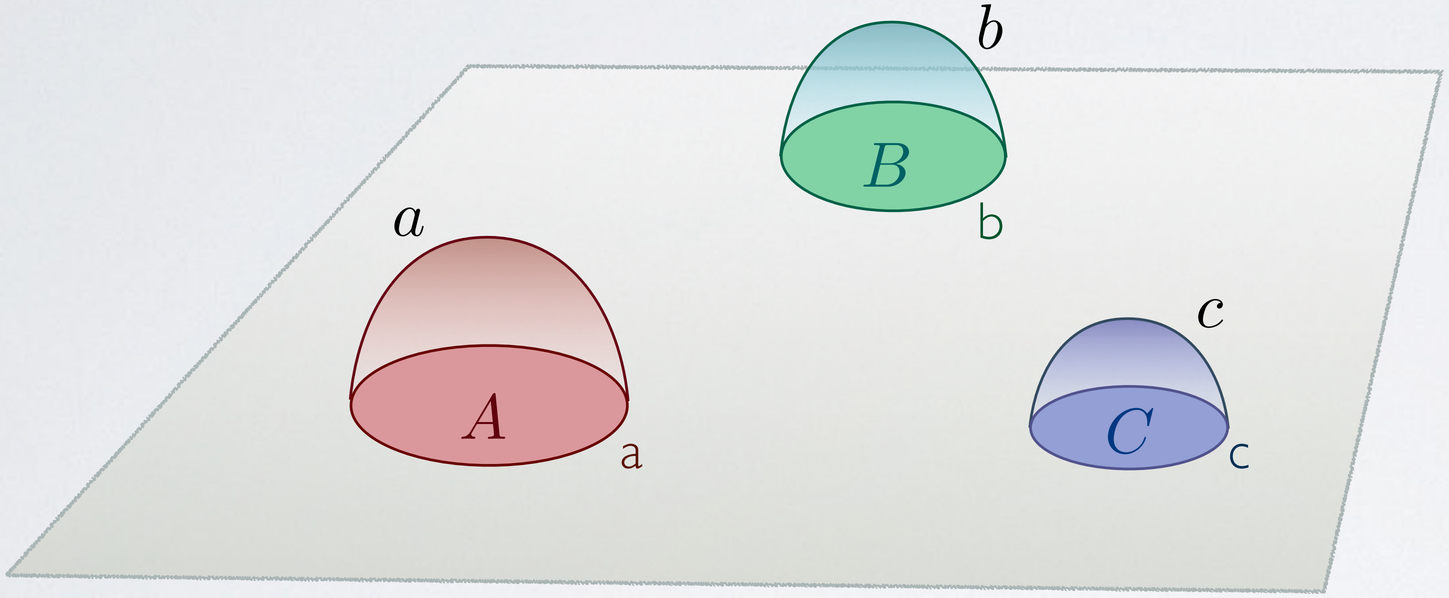
- Construct entropy vector & read off corresponding q relations:

S(.)	A	B	C	AB	AC	BC	ABC
a	1			1	1		1
b		1		1		1	1
c			1		1	1	1

$\rightarrow$ $\underbrace{q_A + q_{AB} + q_{AC} + q_{ABC}}_{\text{all terms involving A}} = 0$

Building up hyperplanes for N=3

- Consider simplest configuration w/ 3 uncorrelated regions
  - 3 entangling surfaces: a,b,c
  - 3 bulk surfaces, called correspondingly a, b, c



- Construct entropy vector & read off corresponding q relations:

S(.)	A	B	C	AB	AC	BC	ABC
a	1			1	1		1
b		1		1		1	1
c			1		1	1	1

$\rightarrow \quad q_A + q_{AB} + q_{AC} + q_{ABC} = 0$

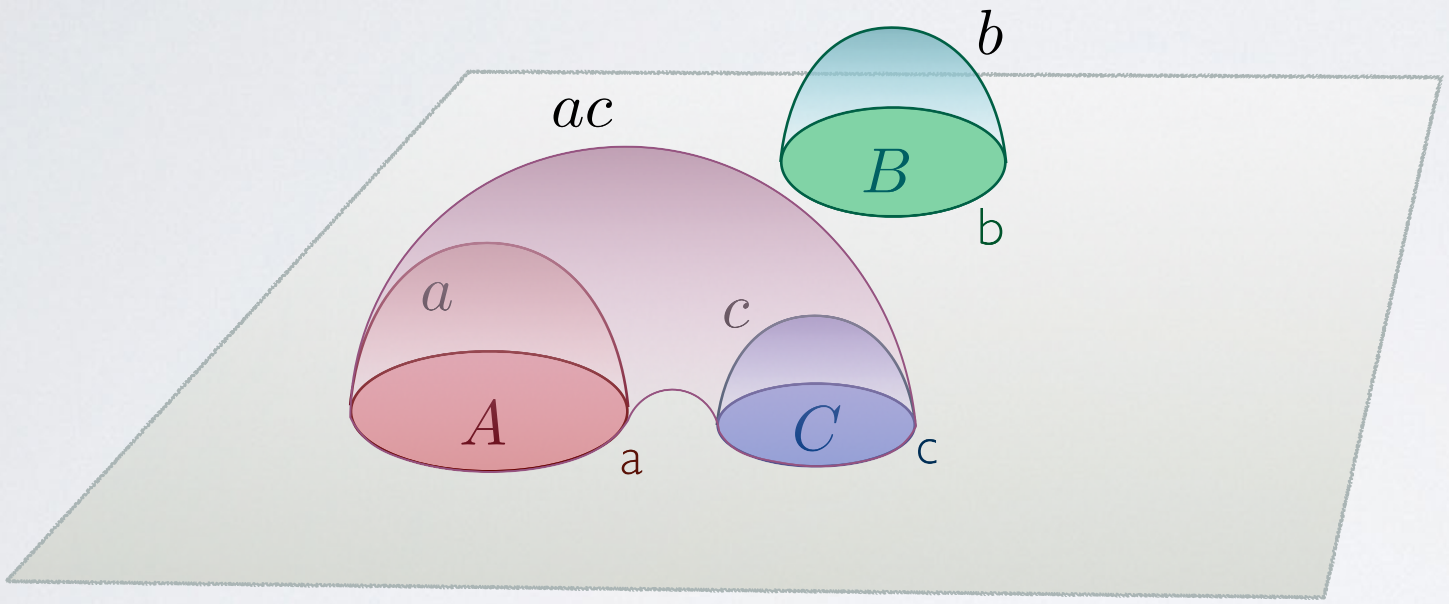
$\rightarrow \quad q_B + q_{AB} + q_{BC} + q_{ABC} = 0$

$\rightarrow \quad q_C + q_{AC} + q_{BC} + q_{ABC} = 0$

- 3 eqns for 7 unknowns ⇒ not sufficient to get a hyperplane…

Building up hyperplanes for N=3

- Add more surfaces by correlating regions (e.g. A & C)
  - still 3 entangling surfaces: a,b,c
  - but now 4 bulk surfaces, a, b, c and extra one, called ac



label by all entangling surfaces the bulk surf. is anchored on

- Gives extra row to entanglement table:

