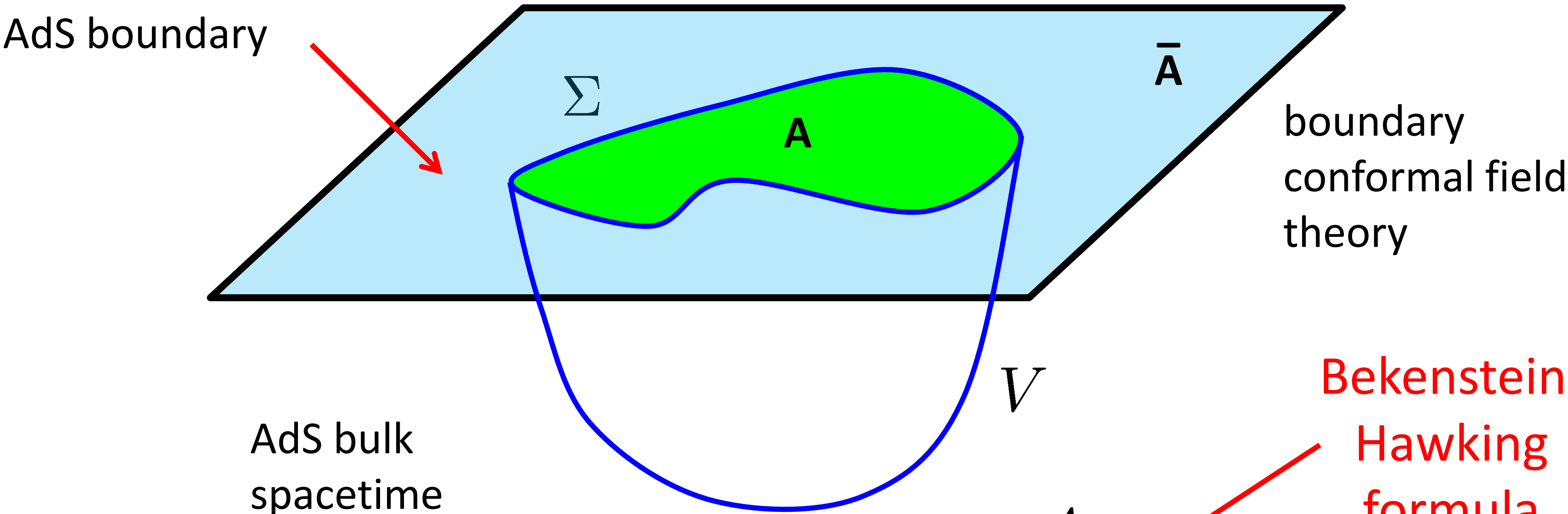
First Law of Complexity

with Bernamonti, Galli, Hernandez, Ruan & Simon
[arXiv:1903.04511]

with Belin & Chen

Holographic Entanglement Entropy:

(Ryu & Takayanagi `06)



Bekenstein-
Hawking
formula

$$S(A) = \min_{\partial V \sim \Sigma} \frac{A_V}{4G_N}$$

- holographic EE is a fruitful forum for bulk-boundary dialogue:

  $\longrightarrow$ new lessons about quantum field theories

  $\longrightarrow$ new lessons about quantum gravity

(Ryu & Takayanagi `06)

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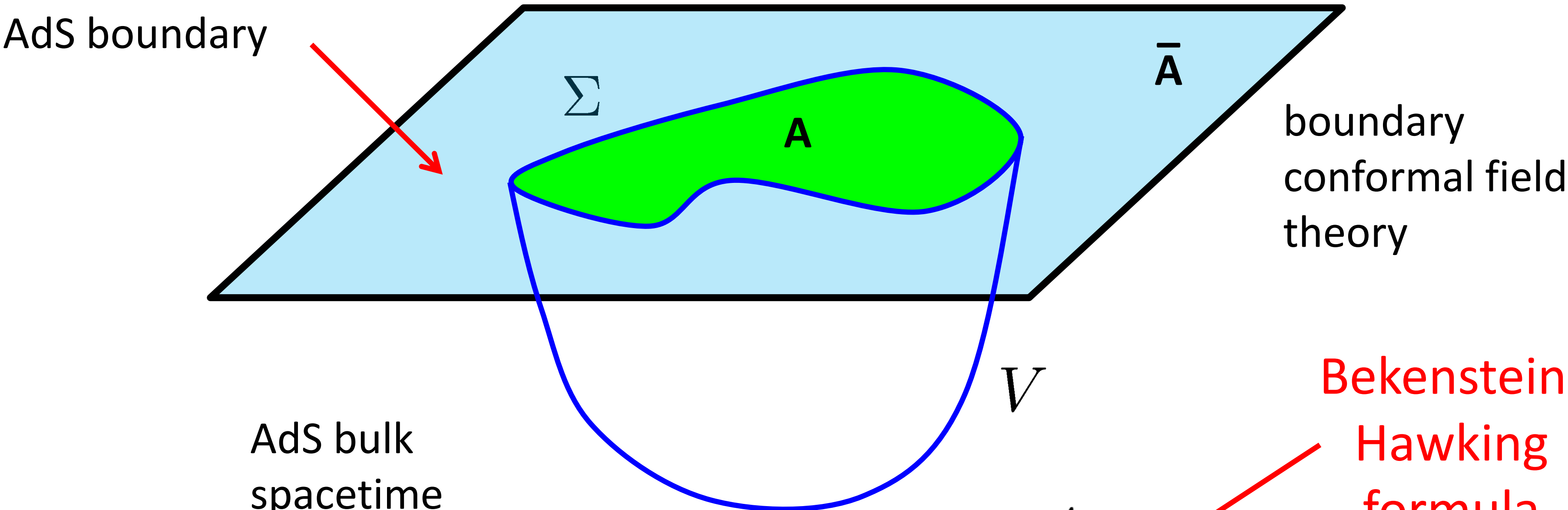

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Spacetime Geometry = Entanglement

(Ryu & Takayanagi `06)

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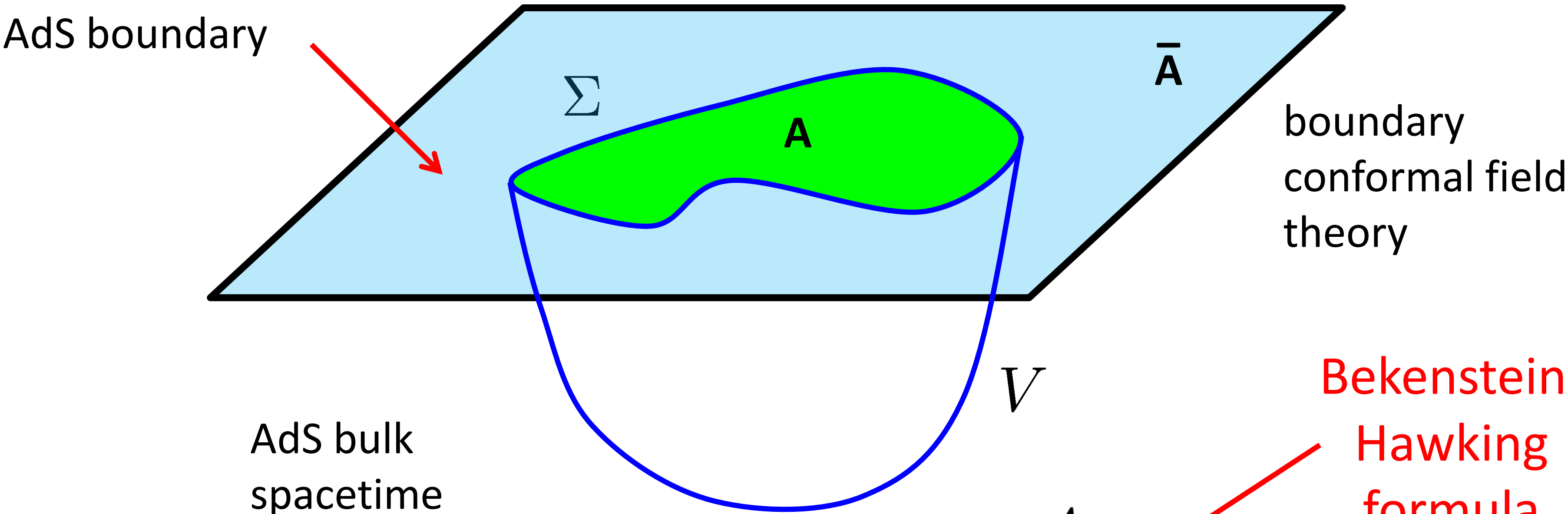

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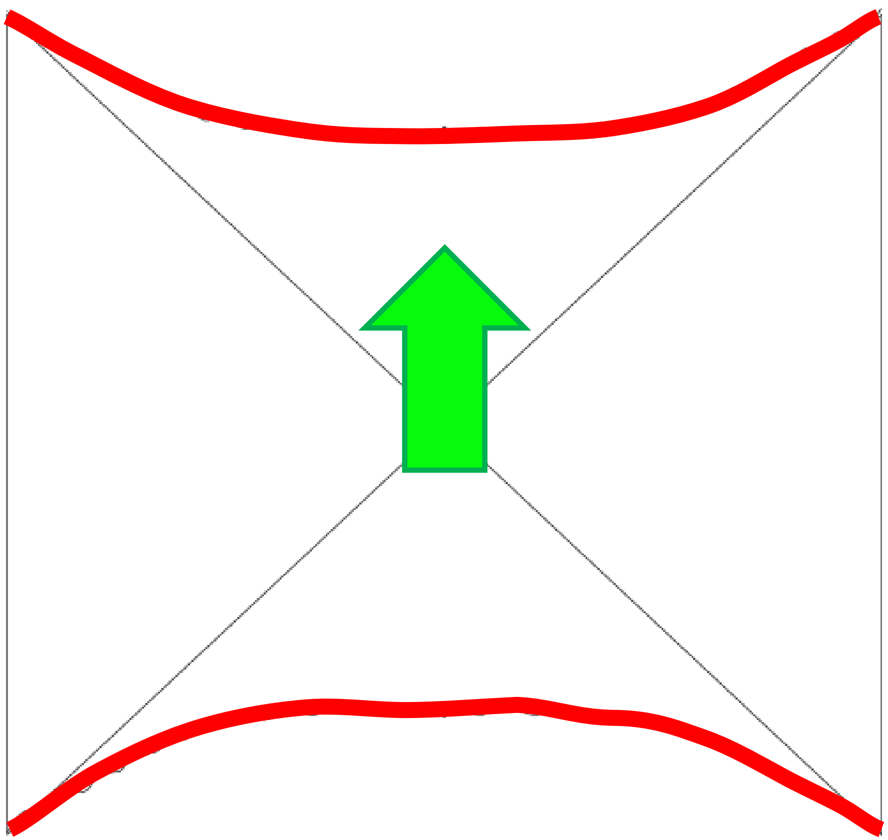

(cf. Hartman & Maldacena)

- recall S_{EE} only probes the eigenvalues of the density matrix

$$\begin{aligned} S_{EE} &= -Tr\left[\rho_A \, \log \rho_A\right] \\ &= -\sum \lambda_i \, \log \lambda_i \end{aligned}$$

- would like a new probe which is "sensitive to phases"

$$|\mathrm{TFD}\rangle \simeq \sum_\alpha e^{-E_\alpha/(2T) - iE_\alpha(t_L+t_R)} \times |E_\alpha\rangle_L |E_\alpha\rangle_R$$

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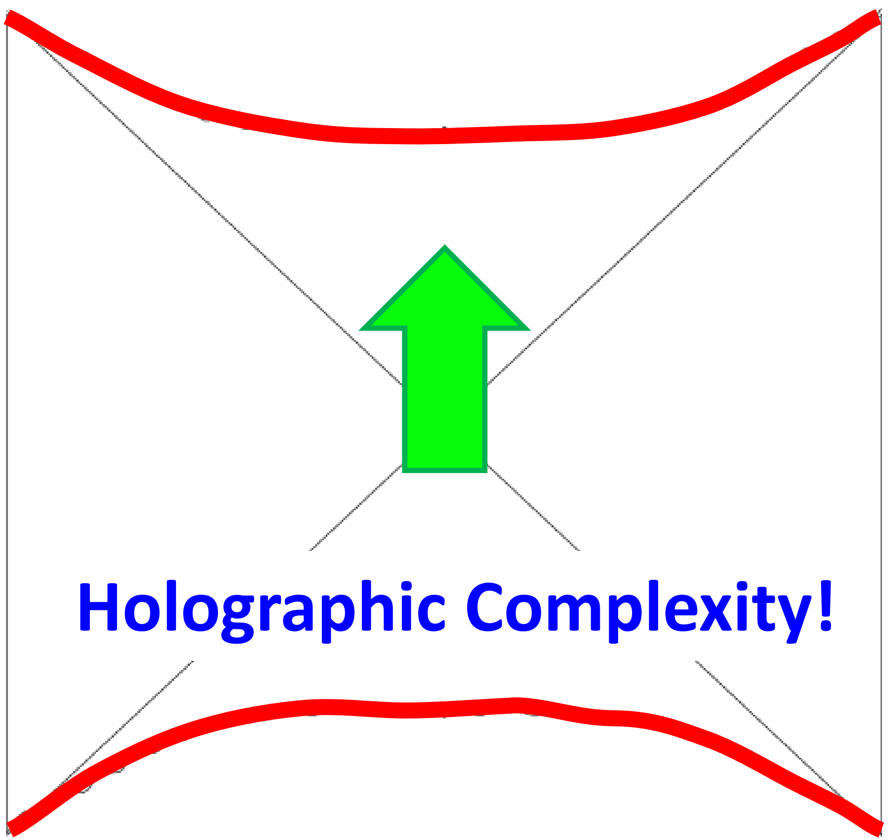

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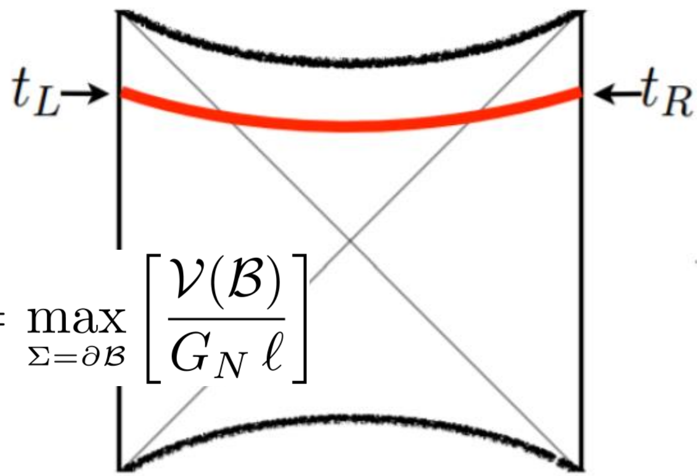
Holographic Complexity: A Tale of Two or more Dualities

- complexity=volume: evaluate proper volume of extremal codim-one surface connecting Cauchy surfaces in boundary theory (cf holo EE)

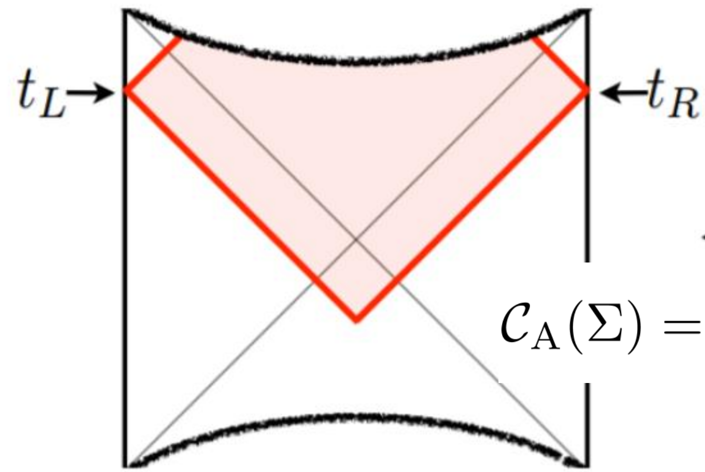
(Stanford & Susskind)

Complexity = Volume



$$\mathcal{C}_{\mathrm{V}}(\Sigma) = \max_{\Sigma=\partial\mathcal{B}} \left[\frac{\mathcal{V}(\mathcal{B})}{G_N\,\ell}\right]$$

Complexity = Action



$$\mathcal{C}_{\mathrm{A}}(\Sigma) = \frac{I_{\mathrm{WDW}}}{\pi\,\hbar}$$

- complexity=action: evaluate gravitational action for Wheeler-DeWitt patch = domain of dependence of bulk time slice connecting boundary Cauchy slices in CFT (Brown, Roberts, Swingle, Susskind & Zhao)

- both of these gravitational “observables” probe the black hole interior (at arbitrarily late times on boundary)

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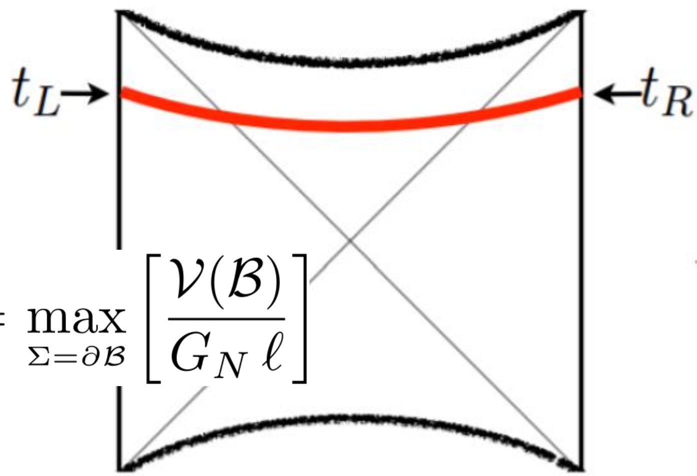

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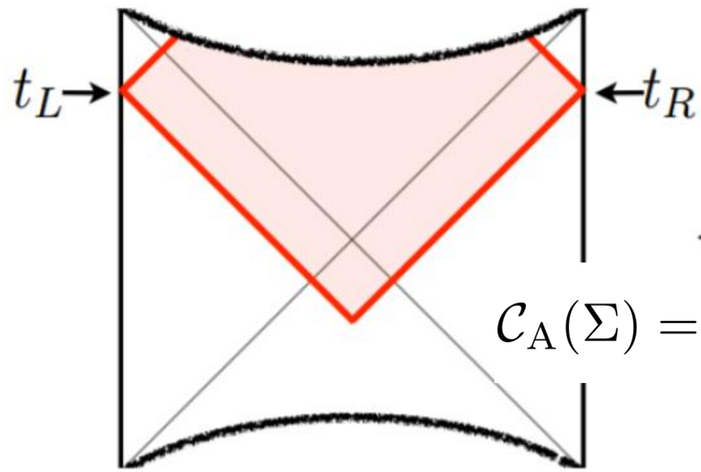

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- complexity=volume**2.0**: evaluate spacetime volume of WDW patch

$$\mathcal{C}'_{\mathrm{V}}(\Sigma) = \frac{V_{\mathrm{WDW}}}{G_N\,\ell^2}$$

(Couch, Fischler & Nguyen)

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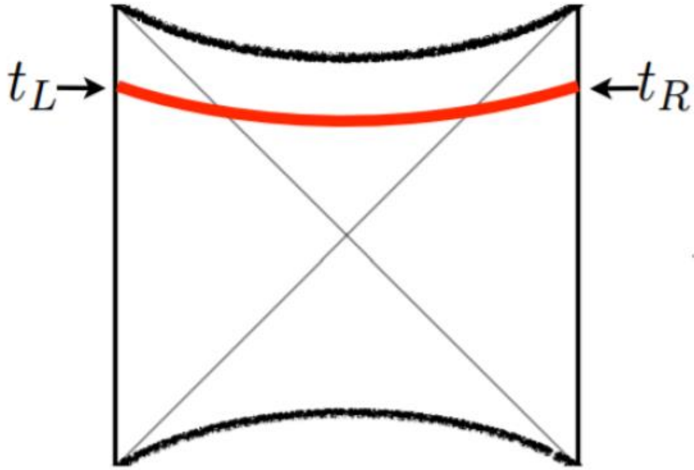

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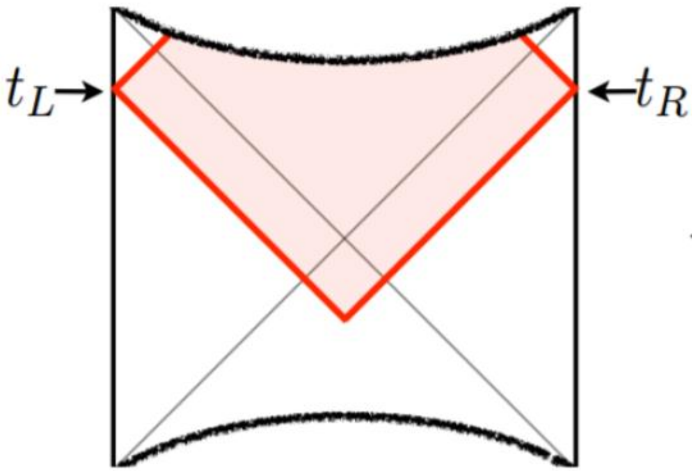

WHY COMPLEXITY??

Holographic Complexity: A Tale of Two Dualities

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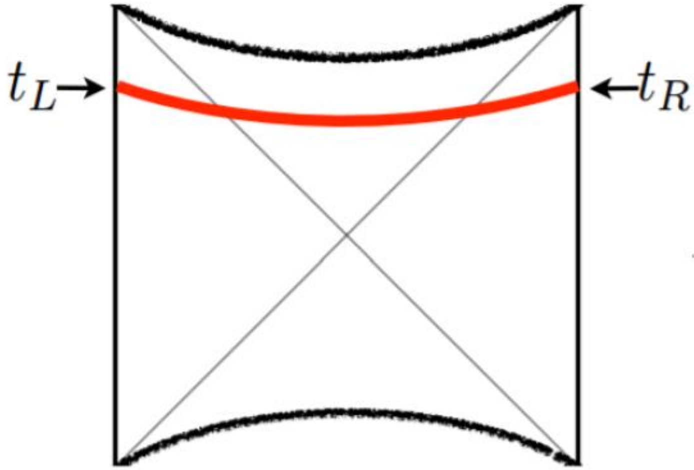

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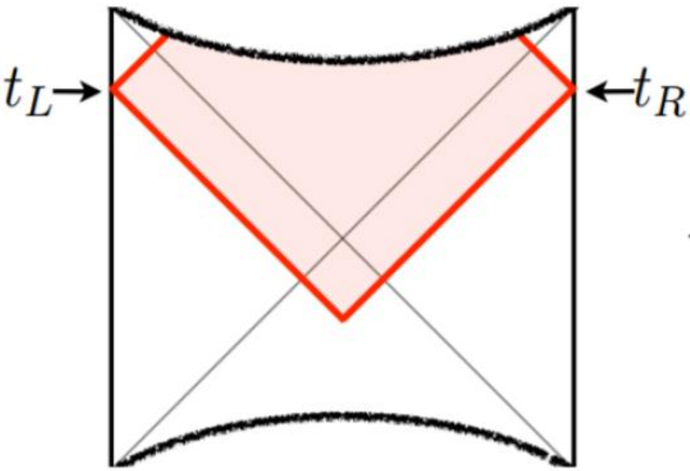

WHY COMPLEXITY??

- connection of complexity=volume to AdS/MERA
- linear growth (at late times)

$(d = \text{boundary dimension})$

$$\left.\frac{d\mathcal{C}_{\rm V}}{dt}\right|_{t\to\infty} = \frac{8\pi}{d-1}\,M \quad \text{(planar)} \qquad\qquad \left.\frac{d\mathcal{C}_{\rm A}}{dt}\right|_{t\to\infty} = \frac{2M}{\pi}$$

- shockwaves and the “switchback effect”

Holographic Complexity: A Tale of Two Dualities

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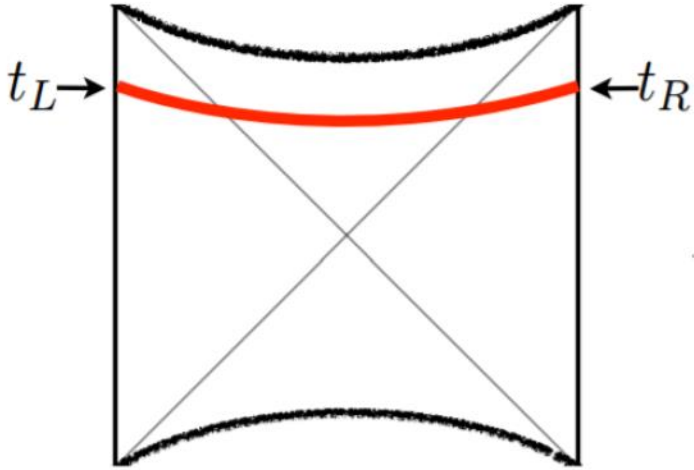

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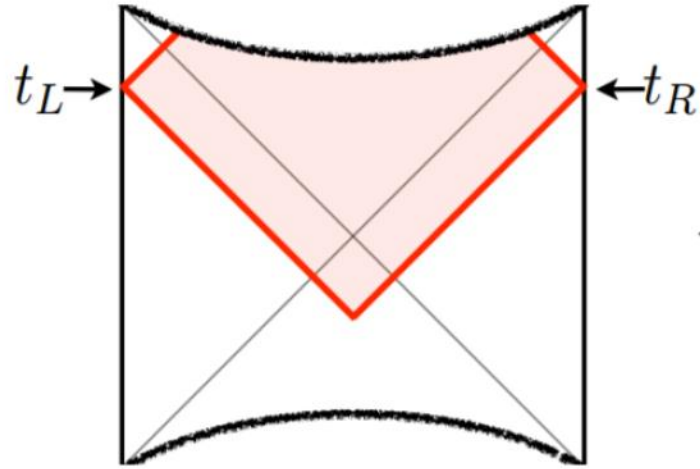

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$$\frac{d\mathcal{C}_{\rm V}}{dt}\Big| \quad \ldots \quad M$$

What does "complexity" mean in a quantum field theory?