Still insufficient for hyperplane…

S(.)	A	B	C	AB	AC	BC	ABC
a	1			1	0		0
b		1		1		1	1
c			1		0	1	0
ac					1		1

→ $q_A + q_{AB} = 0$

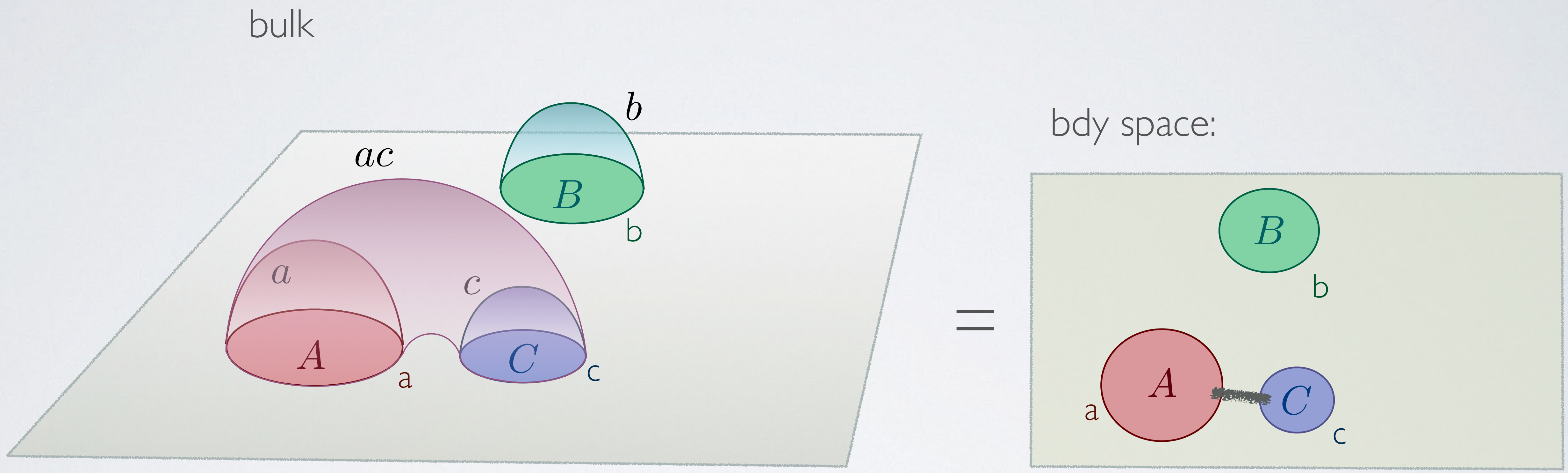
→ $q_B + q_{AB} + q_{BC} + q_{ABC} = 0$

→ $q_C + q_{BC} = 0$

→ $q_{AC} + q_{ABC} = 0$

Building up hyperplanes for N=3

- Introduce notation to denote correlation:

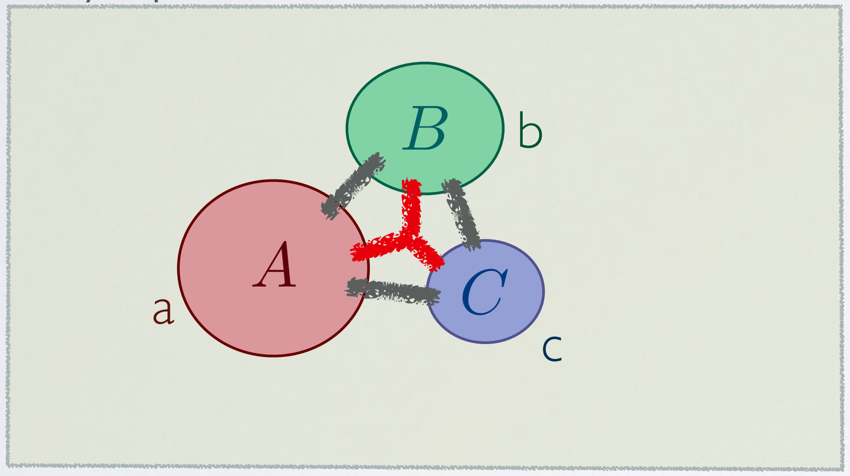


- Depicts a *configuration* in the CFT

Building up hyperplanes for N=3

- Consider fully correlated configuration
  - still 3 entangling surfaces: a,b,c
  - but now 7 bulk surfaces: $a, b, c, ab, ac, bc,$ and abc

bdy space:



→

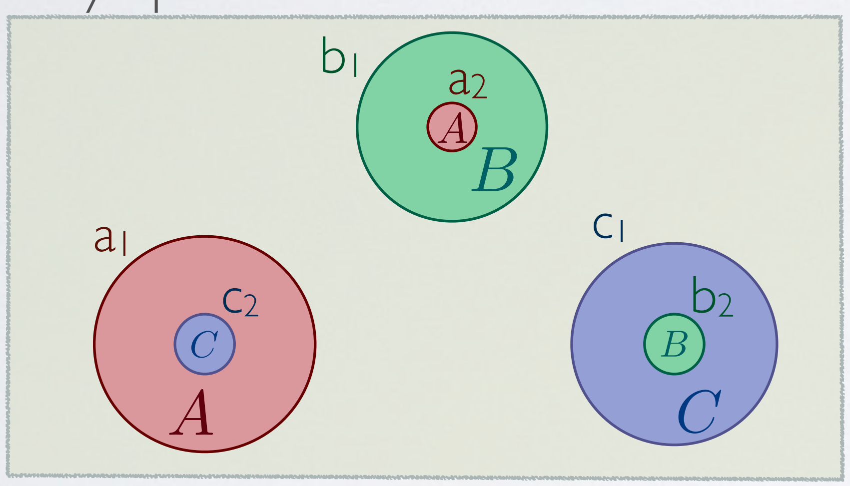
S(.)	A	B	C	AB	AC	BC	ABC
a	1						
b		1					
c			1				
ab				1			
ac					1		
bc						1	
abc							1

- now 7 eqns for 7 unknowns ⇒ all q_X's trivially vanish…

Building up hyperplanes for N=3

- Try correlated configuration w/ 1 less bulk surface:
  - now 6 entangling surfaces: a_1, b_1, c_1, a_2, b_2, and c_2
  - and also 6 bulk surfaces:

bdy space:



S(.)	A	B	C	AB	AC	BC	ABC
a_1	1			1	1		1
a_2	1	1			1	1	
b_1		1		1		1	1
b_2		1	1	1	1		
c_1			1		1	1	1
c_2	1		1	1		1	

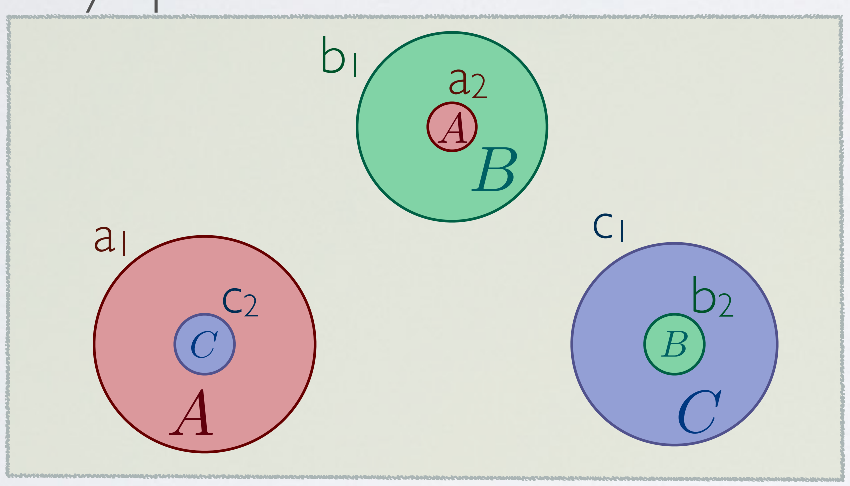
- now we DO get a hyperplane:

Building up hyperplanes for N=3

- Try correlated configuration w/ 1 less bulk surface:
  - now 6 entangling surfaces: a_1, b_1, c_1, a_2, b_2, and c_2
  - and now 6 bulk surfaces:

But we used enveloped regions, i.e. $\mathscr{L}=1$

bdy space:



S(.)	A	B	C	AB	AC	BC	ABC
a_1	1			1	1		1
a_2	1	1			1	1	
b_1		1		1		1	1
b_2		1	1	1	1		
c_1			1		1	1	1
c_2	1		1	1		1	
	+	+	+	-	-	-	+

- now we DO get a hyperplane:

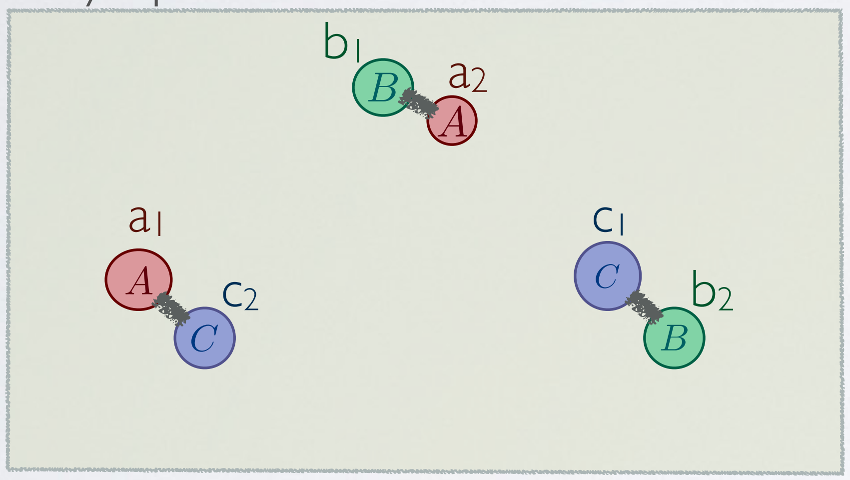
soln. for q's:

- Gives precisely $I_3(A:B:C)=0$ → MMI

Building up hyperplanes for N=3

- We can also do it at $\mathscr{L}$=0, i.e. without enveloping:
  - still 6 entangling surfaces: a_1, b_1, c_1, a_2, b_2, and c_2
  - and 9 bulk surfaces: $a_1,\ b_1,\ c_1,\ a_2,\ b_2,\ c_2,\ a_1c_2,\ b_1a_2,\ c_1b_2$

bdy space:



→

S(.)	A	B	C	AB	AC	BC	ABC
a_1	1			1			
a_2	1				1		
b_1		1				1	
b_2		1		1			
c_1			1		1		
c_2			1			1	
a_1c_2					1		1
b_1a_2				1			1
c_1b_2						1	1
	+	+	+	-	-	-	+

- despite 9 (=#relations) > 7 (=#unknowns), we still DO get a hyperplane:
- Again gives precisely $I_3(A:B:C)=0$ → MMI

Systematizing the search

1. Scan over all configuration classes
   - Consider disjoint regions (can generalize to adjoining as a limit…)
   - (Can abstract any configuration to a graph & identification of non-zero MIs)
   - Organize by enveloping level $\mathscr{L}$

2. Find the basic configuration "building blocks"
   - Start w/ simplest configuration (e.g. minimal # of entangling surfaces) and show when adding complications gives redundant relations

3. Combine building blocks in all possible ways to get hyperplanes
   - Need to build up $D-1$ independent relations between the q's (can be realized by a single configuration)

Canonical form of relations

- e.g. for N=4, $\mathscr{L}$=0: all relations can be rendered into a *canonical form*

Notation:

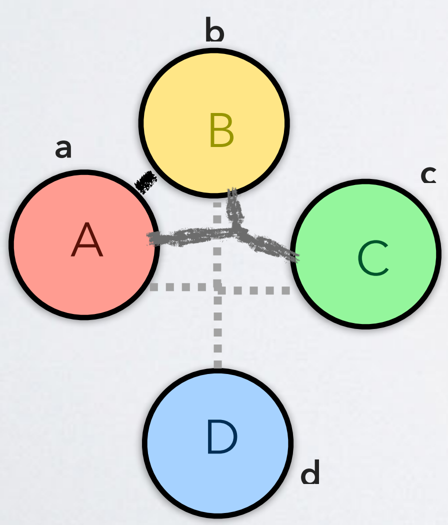
α := vanishing sum of all terms q_x w/ x including all occurrences of A,

i.e. $q_A + q_{AB} + q_{AC} + q_{AD} + q_{ABC} + q_{ABD} + q_{ACD} + q_{ABCD} = 0$

$\alpha\beta$:= vanishing sum of all terms q_x w/ x including all occurrences of A and simultaneously B

$\underline{\alpha}$:= vanishing sum of all terms q_x w/ x NOT including any occurrence of A

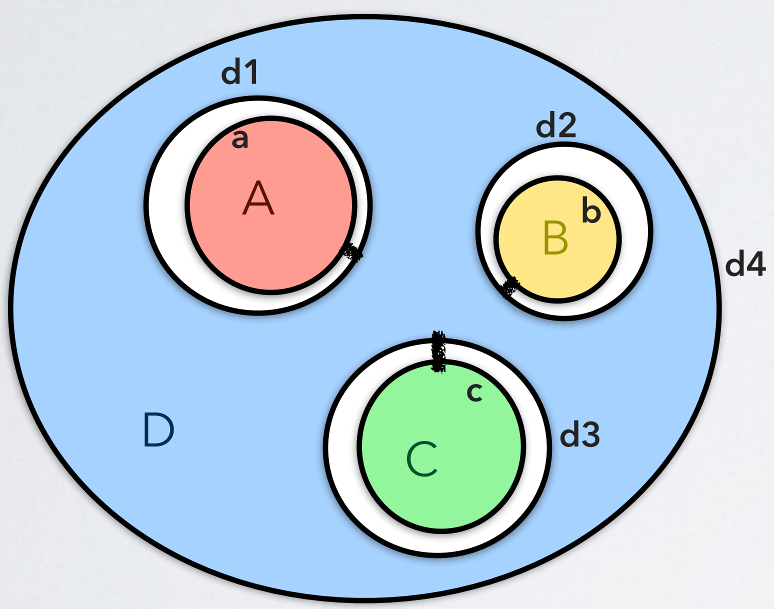
Canonical form is e.g.: α, $\alpha\beta$, $\beta\gamma\delta$, etc. but no exclusions like $\alpha\underline{\beta}$



	A	B	C	D	AB	AC	AD	BC	BD	CD	ABC	ABD	ACD	BCD	ABCD	raw relations	reln operation	new relations
a	1					1	1						1			$\alpha\underline{\beta}$	$+\alpha\beta$	α
b		1						1	1					1		$\beta\underline{\alpha}$	$+\alpha\beta$	β
c			1			1		1		1			1	1		$\gamma(\underline{\alpha\beta})$	$+\alpha\beta\gamma$	γ
d				1			1		1	1		1	1	1		$\delta(\underline{\alpha\beta\gamma})$	$+\alpha\beta\gamma\delta$	δ
ab					1							1				$\alpha\beta\underline{\gamma}$	$+\alpha\beta\gamma$	$\alpha\beta$
abc											1					$\alpha\beta\gamma\underline{\delta}$	$+\alpha\beta\gamma\delta$	$\alpha\beta\gamma$
abcd															1	$\alpha\beta\gamma\delta$		$\alpha\beta\gamma\delta$

Canonical form of relations

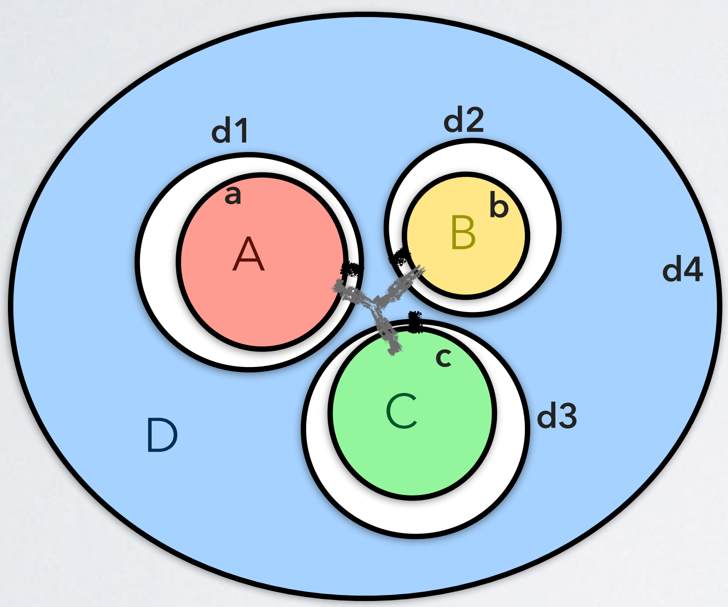
- e.g. for N=4, $\mathscr{L}$=1: i.e. w/ enveloped regions
  all relations can still be rendered into a canonical form



	A	B	C	D	AB	AC	AD	BC	BD	CD	ABC	ABD	ACD	BCD	ABCD	raw relations	operation	new relns
a	1				1	1					1					$\alpha\underline{\delta}$	$+\alpha\delta$	α
b		1			1			1			1					$\beta\underline{\delta}$	$+\beta\delta$	β
c			1			1		1			1					$\gamma\underline{\delta}$	$+\gamma\delta$	γ
d1				1					1	1				1		$\delta\underline{\alpha}$	$+\alpha\delta$	δ
d2				1			1			1			1			$\delta\underline{\beta}$	$+\beta\delta$	δ
d3				1			1		1			1				$\delta\underline{\gamma}$	$+\gamma\delta$	δ
d4				1			1		1	1		1	1	1	1	δ		δ
ad1							1					1	1		1	$\alpha\delta$		$\alpha\ \delta$
bd2									1			1		1	1	$\beta\delta$		$\beta\ \delta$
cd3										1			1	1	1	$\gamma\delta$		$\gamma\ \delta$

Non-canonical form of relations

- e.g. for N=4, $\mathscr{L}$=1:
  enveloped regions w/ tripartite correlation:



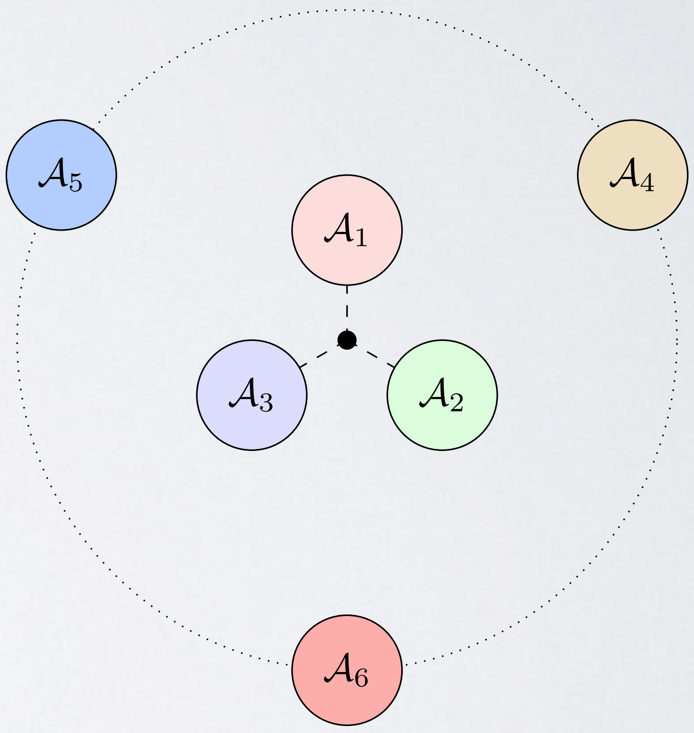
	A	B	C	D	AB	AC	AD	BC	BD	CD	ABC	ABD	ACD	BCD	ABCD	new relations
a	1				1	1										α
b		1			1			1								β
c			1			1		1								γ
d1									1	1				1		δ
d2							1			1			1			δ
d3							1		1			1				δ
d4				1			1		1	1		1	1	1	1	δ
ad1							1					1	1		1	$\alpha\ \delta$
bd2									1			1		1	1	$\beta\ \delta$
cd3										1			1	1	1	$\gamma\ \delta$
abc											1					$\alpha\ \beta\ \gamma\ \underline{\delta}$
d1d2d3				1												$\delta\ \underline{\alpha}\ \underline{\beta}\ \underline{\gamma}$

- the last two terms cannot be converted to canonical form...