- shockwaves and the "switchback effect"

Complexity:

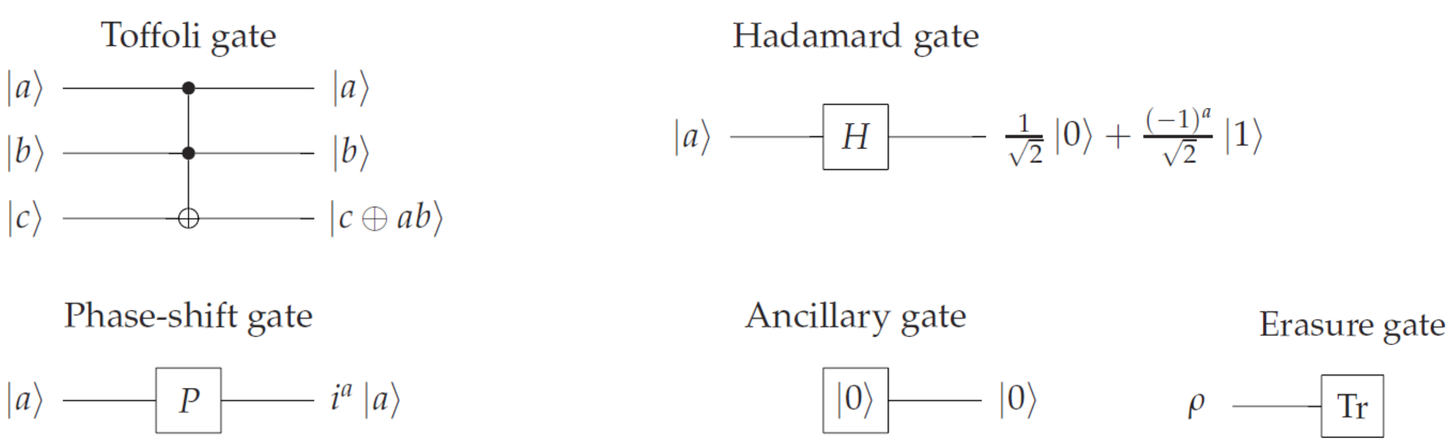
- computational complexity: how difficult is it to implement a task? eg, how difficult is it to prepare a particular quantum state?

- quantum circuit model:

$$|\psi\rangle = U\,|\psi_0\rangle$$

unitary operator built from set of simple gates

simple reference state eg, $|00000\cdots0\rangle$



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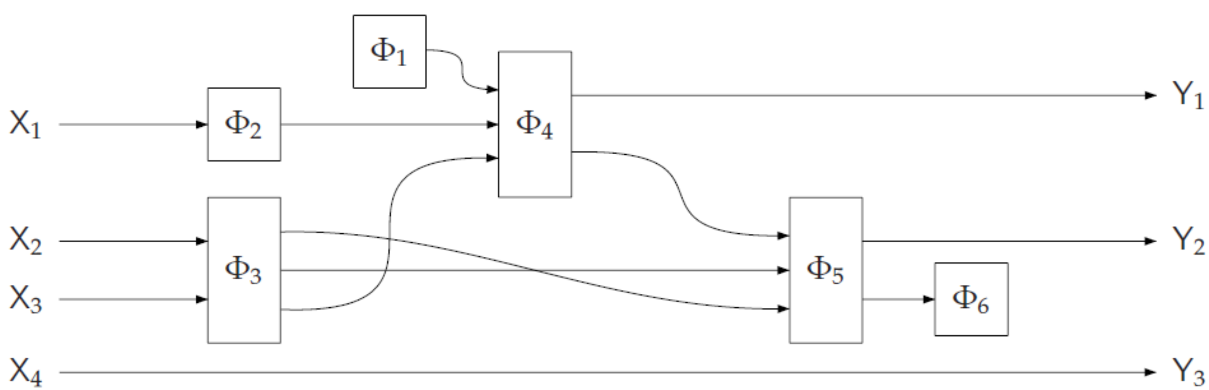

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- **How do we apply these ideas in quantum field theory?**

Quantum Field Theory:

(with Jefferson)

- **free** scalar field theory (in d spacetime dimensions)

$$H = \frac{1}{2}\int d^{d-1}x\left[\pi(x)^2+\vec{\nabla}\phi(x)^2+m^2\phi(x)^2\right]$$

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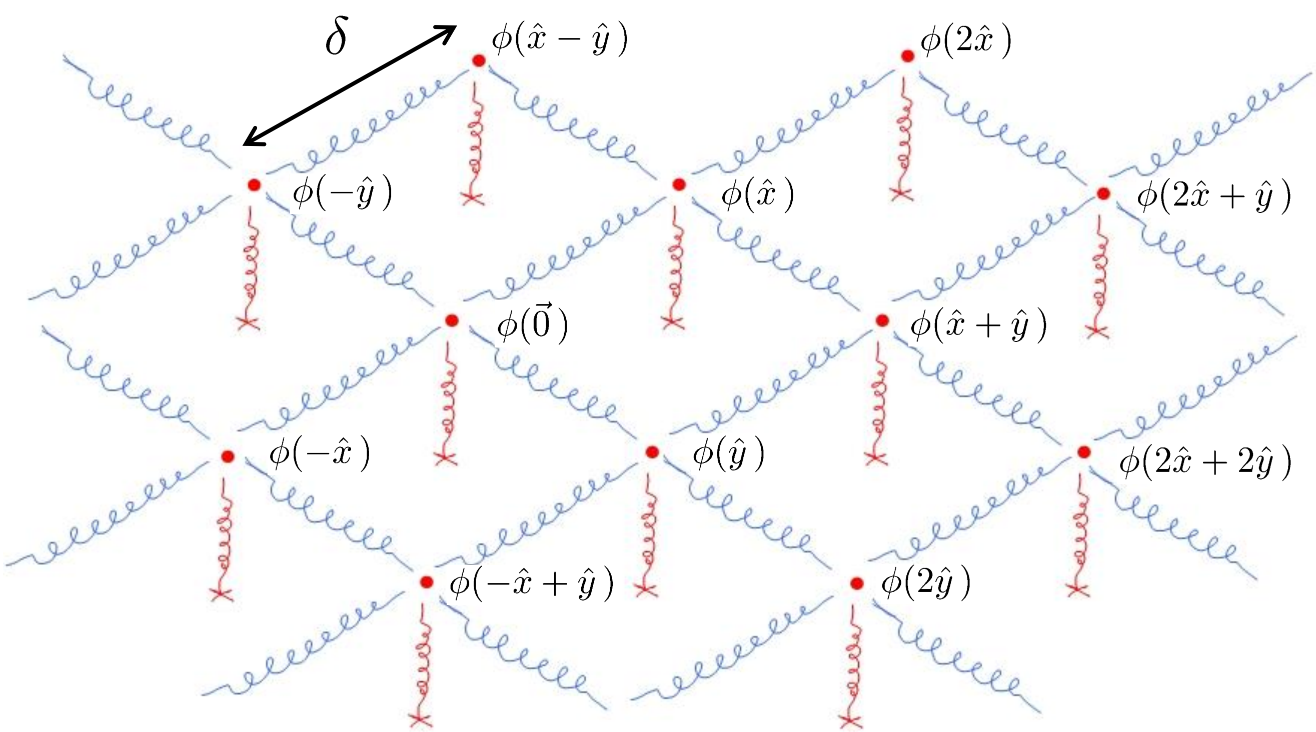
&= \frac{1}{2}\sum_{\vec{n}}\left[\frac{[\pi(\vec{n})^2}{\delta^{d-1}}+\delta^{d-1}\left\{\frac{1}{\delta^2}\sum_i[\phi(\vec{n}\,)-\phi(\vec{n}-\hat{x}_i\,)]^2+m^2\phi(\vec{n}\,)^2\right\}\right]
\end{aligned}
$$



Quantum Field Theory:

(with Jefferson)

- an infinite family of coupled harmonic oscillators

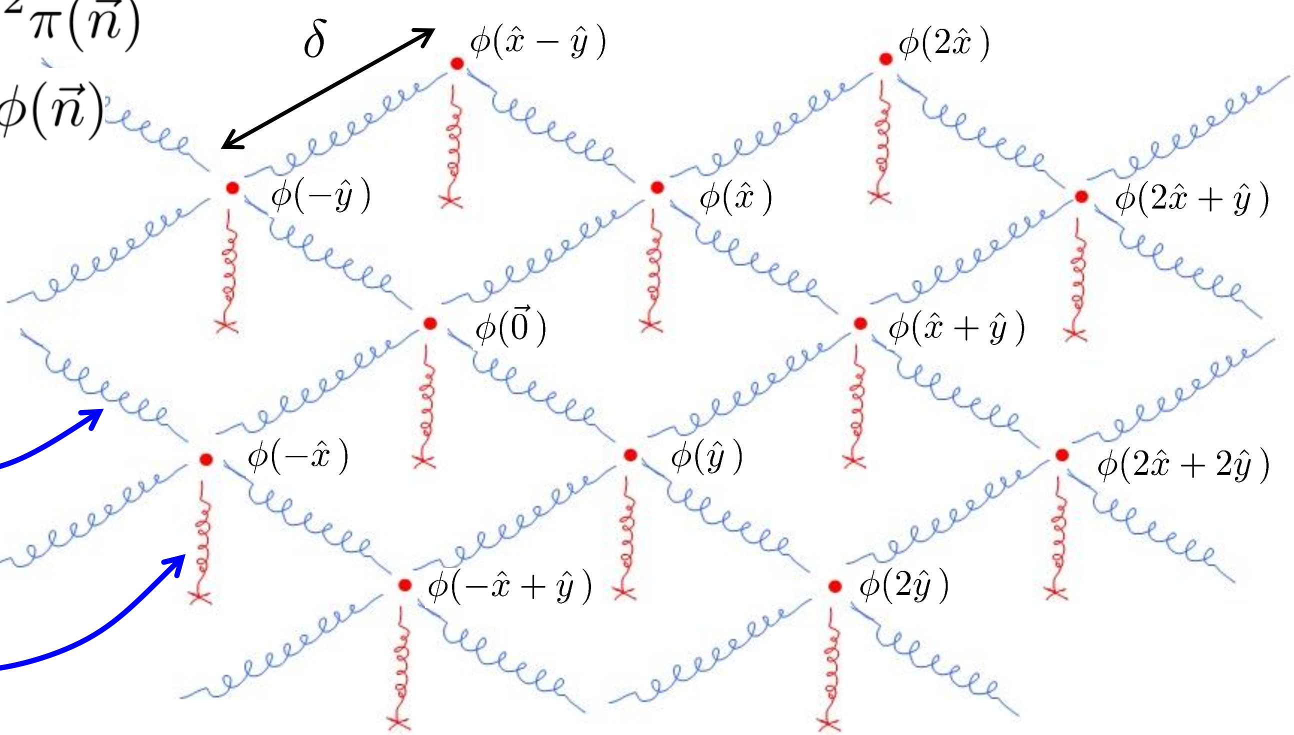
$$H = \frac{1}{2}\int d^{d-1}x\left[\pi(x)^2 + \vec{\nabla}\phi(x)^2 + m^2\phi(x)^2\right]$$

$$= \frac{1}{2}\sum_{\vec{n}}\left[\frac{p(\vec{n})^2}{M} + M\left\{\Omega^2\sum_i [x(\vec{n}\,) - x(\vec{n}-\sigma_i\,)]^2 + \omega^2 x(\vec{n}\,)^2\right\}\right]$$

$p(\vec{n}) = \delta^{-d/2}\pi(\vec{n})$

$x(\vec{n}) = \delta^{d/2}\phi(\vec{n})$

$M = 1/\delta$



$\Omega^2 = 1/\delta^2$

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Gates/Unitaries:

- natural operators: $x_i\,,\quad p_j$ $\qquad [x_i, p_j] = i\,\delta_{ij}$

$$Q_{ij} = \exp[i\epsilon\, x_i\, p_j] \quad (i \neq j)$$ "shift x_j by $\epsilon\, x_i$" (entangling)

$$Q_{ii} = \exp\left[i\frac{\epsilon}{2}(x_i\, p_i + p_i\, x_i)\right] = e^{\epsilon/2}\exp[i\epsilon\, x_i\, p_i]$$ "rescale x_i to $e^{\epsilon} x_i$" (scaling)

infinitesimal parameter: $\epsilon \ll 1$
gates produce small changes

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Target state: $$\psi_T(x_i) \simeq \exp\left[-\frac{1}{2}\sum \omega_{\vec{k}}\, x_{\vec{k}}^2\right]$$

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- have **infinite** number of possible circuits!!

$$\psi_T(x_i) = \cdots\; Q_{11}^{\alpha_{11,2}}\; Q_{22}^{\alpha_{22,1}}\; Q_{21}^{\alpha_{21,1}}\; Q_{11}^{\alpha_{11,1}}\; \psi_R(x_i)$$

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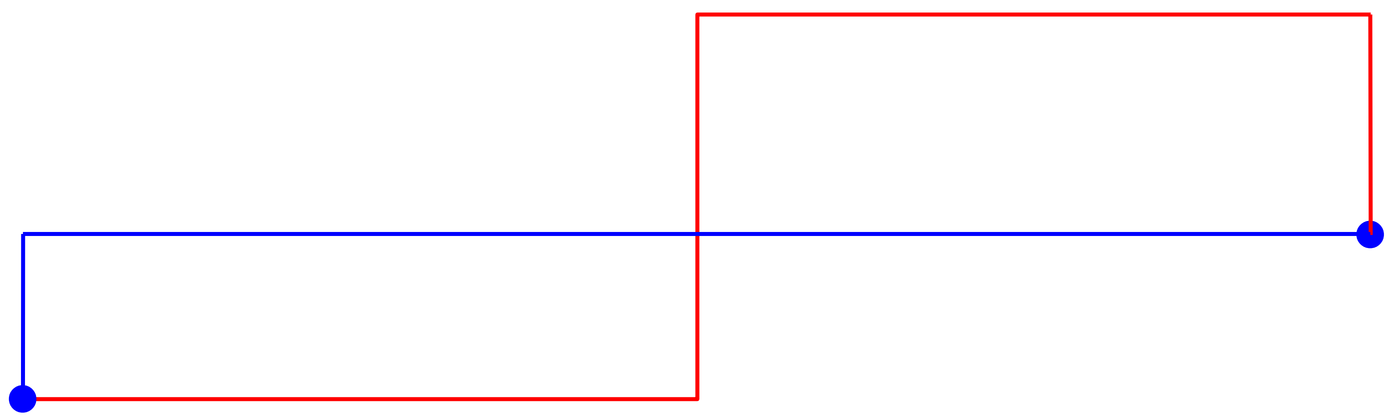
- adapt geometric construction of *Mike Neilsen*