Canonical building blocks

- Consider CFT in vacuum state on $\mathbb{R}^2$
  - Take single disk per color (disjoint from all others)
  - Change position of disks to change pattern of mutual information
- Example:
  - N=6
  - canonical building block $\mathfrak{C}_6^{\circ}[\mathcal{A}_1\mathcal{A}_2\mathcal{A}_3]$
  - → N+1=7 bulk surfaces
  - gives canonical relations

$$\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \ \alpha_1\alpha_2\alpha_3\}$$



- Composing building blocks:
  - Take several widely-separated configurations
  - Resulting q-relations = union of the sets of relations for each block

Classification results for N=3

- Complete $\mathsf{N}=3$ classification
  - I_3 (→ MMI) follows easily (from simple no-enveloping $\mathscr{L}$=0 configuration)

$$I_3(A:B:C) = S_A + S_B + S_C - S_{AB} - S_{AC} - S_{BC} + S_{ABC}$$

  - No additional inequalities can exist (by eqn counting argument)

→ $\mathsf{N}=3$ holographic entropy polyhedron (=holographic entropy cone):

- 3 permutations of SA(1,1) $\quad S_{\mathcal{A}} + S_{\mathcal{B}} \geq S_{\mathcal{AB}}$
- 3 permutations of AL(1,3) $\quad S_{\mathcal{A}} + S_{\mathcal{ABC}} \geq S_{\mathcal{BC}}$
- 1 (permutation symmetric) MMI $\quad S_{\mathcal{AB}} + S_{\mathcal{BC}} + S_{\mathcal{AC}} \geq S_{\mathcal{A}} + S_{\mathcal{B}} + S_{\mathcal{C}} + S_{\mathcal{ABC}}$

⇒ 7 facets

Classification results for N=4

- New $\mathsf{N} = 4$ information quantities (all sign-indefinite)
  - At $\mathscr{L}$=0:

$$I_4^{ABCD} = S_A + S_B + S_C + S_D - S_{AB} - S_{AC} - S_{AD} - S_{BC} - S_{BD} - S_{CD} + \\ + S_{ABC} + S_{ABD} + S_{ACD} + S_{BCD} - S_{ABCD}$$

  - At $\mathscr{L}$=1:

written more compactly in terms of $\mathbf{I}_\mathsf{n}$'s:

$$\mathbf{Q}_4^{(1)} = S_{\mathcal{A}} - S_{\mathcal{B}} - S_{\mathcal{AC}} - S_{\mathcal{AD}} + S_{\mathcal{BC}} + S_{\mathcal{BD}} + S_{\mathcal{ACD}} - S_{\mathcal{BCD}} \qquad = \mathbf{I}_3^{\mathcal{ACD}} - \mathbf{I}_3^{\mathcal{BCD}}$$

$$\mathbf{Q}_4^{(2)} = S_{\mathcal{A}} - S_{\mathcal{AB}} - S_{\mathcal{AC}} - S_{\mathcal{AD}} + S_{\mathcal{ABC}} + S_{\mathcal{ABD}} + S_{\mathcal{ACD}} - S_{\mathcal{ABCD}} \qquad = -\mathbf{I}_3^{\mathcal{BCD}} + \mathbf{I}_4^{\mathcal{ABCD}}$$

$$\mathbf{Q}_4^{(4)} = 2S_{\mathcal{A}} + S_{\mathcal{B}} - 2S_{\mathcal{AB}} - S_{\mathcal{AC}} - S_{\mathcal{AD}} + S_{\mathcal{CD}} + S_{\mathcal{ABC}} + S_{\mathcal{ABD}} - S_{\mathcal{BCD}}$$

$$= \mathbf{I}_3^{\mathcal{ABC}} + \mathbf{I}_3^{\mathcal{ABD}} - \mathbf{I}_3^{\mathcal{BCD}}$$

$$\mathbf{Q}_4^{(5)} = S_{\mathcal{A}} + S_{\mathcal{BC}} + S_{\mathcal{BD}} + S_{\mathcal{CD}} - S_{\mathcal{ABC}} - S_{\mathcal{ABD}} - S_{\mathcal{ACD}} - 2S_{\mathcal{BCD}} + 2S_{\mathcal{ABCD}}$$

$$= \mathbf{I}_3^{\mathcal{ABC}} + \mathbf{I}_3^{\mathcal{ABD}} + \mathbf{I}_3^{\mathcal{ACD}} - \mathbf{I}_4^{\mathcal{ABCD}}$$

$$\mathbf{Q}_4^{(6)} = 3S_{\mathcal{A}} - 2S_{\mathcal{AB}} - 2S_{\mathcal{AC}} - 2S_{\mathcal{AD}} + S_{\mathcal{BC}} + S_{\mathcal{BD}} + S_{\mathcal{CD}} + S_{\mathcal{ABC}} + S_{\mathcal{ABD}} + S_{\mathcal{ACD}} - 2S_{\mathcal{BCD}}$$

$$= \mathbf{I}_3^{\mathcal{ABC}} + \mathbf{I}_3^{\mathcal{ABD}} + \mathbf{I}_3^{\mathcal{ACD}} - 2\mathbf{I}_3^{\mathcal{BCD}}$$

$$\mathbf{Q}_4^{(7)} = S_{\mathcal{AB}} + S_{\mathcal{AC}} + S_{\mathcal{AD}} + S_{\mathcal{BC}} + S_{\mathcal{BD}} + S_{\mathcal{CD}} \\ - 2S_{\mathcal{ABC}} - 2S_{\mathcal{ABD}} - 2S_{\mathcal{ACD}} - 2S_{\mathcal{BCD}} + 3S_{\mathcal{ABCD}} = \mathbf{I}_3^{\mathcal{ABC}} + \mathbf{I}_3^{\mathcal{ABD}} + \mathbf{I}_3^{\mathcal{ACD}} + \mathbf{I}_3^{\mathcal{BCD}} - 3\mathbf{I}_4^{\mathcal{ABCD}}$$

Classification results for N=4

- New $\mathsf{N}=4$ information quantities (all sign-indefinite)
  - At $\mathscr{L}$=0:

$$I_4^{ABCD} = S_A + S_B + S_C + S_D - S_{AB} - S_{AC} - S_{AD} - S_{BC} - S_{BD} - S_{CD} + S_{ABC} + S_{ABD} + S_{ACD} + S_{BCD} - S_{ABCD}$$

  - At $\mathscr{L}$=1: (pairwise related by purifications)

$$\mathbf{Q}_4^{(1)} = S_{\mathcal{A}} - S_{\mathcal{B}} - S_{\mathcal{AC}} - S_{\mathcal{AD}} + S_{\mathcal{BC}} + S_{\mathcal{BD}} + S_{\mathcal{ACD}} - S_{\mathcal{BCD}} = \mathbf{I}_3^{\mathcal{ACD}} - \mathbf{I}_3^{\mathcal{BCD}}$$

$$\mathbf{Q}_4^{(2)} = S_{\mathcal{A}} - S_{\mathcal{AB}} - S_{\mathcal{AC}} - S_{\mathcal{AD}} + S_{\mathcal{ABC}} + S_{\mathcal{ABD}} + S_{\mathcal{ACD}} - S_{\mathcal{ABCD}} = -\mathbf{I}_3^{\mathcal{BCD}} + \mathbf{I}_4^{\mathcal{ABCD}}$$

$$\mathbf{Q}_4^{(4)} = 2S_{\mathcal{A}} + S_{\mathcal{B}} - 2S_{\mathcal{AB}} - S_{\mathcal{AC}} - S_{\mathcal{AD}} + S_{\mathcal{CD}} + S_{\mathcal{ABC}} + S_{\mathcal{ABD}} - S_{\mathcal{BCD}} = \mathbf{I}_3^{\mathcal{ABC}} + \mathbf{I}_3^{\mathcal{ABD}} - \mathbf{I}_3^{\mathcal{BCD}}$$