Nielsen [arXiv:0502070]; Neilsen et al [arXiv:0603161]; Neilsen & Dowling [arXiv:0701004]

- circuits distinguish constructions of desired transformation U_{TR} with sequences of states, ie, paths in space of states between ψ_R and ψ_T



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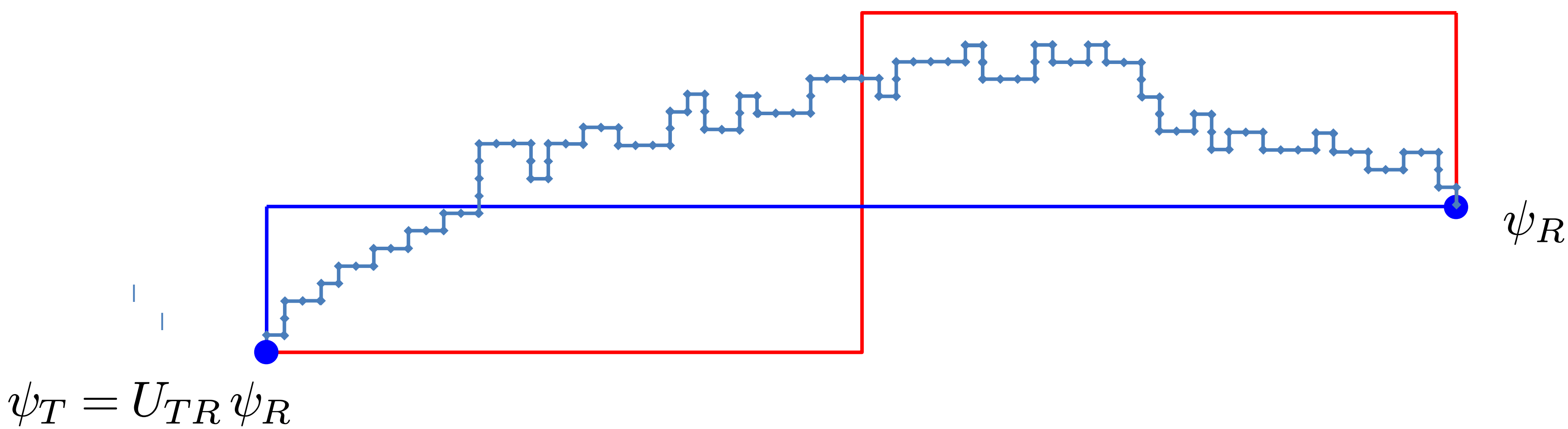

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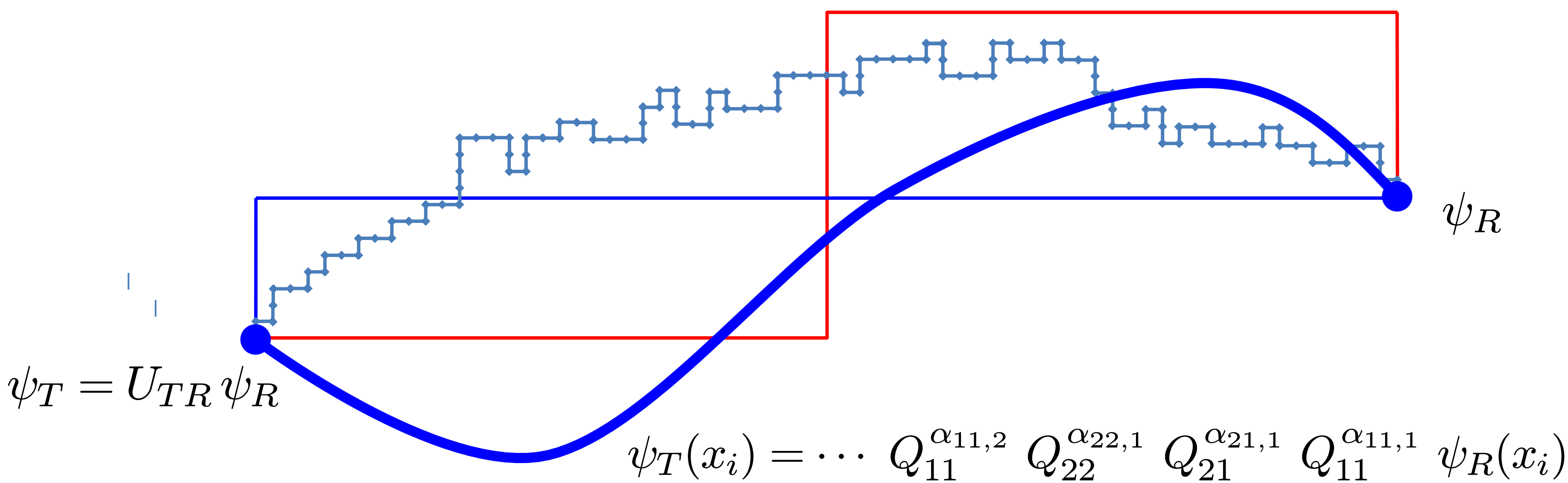
- tuning $\epsilon \to 0$ allows for smaller "steps" (recall $Q_{ij} = \exp[i\epsilon\, x_i\, p_j]$)
- as sequences of gates becomes more and more involved with shorter and shorter steps, paths approach smooth continuous trajectories



Nielsen approach:

- in order to optimize circuit easier to work with smooth functions on a smooth space (rather than with discrete gates)

$$\psi_T(x_i) = U_{TR}\,\psi_R(x_i) \qquad \text{with} \qquad U_{TR} = \mathcal{P}\exp\left[\int_0^1 ds\; Y^I(s)\,\mathcal{O}_I\right]$$

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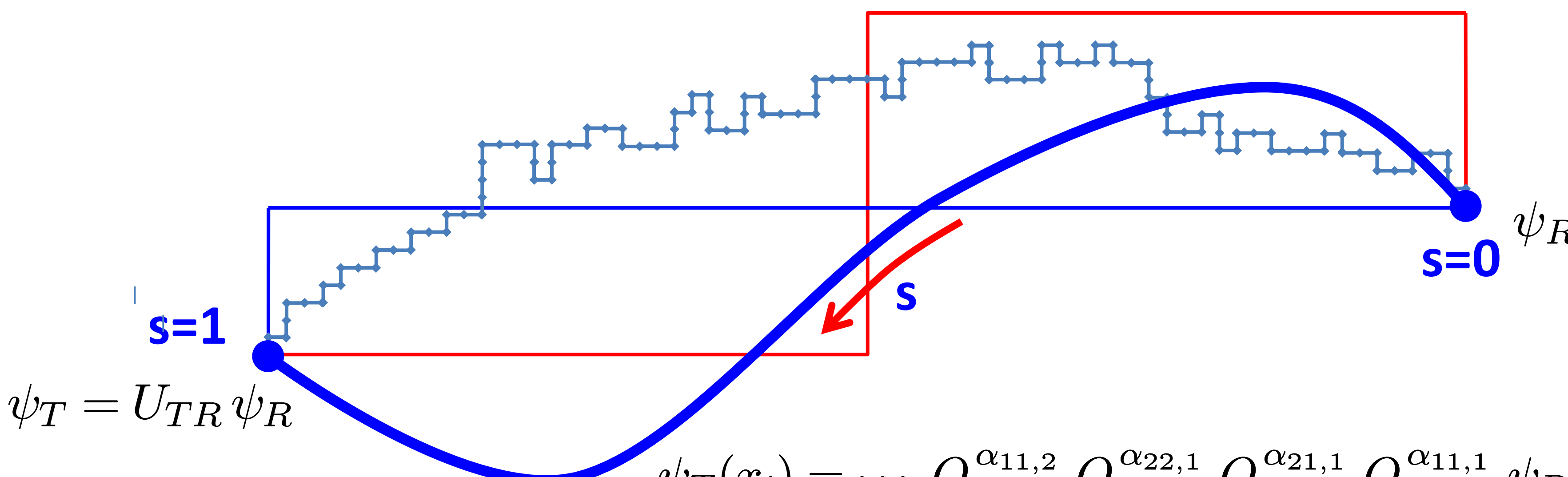

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right-to-left s: position label

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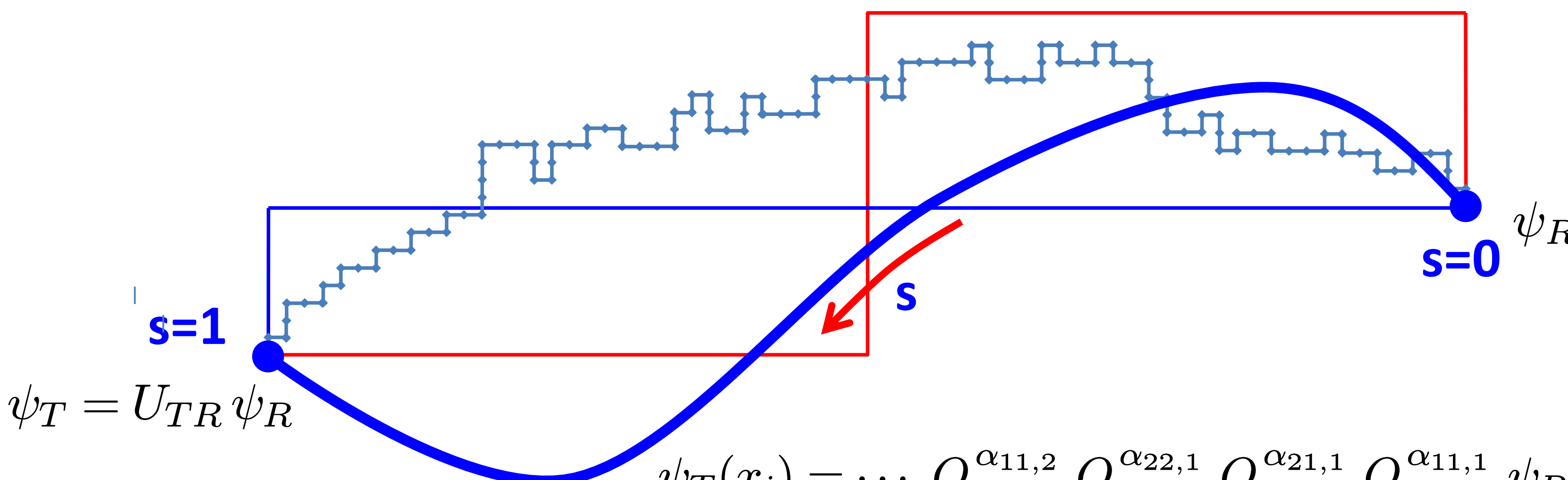

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$$U(s) = \mathcal{P}\exp\left[\int_0^s d\tilde{s}\; Y^I(\tilde{s})\,M_I\right] \quad \text{where} \quad U(s=0) = 1\,, \quad U(s=1) = U_{TR}$$

velocity: $Y^I(s) = \mathrm{Tr}\left[\partial_s U(s)\,U^{-1}(s)\,M_I\right]$

- alternatively, trajectories in space of states: $\psi(x_i;s) = U(s)\,\psi_R(x_i)$

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- geometry of “states” versus geometry of “unitaries”

Chapman, Heller, Marrochio & Pastawski

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- analogy with particle motion determined by minimizing classical action

⟶ minimizing the cost function: $\mathcal{D} = \int_0^1 ds \sum_I |Y^I(s)|$

[F_1]

⟶ extremal path $U(s)$ is geodesic in a Finsler geometry

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- analogy with particle motion determined by minimizing classical action

$\longrightarrow$ minimizing the cost function: $\mathcal{D} = \int_0^1 ds \sqrt{\sum_{IJ} \delta_{IJ}\, Y^I(s) Y^J(s)}$

[$F_1 \rightarrow F_2$]

$\longrightarrow$ extremal path $U(s)$ is geodesic in a Riemannian geometry

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velocity: $Y^I(s) = \mathrm{Tr}\left[\partial_s U(s)\, U^{-1}(s)\, M_I \right]$

- analogy with particle motion determined by minimizing classical action

⟶ minimizing the cost function: $\mathcal{D} = \int_0^1 ds \sqrt{\sum_{IJ} \underline{g_{IJ}}\, Y^I(s) Y^J(s)}$

$[\, F_1 \rightarrow F_2 \rightarrow F_q \,]$

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Calculate, Calculate, Calculate, . . .