$$\mathbf{Q}_4^{(5)} = S_{\mathcal{A}} + S_{\mathcal{BC}} + S_{\mathcal{BD}} + S_{\mathcal{CD}} - S_{\mathcal{ABC}} - S_{\mathcal{ABD}} - S_{\mathcal{ACD}} - 2S_{\mathcal{BCD}} + 2S_{\mathcal{ABCD}} = \mathbf{I}_3^{\mathcal{ABC}} + \mathbf{I}_3^{\mathcal{ABD}} + \mathbf{I}_3^{\mathcal{ACD}} - \mathbf{I}_4^{\mathcal{ABCD}}$$

$$\mathbf{Q}_4^{(6)} = 3S_{\mathcal{A}} - 2S_{\mathcal{AB}} - 2S_{\mathcal{AC}} - 2S_{\mathcal{AD}} + S_{\mathcal{BC}} + S_{\mathcal{BD}} + S_{\mathcal{CD}} + S_{\mathcal{ABC}} + S_{\mathcal{ABD}} + S_{\mathcal{ACD}} - 2S_{\mathcal{BCD}} = \mathbf{I}_3^{\mathcal{ABC}} + \mathbf{I}_3^{\mathcal{ABD}} + \mathbf{I}_3^{\mathcal{ACD}} - 2\mathbf{I}_3^{\mathcal{BCD}}$$

$$\mathbf{Q}_4^{(7)} = S_{\mathcal{AB}} + S_{\mathcal{AC}} + S_{\mathcal{AD}} + S_{\mathcal{BC}} + S_{\mathcal{BD}} + S_{\mathcal{CD}} - 2S_{\mathcal{ABC}} - 2S_{\mathcal{ABD}} - 2S_{\mathcal{ACD}} - 2S_{\mathcal{BCD}} + 3S_{\mathcal{ABCD}} = \mathbf{I}_3^{\mathcal{ABC}} + \mathbf{I}_3^{\mathcal{ABD}} + \mathbf{I}_3^{\mathcal{ACD}} + \mathbf{I}_3^{\mathcal{BCD}} - 3\mathbf{I}_4^{\mathcal{ABCD}}$$

Classification results for all N

- Complete classification for disjoint $\mathscr{L}$=0 configurations, ∀ N:
  - The $\mathbf{I}_{\mathsf{n}}$-**Theorem:**
    the only information quantities we can get for disjoint $\mathscr{L}$=0 configurations are the $\mathbf{I}_{\mathsf{n}}$'s.
  - ⇒ the only genuinely N-partite information quantity is $\mathbf{I}_{\mathsf{N}}$

$$
\begin{aligned}
I_{\mathsf{n}}(A_{\ell_1} : A_{\ell_2} : \ldots : A_{\ell_{\mathsf{n}}}) = &\; S_{\ell_1} + S_{\ell_2} + \cdots + S_{\ell_{\mathsf{n}}} \\
&- S_{\ell_1 \ell_2} - S_{\ell_1 \ell_3} - \cdots - S_{\ell_{\mathsf{n}-1} \ell_{\mathsf{n}}} \\
&+ S_{\ell_1 \ell_2 \ell_3} + \cdots + (-1)^{\mathsf{n}+1} S_{\ell_1 \ell_2 \ldots \ell_{\mathsf{n}}}
\end{aligned}
$$

- Using $\mathscr{L}$=1 building block w/ single non-canonical constraint
  - We obtain another infinite family of information quantities:

$$
\begin{aligned}
J_{\mathsf{n}}(A_{\ell_1} : A_{\ell_2} : \ldots : A_{\ell_{\mathsf{n}}}) = &\; S_{\ell_1 \ell_2} + S_{\ell_1 \ell_3} + \cdots + S_{\ell_{\mathsf{n}-1} \ell_{\mathsf{n}}} \\
&- 2 S_{\ell_1 \ell_2 \ell_3} - \cdots + (-1)^{\mathsf{n}} (\mathsf{n} - 1)\, S_{\ell_1 \ell_2 \ldots \ell_{\mathsf{n}}}
\end{aligned}
$$

Nicer representations for relations

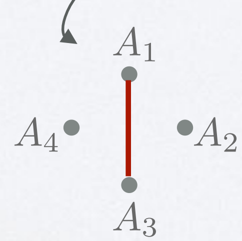
[He, Headrick, VH '19]

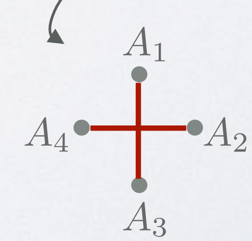
- S-basis
  - Specify individual entropies of all composite subsystems
    - e.g. for N=3, entropy vector = $\vec{S} = \{S_1,\ S_2,\ S_3,\ S_{12},\ S_{13},\ S_{23},\ S_{123}\}$
- I-basis
  - Specify multipartite information for all sets of single-color subsystems
    - e.g. for N=3, entropy vector = $\vec{S} = \{I_1,\ I_2,\ I_3,\ I_{12},\ I_{13}, I_{23},\ I_{123}\}_{[I]}$

      notational shorthand: $\equiv S(A_1)$ $\equiv I_2(A_1 : A_2)$ $\equiv I_3(A_1 : A_2 : A_3)$
- K-basis
  - Specify perfect tensor ∀ even sets of single-color subsystems + purifier
    - e.g. for N=3, $\{A_1, A_2, A_3\}$ A_4

$$\vec{S} = \{K_{12},\ K_{13},\ K_{14},\ K_{23},\ K_{24},\ K_{34},\ K_{1234}\}_{[K]}$$

K's = coefficients of





Nicer representations for N<4 relations

N	Relation	S-basis	I-basis	K-basis	primitive:
2,3	$\mathrm{SA}_{(1,1)}$	$\mathtt{S}_1 + \mathtt{S}_2 - \mathtt{S}_{12}$	$\mathtt{I}_{12}$	$\mathtt{K}_{12}^{(\mathtt{N})}$	←
3	$\mathrm{SA}_{(1,2)}$	$\mathtt{S}_1 + \mathtt{S}_{23} - \mathtt{S}_{123}$	$\mathtt{I}_{12} + \mathtt{I}_{13} - \mathtt{I}_{123}$	$\mathtt{K}_{12}^{(3)} + \mathtt{K}_{13}^{(3)} + \mathtt{K}_{1234}^{(3)}$	
2	$\mathrm{AL}_{(1,2)}$	$\mathtt{S}_1 + \mathtt{S}_{12} - \mathtt{S}_2$	$2\mathtt{I}_1 - \mathtt{I}_{12}$	$\mathtt{K}_{13}^{(2)}$	←
3	$\mathrm{AL}_{(1,2)}$	$\mathtt{S}_1 + \mathtt{S}_{12} - \mathtt{S}_2$	$2\mathtt{I}_1 - \mathtt{I}_{12}$	$\mathtt{K}_{13}^{(3)} + \mathtt{K}_{14}^{(3)} + \mathtt{K}_{1234}^{(3)}$	
3	$\mathrm{AL}_{(1,3)}$	$\mathtt{S}_1 + \mathtt{S}_{123} - \mathtt{S}_{23}$	$2\mathtt{I}_1 - \mathtt{I}_{12} - \mathtt{I}_{13} + \mathtt{I}_{123}$	$\mathtt{K}_{14}^{(3)}$	←
3	$\mathrm{AL}_{(2,3)}$	$\mathtt{S}_{12} + \mathtt{S}_{123} - \mathtt{S}_3$	$2\mathtt{I}_1 + 2\mathtt{I}_2 - 2\mathtt{I}_{12} - \mathtt{I}_{13}$ $-\mathtt{I}_{23} + \mathtt{I}_{123}$	$\mathtt{K}_{14}^{(3)} + \mathtt{K}_{24}^{(3)} + \mathtt{K}_{1234}^{(3)}$	
3	$\mathrm{SSA}_{(2,2)}$	$\mathtt{S}_{12} + \mathtt{S}_{23} - \mathtt{S}_2 - \mathtt{S}_{123}$	$\mathtt{I}_{13} - \mathtt{I}_{123}$	$\mathtt{K}_{13}^{(3)} + \mathtt{K}_{1234}^{(3)}$	
3	$\mathrm{WM}_{(2,2)}$	$\mathtt{S}_{12} + \mathtt{S}_{23} - \mathtt{S}_1 - \mathtt{S}_3$	$2\mathtt{I}_2 - \mathtt{I}_{12} - \mathtt{I}_{23}$	$\mathtt{K}_{24}^{(3)} + \mathtt{K}_{1234}^{(3)}$	
3	$\mathrm{MMI}_{(1,1,1)}$	$-\mathtt{S}_1 - \mathtt{S}_2 - \mathtt{S}_3 + \mathtt{S}_{12}$ $+\mathtt{S}_{13} + \mathtt{S}_{23} - \mathtt{S}_{123}$	$-\mathtt{I}_{123}$	$\mathtt{K}_{1234}^{(3)}$	←