(interesting geometry & interesting geodesics)

Nielsen approach:

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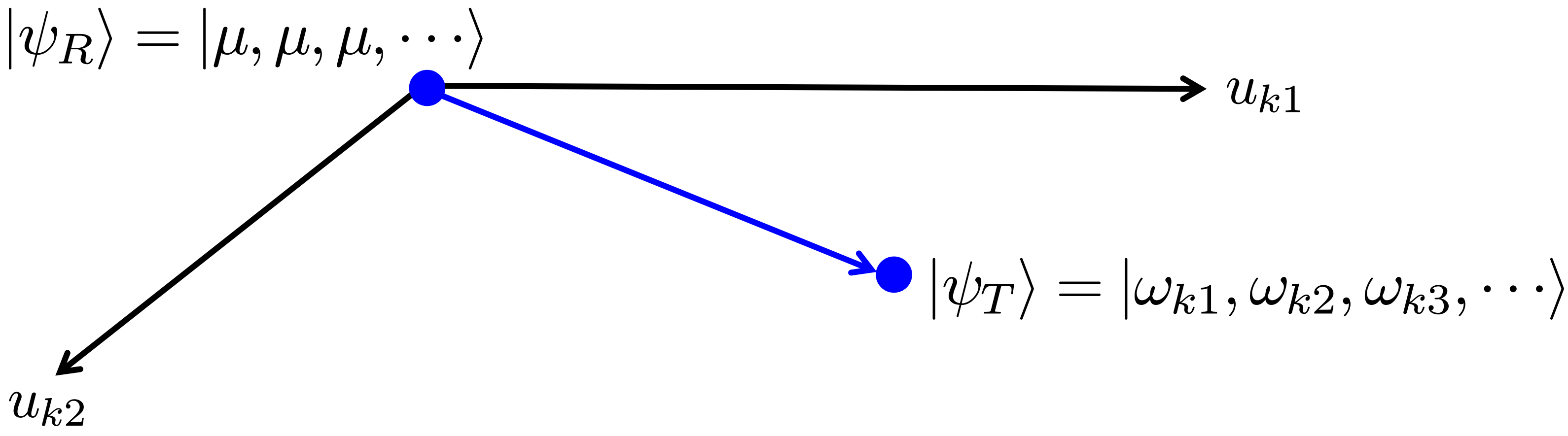
$\longrightarrow$ expressed in terms of normal modes, both target state and reference state are products of decoupled Gaussians!

$\longrightarrow$ **optimal path/circuit simply consists of squeezing/scaling each of the normal modes independently!!**



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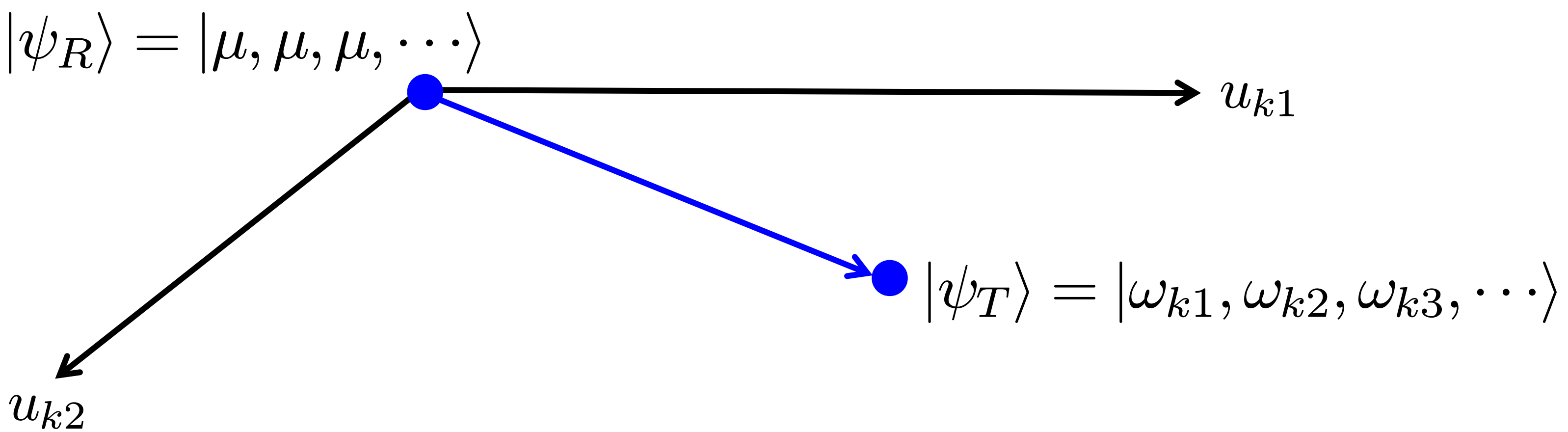
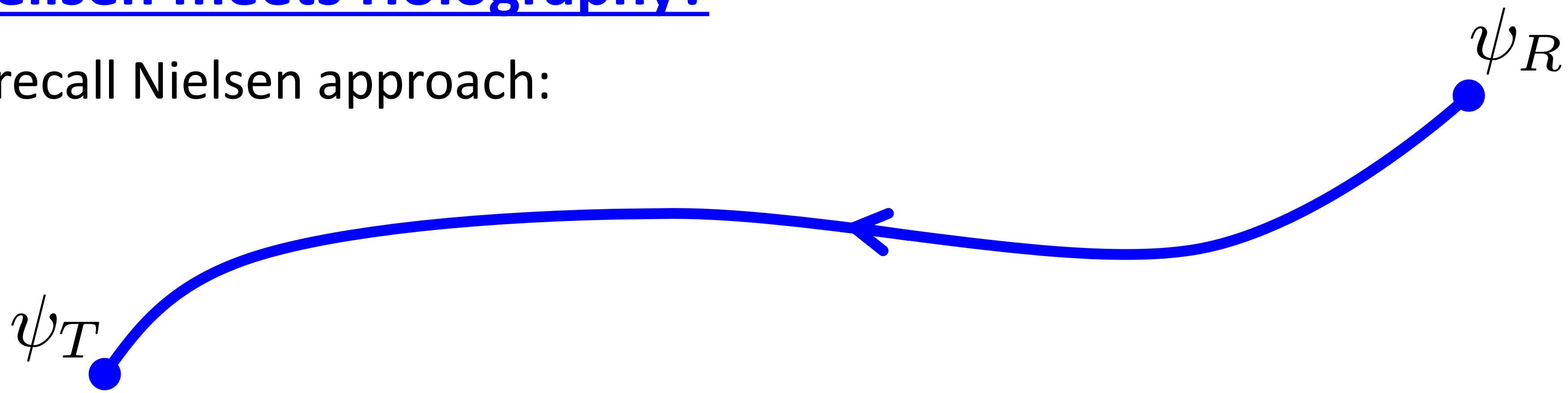


- using cost function: $\mathcal{D} = \int_0^1 ds \sum_{IJ} \delta_{IJ}\, Y^I(s) Y^J(s)$

$$\longrightarrow \quad \mathcal{C}_{vac} = \frac{1}{2}\sum \log^2\left[\omega_{\vec{k}}/\mu\right] \qquad \sim \frac{V}{\delta^{d-1}} \log^2[\delta\,\mu]$$

Neilsen meets Holography?

- recall Nielsen approach:



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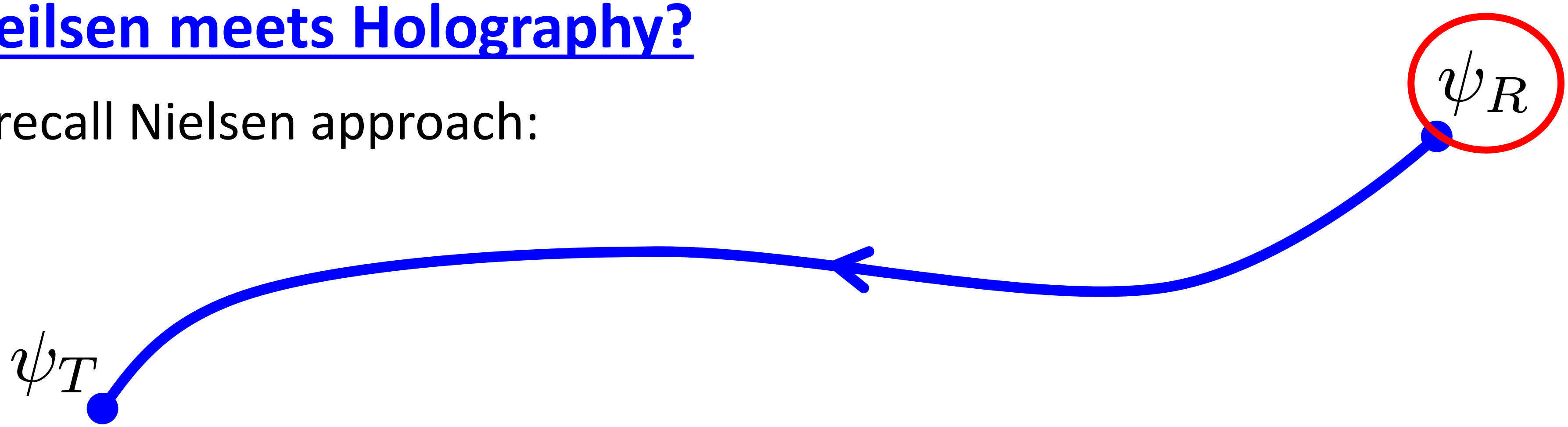

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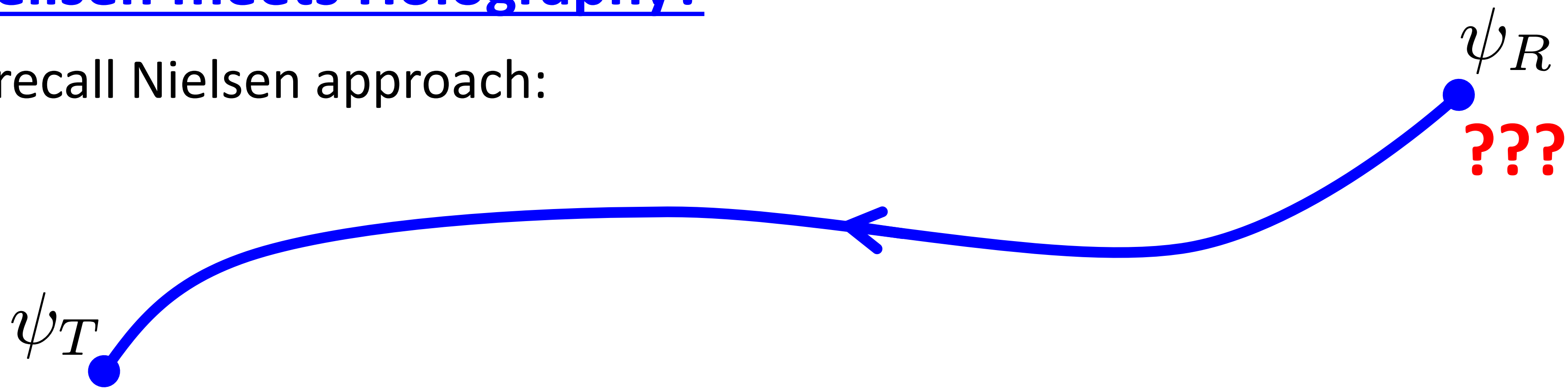

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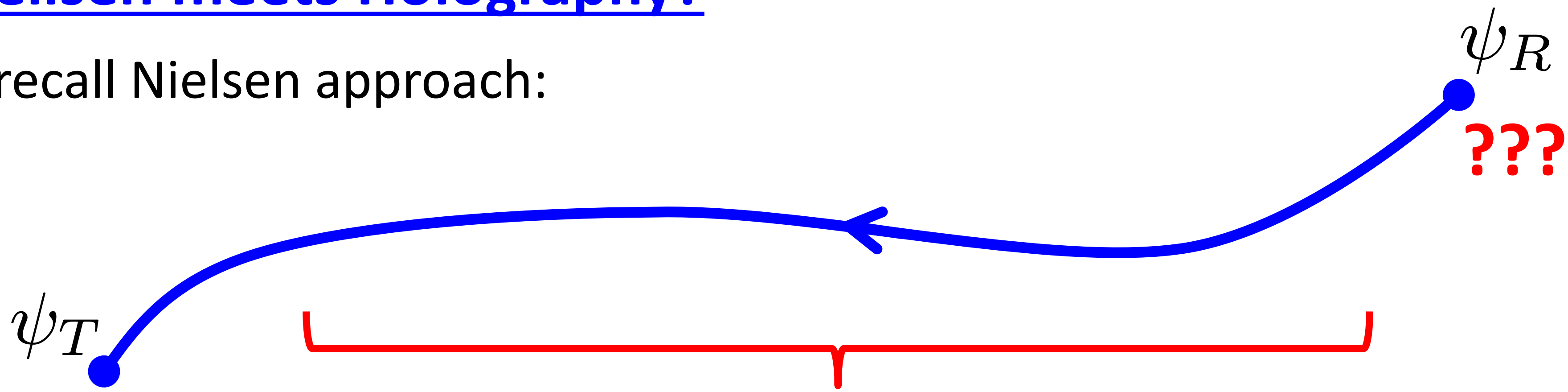

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→ Virasoro generators? primary operators? coadjoint orbits?

eg, Caputa & Magan

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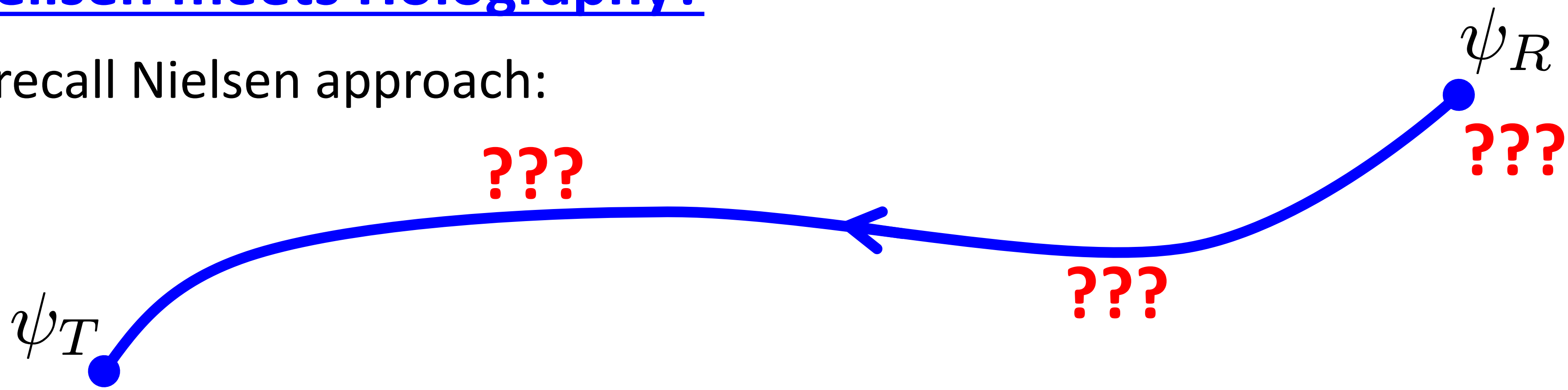

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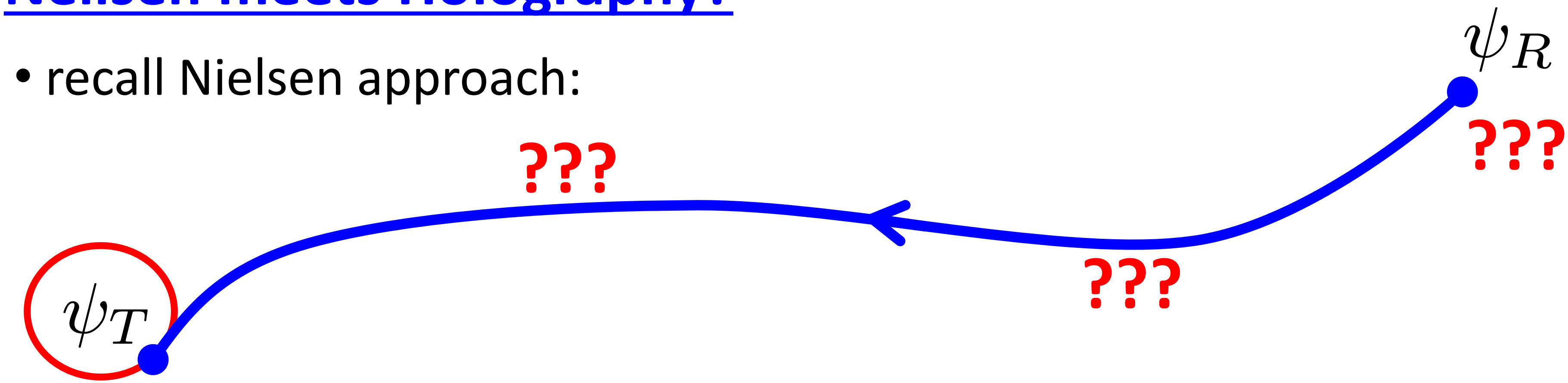

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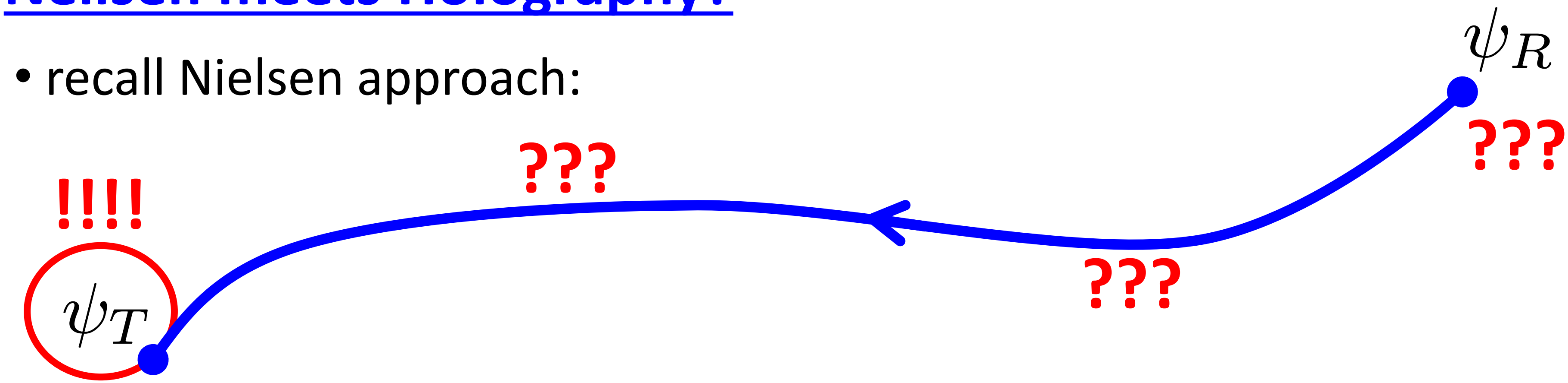

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eg, Caputa & Magan

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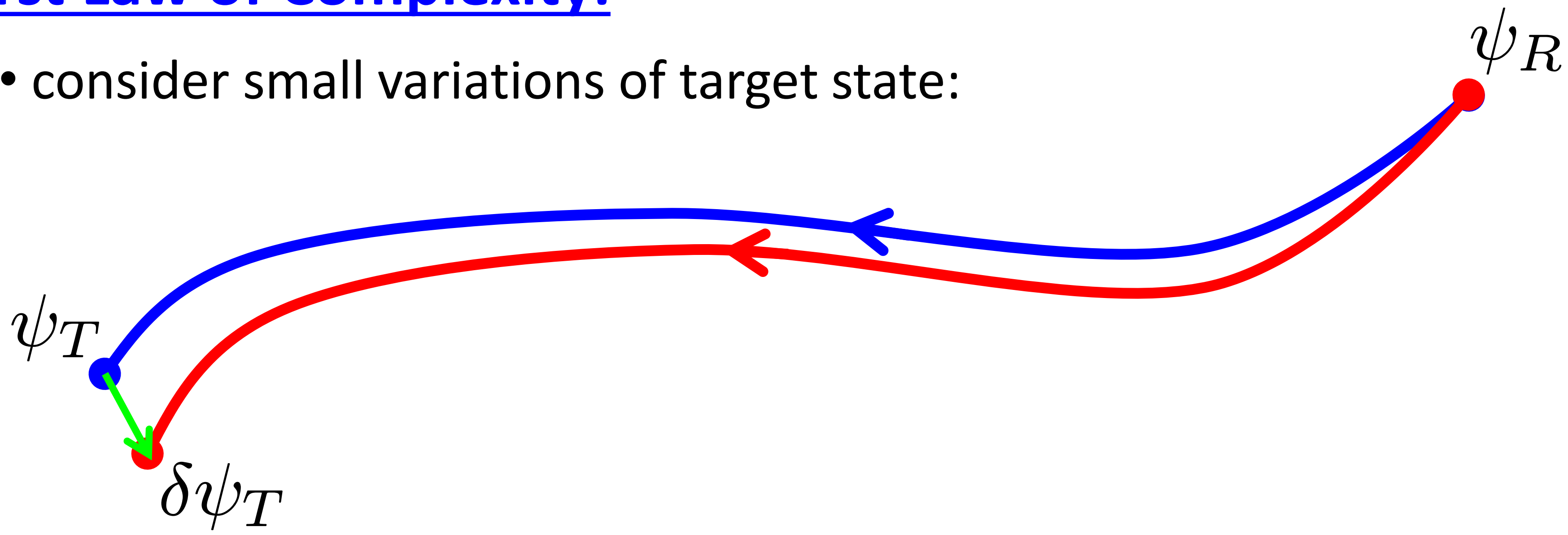
⟶ **good place to focus our attention!**

First Law of Complexity:

Bernamonti, Galli, Hernandez, RCM, Ruan & Simon

- consider small variations of target state:



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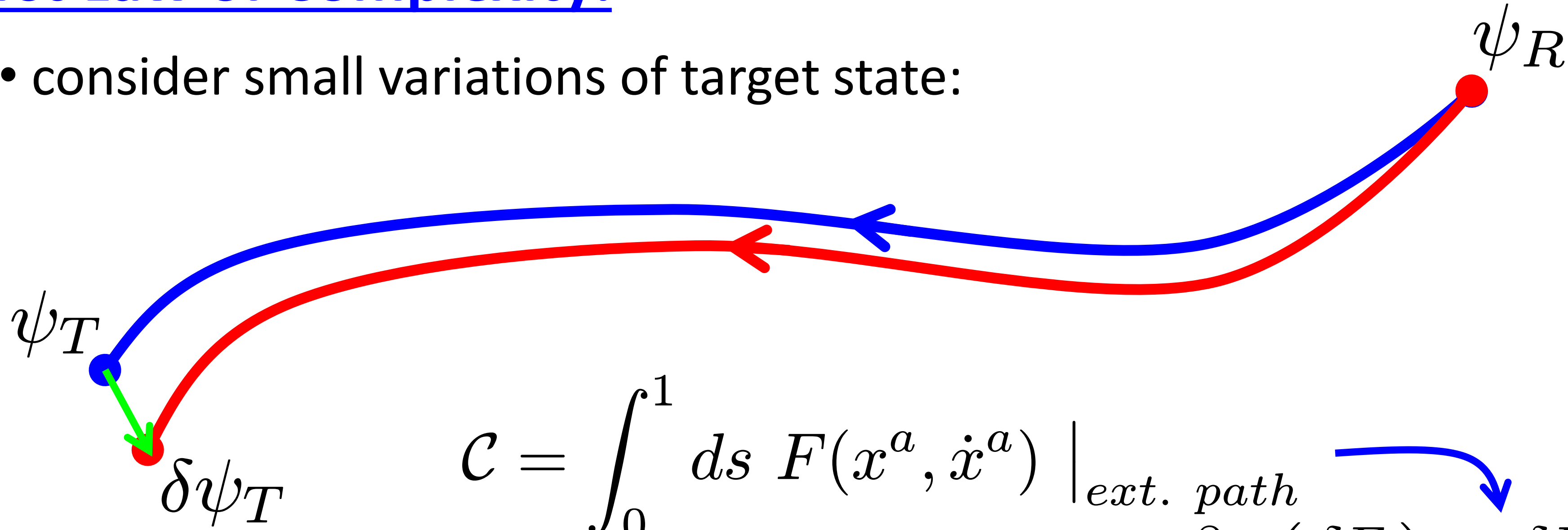

$$\mathcal{C} = \int_0^1 ds\ F(x^a, \dot{x}^a)\ \Big|_{ext.\ path}$$

- analog of particle action evaluated on-shell: $0 = -\frac{\partial}{\partial s}\left(\frac{\delta F}{\delta \dot{x}^a}\right) + \frac{\delta F}{\delta x^a}$

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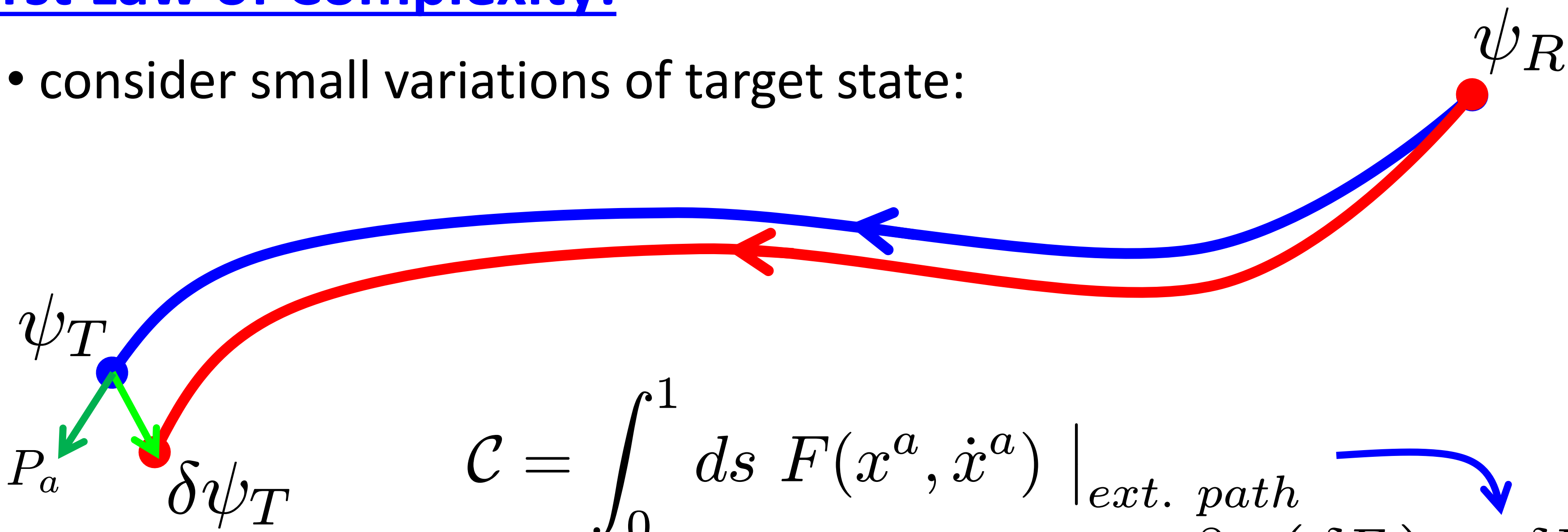

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- endpoint contribution at $s = 1$ $\longrightarrow$ bulk variation vanishes by eq. of motion and chose to fix $\delta\psi_R = 0$ at $s = 0$