- Observations:
  - K-basis representation is more (or as) compact than S-basis one
  - K-basis representation is most compact for primitive quantities
  - K-basis representation for all $Q \geq 0$ has all coefficients ≥ 0

Nicer representations for N=5 relations

Relation	Basis	Primitive Information Quantity (all univ. holographic inequalities)
$\mathrm{SA}_{(1,1)}$	S (3)	$S_1 + S_2 - S_{12}$
	I (1)	I_{12}
	K (1)	$K^{(5)}_{12}$
$\mathrm{MMI}_{(1,1,1)}$	S (7)	$-S_1 - S_2 - S_3 + S_{12} + S_{13} + S_{23} - S_{123}$
	I (1)	$-I_{123}$
	K (3)	$K^{(5)}_{1234} + K^{(5)}_{1235} + K^{(5)}_{1236}$
$\mathrm{MMI}_{(1,2,2)}$	S (7)	$-S_1 - S_{23} - S_{45} + S_{123} + S_{145} + S_{2345} - S_{12345}$
	I (9)	$-I_{124} - I_{125} - I_{134} - I_{135} + I_{1234} + I_{1235} + I_{1245} + I_{1345} - I_{12345}$
	K (5)	$K^{(5)}_{1246} + K^{(5)}_{1256} + K^{(5)}_{1346} + K^{(5)}_{1356} + K^{(5)}_{123456}$
$Q^{(5)}_1$	S (11)	$-S_{12} - S_{23} - S_{34} - S_{45} - S_{15} + S_{123} + S_{234} + S_{345} + S_{145} + S_{125} - S_{12345}$
	I (11)	$-I_{124} - I_{134} - I_{135} - I_{235} - I_{245} + I_{1234} + I_{1235} + I_{1245} + I_{1345} + I_{2345} - I_{12345}$
	K (6)	$K^{(5)}_{1246} + K^{(5)}_{1346} + K^{(5)}_{1356} + K^{(5)}_{2356} + K^{(5)}_{2456} + 2K^{(5)}_{123456}$
$Q^{(5)}_2$	S (16)	$-S_{12} - S_{13} - S_{14} - S_{23} - S_{25} - S_{45} + 2S_{123} + S_{124} + S_{125} + S_{134} + S_{145} + S_{235} + S_{245}$ $-S_{1234} - S_{1235} - S_{1245}$
	I (7)	$-I_{124} - I_{125} - I_{135} - I_{234} + I_{1234} + I_{1235} + I_{1245}$
	K (7)	$K^{(5)}_{1246} + K^{(5)}_{1256} + K^{(5)}_{1345} + K^{(5)}_{1356} + K^{(5)}_{2345} + K^{(5)}_{2346} + 3K^{(5)}_{123456}$
$Q^{(5)}_3$	S (19)	$-S_{12} - S_{13} - S_{14} - S_{25} - S_{35} - S_{45} + S_{123} + S_{124} + S_{125} + S_{134} + S_{135} + S_{145} + S_{235}$ $+S_{245} + S_{345} - S_{234} - S_{1235} - S_{1245} - S_{1345}$
	I (7)	$-I_{125} - I_{135} - I_{145} - I_{234} + I_{1235} + I_{1245} + I_{1345}$
	K (7)	$K^{(5)}_{1234} + K^{(5)}_{1256} + K^{(5)}_{1356} + K^{(5)}_{1456} + K^{(5)}_{2345} + K^{(5)}_{2346} + 3K^{(5)}_{123456}$
$Q^{(5)}_4$	S (16)	$-S_2 - S_3 - S_4 - S_5 - S_{12} - S_{13} + S_{23} + S_{45} + S_{123} + S_{124} + S_{125} + S_{134} + S_{135} - S_{145}$ $-S_{1234} - S_{1235}$
	I (6)	$-I_{123} - I_{145} - I_{234} - I_{235} + I_{1234} + I_{1235}$
	K (8)	$K^{(5)}_{1236} + K^{(5)}_{1245} + K^{(5)}_{1345} + K^{(5)}_{1456} + 2K^{(5)}_{2345} + K^{(5)}_{2346} + K^{(5)}_{2356} + 2K^{(5)}_{123456}$
$Q^{(5)}_5$	S (22)	$-2S_{12} - 2S_{13} - S_{14} - S_{15} - S_{23} - 2S_{24} - 2S_{35} - S_{45} + 3S_{123} + 3S_{124} + S_{125} + S_{134}$ $+3S_{135} + S_{145} + S_{234} + S_{235} + S_{245} + S_{345} - 2S_{1234} - 2S_{1235} - S_{1245} - S_{1345}$
	I (10)	$-I_{123} - 2I_{125} - 2I_{134} - I_{145} - I_{234} - I_{235} + 2I_{1234} + 2I_{1235} + I_{1245} + I_{1345}$
	K (10)	$K^{(5)}_{1236} + K^{(5)}_{1245} + 2K^{(5)}_{1256} + K^{(5)}_{1345} + 2K^{(5)}_{1346} + K^{(5)}_{1456} + 2K^{(5)}_{2345} + K^{(5)}_{2346} + K^{(5)}_{2356} + 6K^{(5)}_{123456}$