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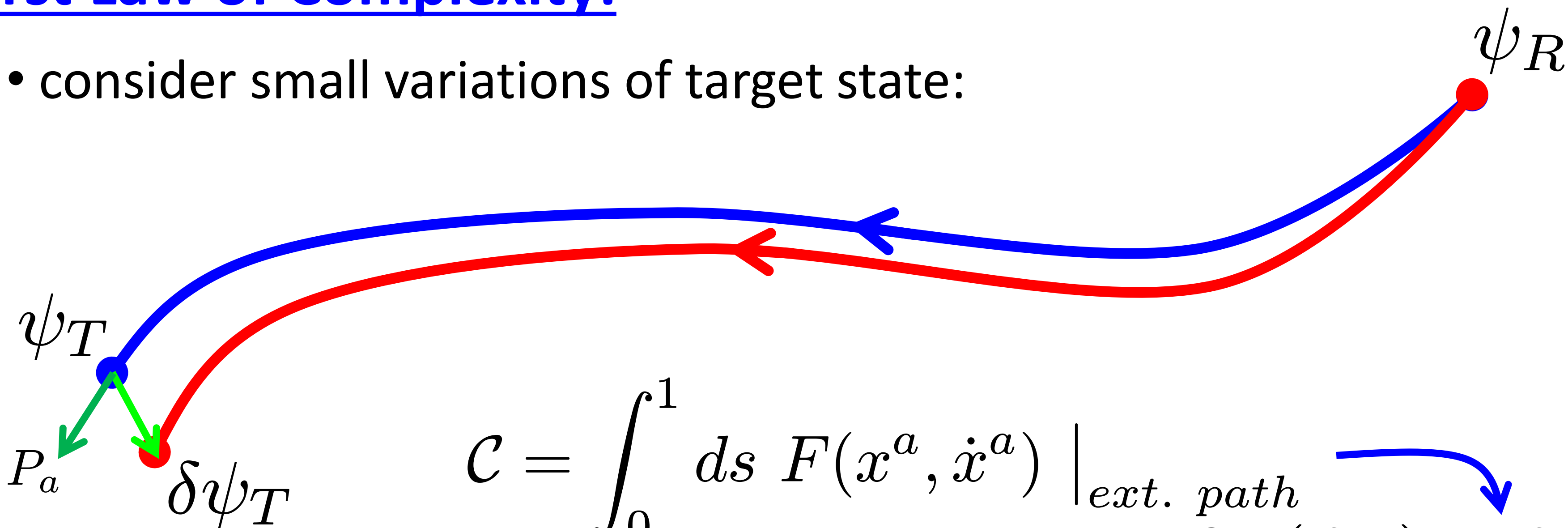

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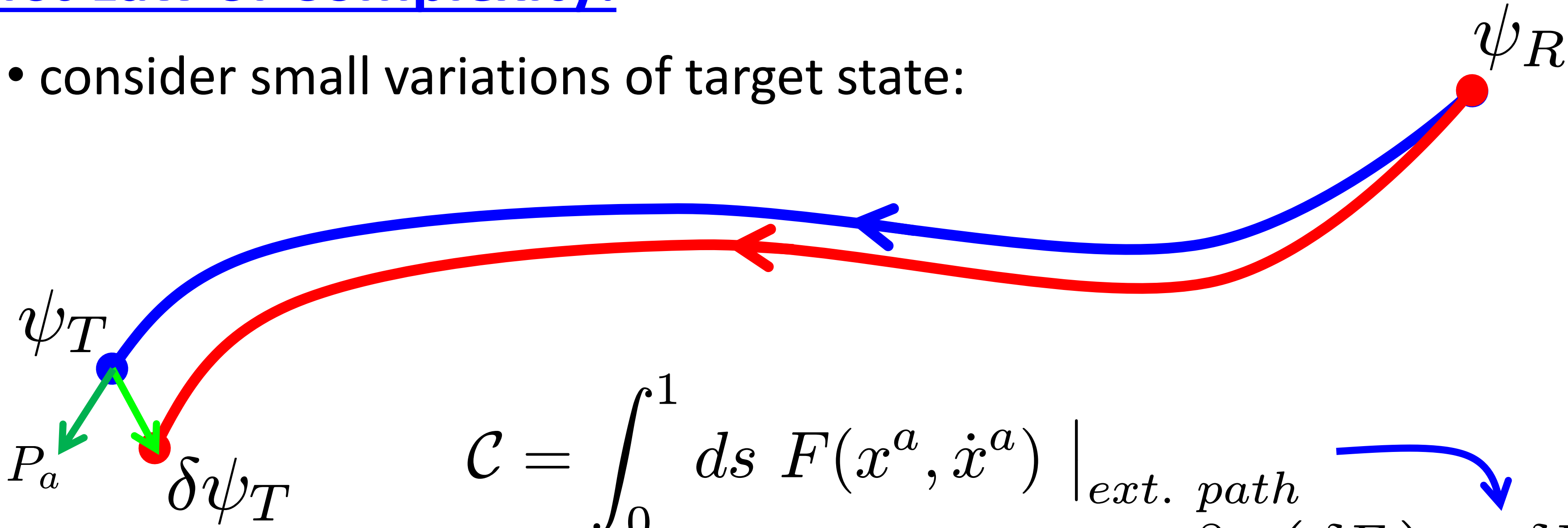

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$$\delta\mathcal{C} = \cancelto{0}{\delta x^a\, P_a\Big|_{s=1}} + \frac{1}{2}\,\delta x^a\,\delta P_a\Big|_{s=1}$$

- keep 2nd order term for $\delta x^a \perp P_a$

⟶ still a boundary contribution!

$$\delta P_a = \delta x^b\,\frac{\delta^2 F}{\delta x^b \delta \dot{x}^a} + \delta \dot{x}^b\,\frac{\delta^2 F}{\delta \dot{x}^b \delta \dot{x}^a}$$

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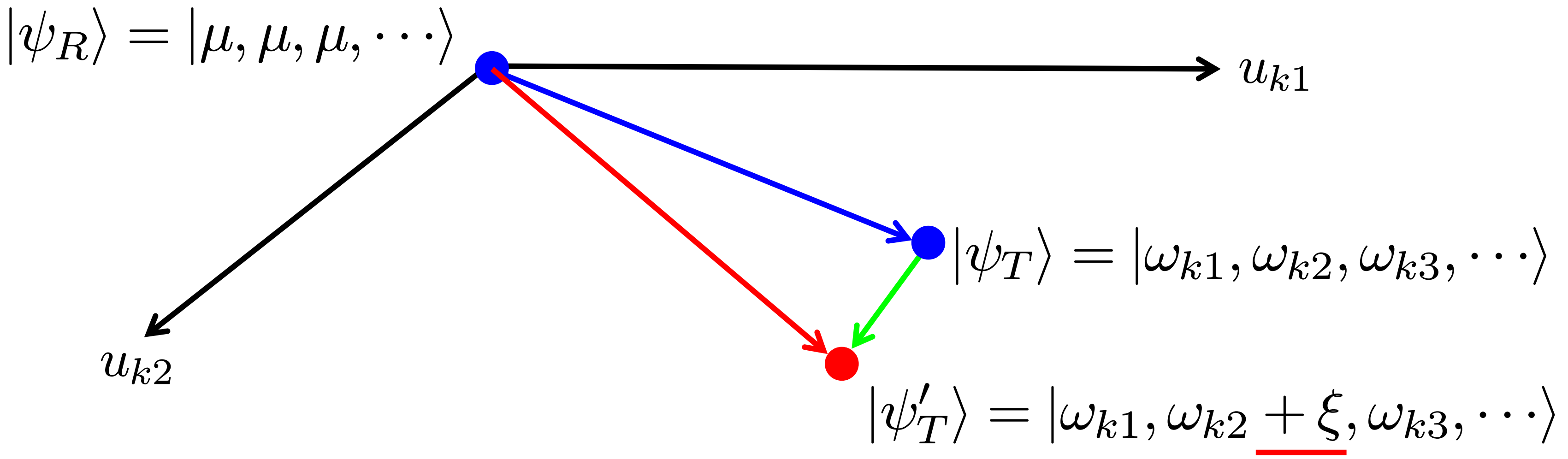
- consider small variations of target state **in free scalar theory**:

$\longrightarrow$ **squeezed state:** apply stronger scaling for some modes



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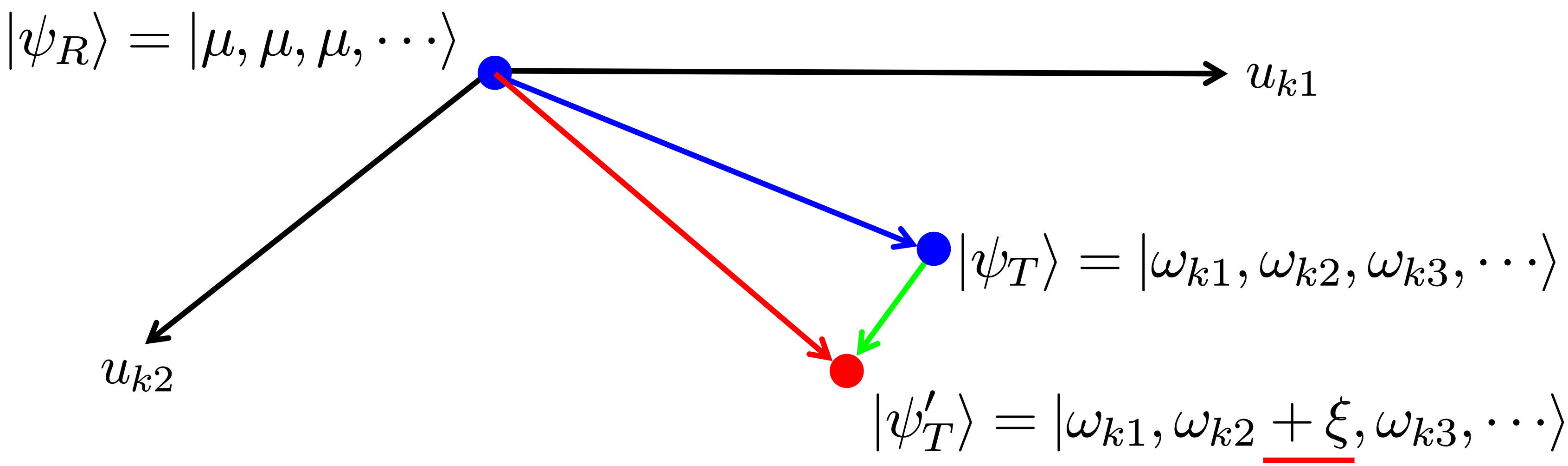

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$$\mathcal{D} = \int_0^1 ds \sum_{IJ} \delta_{IJ} Y^I(s) Y^J(s) \longrightarrow \delta\mathcal{C} = \underbrace{\xi}_{\delta x} \times \underbrace{\frac{\log[\omega_{k2}/\mu]}{\omega_{k2}}}_{P\big|_{s=1}}$$

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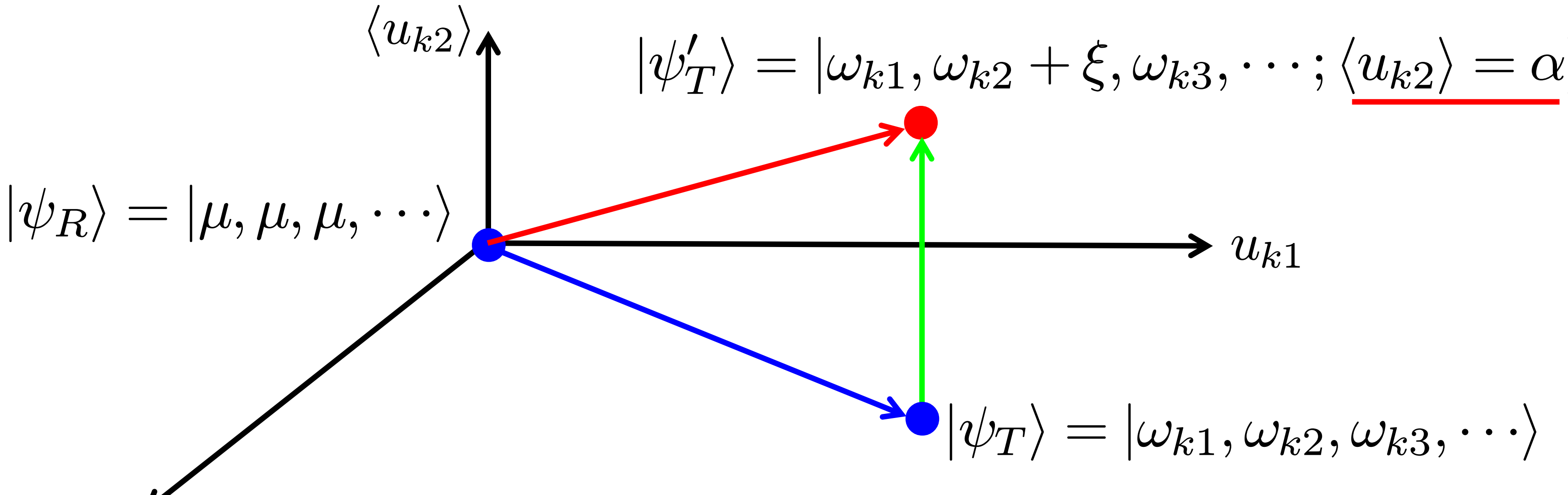

→ **coherent state:** apply "translation" gate

$Q_{0i} = \exp[i\epsilon\, x_0\, p_i]$ for some modes

(Guo, Hernandez, RM & Ruan)