Nicer representations for N=4 relations

Relation	Basis	Primitive Information Quantity
$\mathtt{I}_4$	S (15)	$\mathtt{S}_1+\mathtt{S}_2+\mathtt{S}_3+\mathtt{S}_4-\mathtt{S}_{12}-\mathtt{S}_{13}-\mathtt{S}_{14}-\mathtt{S}_{23}-\mathtt{S}_{24}-\mathtt{S}_{34}+\mathtt{S}_{123}+\mathtt{S}_{124}+\mathtt{S}_{134}+\mathtt{S}_{234}-\mathtt{S}_{1234}$
	I (1)	$\mathtt{I}_{1234}$
	K (1)	$-2\mathtt{K}^{(4)}_{1234}$
$Q^{(4)}_1$	S (8)	$\mathtt{S}_1-\mathtt{S}_2-\mathtt{S}_{13}-\mathtt{S}_{14}+\mathtt{S}_{23}+\mathtt{S}_{24}+\mathtt{S}_{134}-\mathtt{S}_{234}$
	I (2)	$\mathtt{I}_{134}-\mathtt{I}_{234}$
	K (2)	$-\mathtt{K}^{(4)}_{1345}+\mathtt{K}^{(4)}_{2345}$
$Q^{(4)}_2$	S (8)	$\mathtt{S}_1-\mathtt{S}_{12}-\mathtt{S}_{13}-\mathtt{S}_{14}+\mathtt{S}_{123}+\mathtt{S}_{124}+\mathtt{S}_{134}-\mathtt{S}_{1234}$
	I (2)	$-\mathtt{I}_{234}+\mathtt{I}_{1234}$
	K (2)	$-\mathtt{K}^{(4)}_{1234}+\mathtt{K}^{(4)}_{2345}$
$Q^{(4)}_4$	S (9)	$2\mathtt{S}_1+\mathtt{S}_2-2\mathtt{S}_{12}-\mathtt{S}_{13}-\mathtt{S}_{14}+\mathtt{S}_{34}+\mathtt{S}_{123}+\mathtt{S}_{124}-\mathtt{S}_{234}$
	I (3)	$\mathtt{I}_{123}+\mathtt{I}_{124}-\mathtt{I}_{234}$
	K (4)	$-\mathtt{K}^{(4)}_{1234}-\mathtt{K}^{(4)}_{1235}-\mathtt{K}^{(4)}_{1245}+\mathtt{K}^{(4)}_{2345}$
$Q^{(4)}_5$	S (9)	$\mathtt{S}_1+\mathtt{S}_{23}+\mathtt{S}_{24}+\mathtt{S}_{34}-\mathtt{S}_{123}-\mathtt{S}_{124}-\mathtt{S}_{134}-2\mathtt{S}_{234}+2\mathtt{S}_{1234}$
	I (4)	$\mathtt{I}_{123}+\mathtt{I}_{124}+\mathtt{I}_{134}-2\mathtt{I}_{1234}$
	K (4)	$\mathtt{K}^{(4)}_{1234}-\mathtt{K}^{(4)}_{1235}-\mathtt{K}^{(4)}_{1245}-\mathtt{K}^{(4)}_{1345}$
$Q^{(4)}_6$	S (11)	$3\mathtt{S}_1-2\mathtt{S}_{12}-2\mathtt{S}_{13}-2\mathtt{S}_{14}+\mathtt{S}_{23}+\mathtt{S}_{24}+\mathtt{S}_{34}+\mathtt{S}_{123}+\mathtt{S}_{124}+\mathtt{S}_{134}-2\mathtt{S}_{234}$
	I (4)	$\mathtt{I}_{123}+\mathtt{I}_{124}+\mathtt{I}_{134}-2\mathtt{I}_{234}$
	K (5)	$-\mathtt{K}^{(4)}_{1234}-\mathtt{K}^{(4)}_{1235}-\mathtt{K}^{(4)}_{1245}-\mathtt{K}^{(4)}_{1345}+2\mathtt{K}^{(4)}_{2345}$
$Q^{(4)}_7$	S (11)	$\mathtt{S}_{12}+\mathtt{S}_{13}+\mathtt{S}_{14}+\mathtt{S}_{23}+\mathtt{S}_{24}+\mathtt{S}_{34}-2\mathtt{S}_{123}-2\mathtt{S}_{124}-2\mathtt{S}_{134}-2\mathtt{S}_{234}+3\mathtt{S}_{1234}$
	I (5)	$\mathtt{I}_{123}+\mathtt{I}_{124}+\mathtt{I}_{134}+\mathtt{I}_{234}-3\mathtt{I}_{1234}$
	K (5)	$2\mathtt{K}^{(4)}_{1234}-\mathtt{K}^{(4)}_{1235}-\mathtt{K}^{(4)}_{1245}-\mathtt{K}^{(4)}_{1345}-\mathtt{K}^{(4)}_{2345}$

- All these are sign-**in**definite.
- K-basis expressions now (mostly) have mixes signs…

Advantages of K basis:

[He, Headrick, VH '19]

- K-basis manifests larger symmetry (S_{N+1}) than the S and I bases.

- Any non-negative information quantity necessarily has non-negative coefficients when expressed in the K-basis.

- All (even-party) perfect tensors are extremal rays of holographic entropy cone — and in fact also the full quantum entropy cone.

- **Q**: Is the K-basis the best we can do?

Sieve

- Original purpose
  - Constrain holographic entropy polyhedron by ruling out information quantities which can have either sign (depending on configuration)
- Basic idea
  - Determine sign of the information quantity for canonical building block
  - Determine sign for local purification of canonical building block
  - If signs differ, reject the quantity from holographic entropy polyhedron
  - Remaining quantities give us candidates for hol. ent. polyhedron
- In practice
  - The sieve is actually far more powerful!
  - Start abstractly by constraints on coefficients so as to avoid rejection
  - Consider extremal rays of the coefficient space cone
  - These give candidate universal holographic inequalities

Summary

- Utility of proto-entropy & hyperplane arrangement
  - Saturation of universal holographic inequalities by 'cancelling surfaces'
  - ⇒ cutoff-independent (even for single CFT)
  - Logic of construction is independent of N
  - Automatically avoids generating redundant relations (such as SSA)
- Utility of sieve
  - Surprisingly powerful in constraining holographic entropy polyhedron
    - provides an inner bound, even without any knowledge of the arrangement
  - We can recover the entropy polyhedron for N=5 without evoking the building blocks
- Conjecture: “RT cone = HRT cone”
  - Since cancelation of surfaces works equally well for time-dependence

Summary

- Results so far:
  - Recovers holographic entropy cone for N=2 and N=3
  - At N=4, no new universal holographic inequalities, but 4 extra IQs
  - At N=5, sieve recovers all proven inequalities for holographic entropy cone (along with candidate $-I_5 \geq 0$ which is not a true inequality); explicit construction [Hernandez-Cuenca] confirms only 5 new inequalities.

  - For general N:
    - Complete classification for $\mathscr{L}$=0 simple configurations:
      ⇨ the I_n-Thm: the only IQs we can get are I_n's for n ≤ N
    - New infinite family of information quantities for $\mathscr{L}$=1

- Outlook
  - Seems promising: expect only few new inequalities at each N…

Open Questions

- Efficiency of generating holographic entropy arrangement $\forall$ N
  - Obtaining/characterizing complete set of building blocks
  - Relating arrangements for different N
- Obtaining the holographic entropy polyhedron $\forall$ N
  - Efficiency of the sieve
  - Proving candidate inequalities
- Interpreting information quantities and universal holographic inequalities
  - Structure of density matrix
  - Role of hyperplane arrangement beyond holography (in general QFT)
  - New insights into the entanglement structures of holographic CFTs w/ geometric states
  - Relation to Mutual Information Arrangement

Stay Tuned…


Thank you