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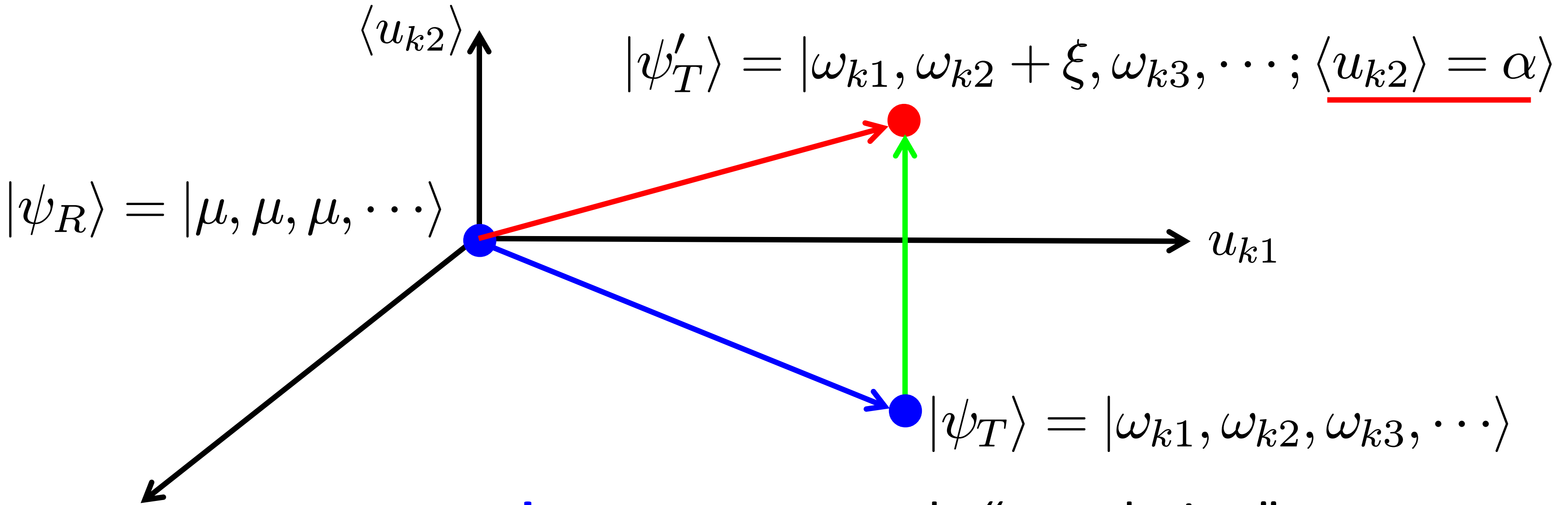

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Bernamonti, Galli, Hernandez, RCM, Ruan & Simon

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- "quantum circuit" builds "quantum gravity" state, with semi-classical description: quantum fields/strings in a classical spacetime geometry

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Note: $\hat{\mathcal{O}}_\Delta(t, \theta_i) \sim \lim_{r\to\infty} \dfrac{\hat{\Phi}(t, r, \theta_i)}{r^\Delta}$

AdS/CFT correspondence is a dictionary providing two languages describing a single set of physical phenomena!

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AdS

first back-reaction

$$\phi = \phi^{(1)} + \phi^{(3)} + \phi^{(5)} + \cdots$$

initial configuration

Bernamonti, Galli, Hernandez, RCM, Ruan & Simon

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mode amplitude

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$\ell = 0,\ m = 0$ spherical symmetry

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  coherent state: turn on classical scalar in AdS (with small amplitude)
- working perturbatively in amplitude (or Newton's constant G_N)

$$g_{ab} = \underbrace{g^{(0)}_{ab}}_{\text{AdS}} + \underbrace{g^{(2)}_{ab}}_{\text{first back-reaction}} \cancel{+ g^{(4)}_{ab} + \cdots}$$

$$\phi = \underbrace{\phi^{(1)}}_{\text{initial configuration}} \cancel{+ \phi^{(3)} + \phi^{(5)} + \cdots}$$

- choose, eg, AdS$_4$ (d=3), m=0 ($\Delta = 3$), and profile is eigenmode:

$$\phi^{(1)} = \varepsilon \sum \big[\underbrace{\alpha_n}_{\text{mode amplitude}} \ \exp(-i\omega_n t)\ u_n(r) + c.c.\,\big]$$

spherical symmetry

- evaluate $g^{(2)}_{ab}$ and, eg, evaluate change in gravitational action

Calculate, Calculate, Calculate, . . .

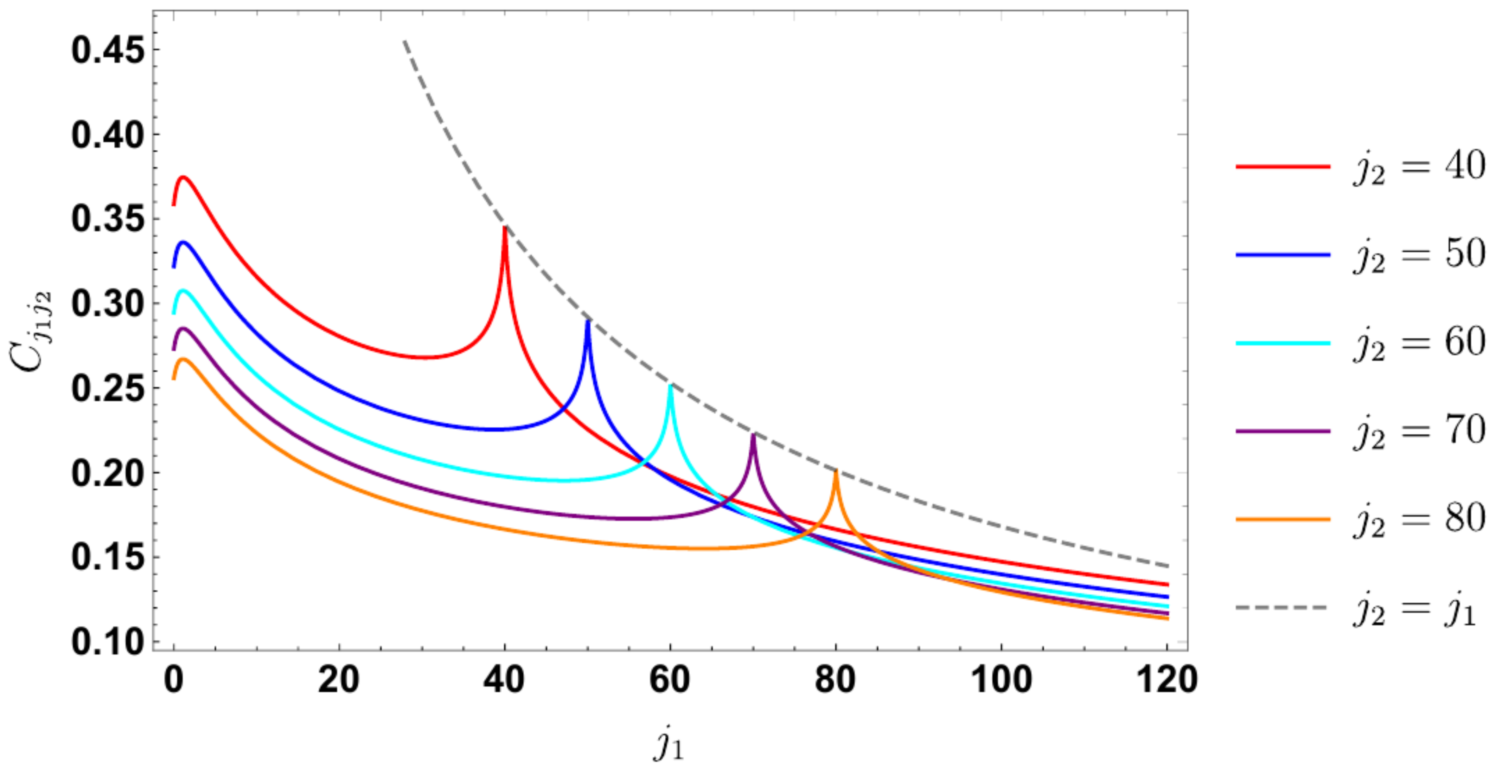
First Law of Complexity:

Bernamonti, Galli, Hernandez, RCM, Ruan & Simon

- change in complexity: $$\delta\mathcal{C}_A = \frac{\varepsilon^2}{\pi^2}\sum_{j_1,j_2}\alpha_{j_1}\,\alpha_{j_2}\,C_{j_1,j_2}$$

$$C_{j_1,j_2} = \sqrt{\frac{(j_1+\frac{3}{2})(j_2+\frac{3}{2})}{(j_1+1)\,(j_1+2)\,(j_2+1)\,(j_2+2)}}\left(H_{j_1+\frac{1}{2}}+H_{j_1+\frac{3}{2}}+H_{j_2+\frac{1}{2}}+H_{j_2+\frac{3}{2}}-H_{j_1+j_2+\frac{5}{2}}-H_{j_1-j_2-\frac{1}{2}}-2+4\log 2\right)$$

$$\xrightarrow[j_1=j_2\to\infty]{} \frac{3\varepsilon^2\alpha^2}{\pi}\,\frac{\log(2j_1)}{j_1}+\mathcal{O}\left(\frac{\alpha^2}{j_1}\right)$$



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- variation is 2nd order $\longrightarrow$ $\delta x^a \perp P_a$
- independent of scales, eg, AdS scale, volume or cut-off δ
- for cost function $F(x^a,\dot{x}^a) = g_{ab}\,\dot{x}^a\,\dot{x}^b$, then $C_{j_1,j_2} \sim g_{j_1 j_2}\big|_{s=1}$
- compare to coherent state in free scalar theory in AdS

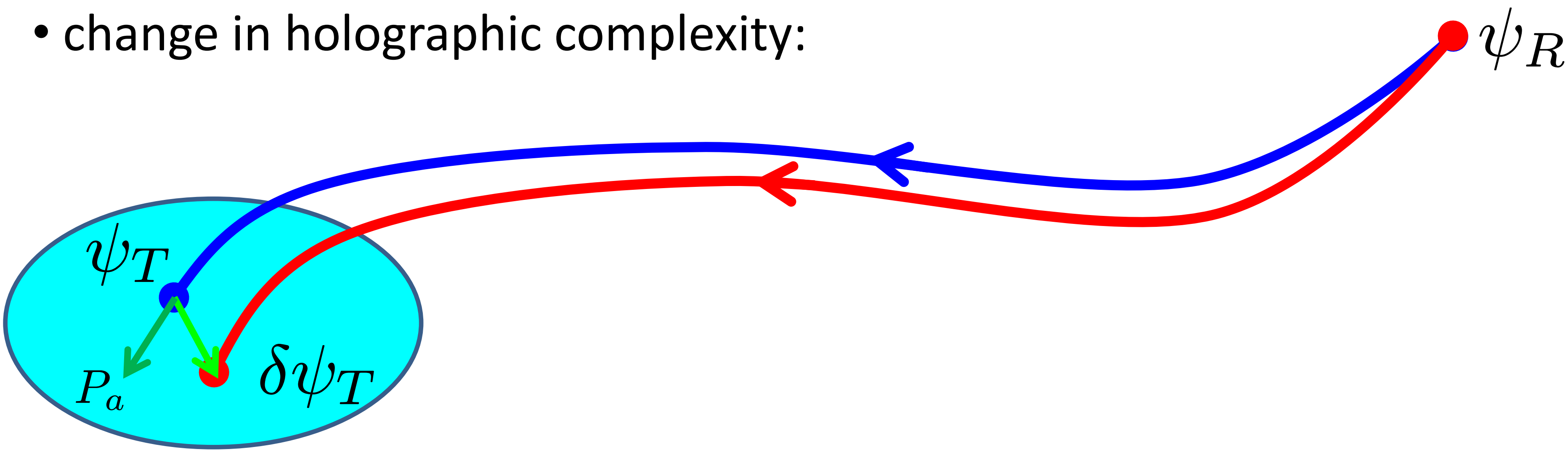
$$\kappa = 2\ :\quad C_{j_1,j_2} \xrightarrow[j\to\infty]{} \delta_{j_1\,j_2}\,\frac{\alpha^2}{j_1}\,\frac{\mu R}{\mu^2 x_0^2}\,\log\left(\frac{2j_1}{\mu R}\right)$$

- compares well(?) to holographic result but in holography, scales must conspire, eg, $\mu\, x_0 \sim 1 \sim \mu\, R$

First Law of Complexity:

Bernamonti, Galli, Hernandez, RCM, Ruan & Simon

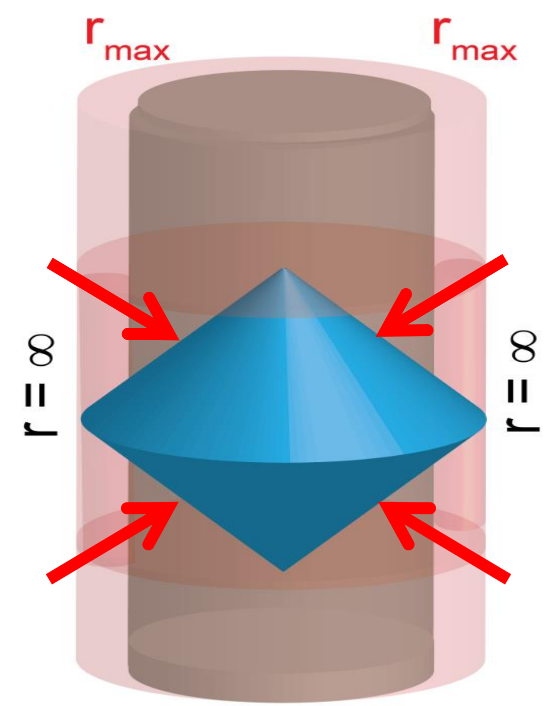
• change in holographic complexity:



• variation is a boundary term that comes from **the end of the circuit**

• $\delta\mathcal{C}_A$ comes only from boundary of WDW patch ⟶

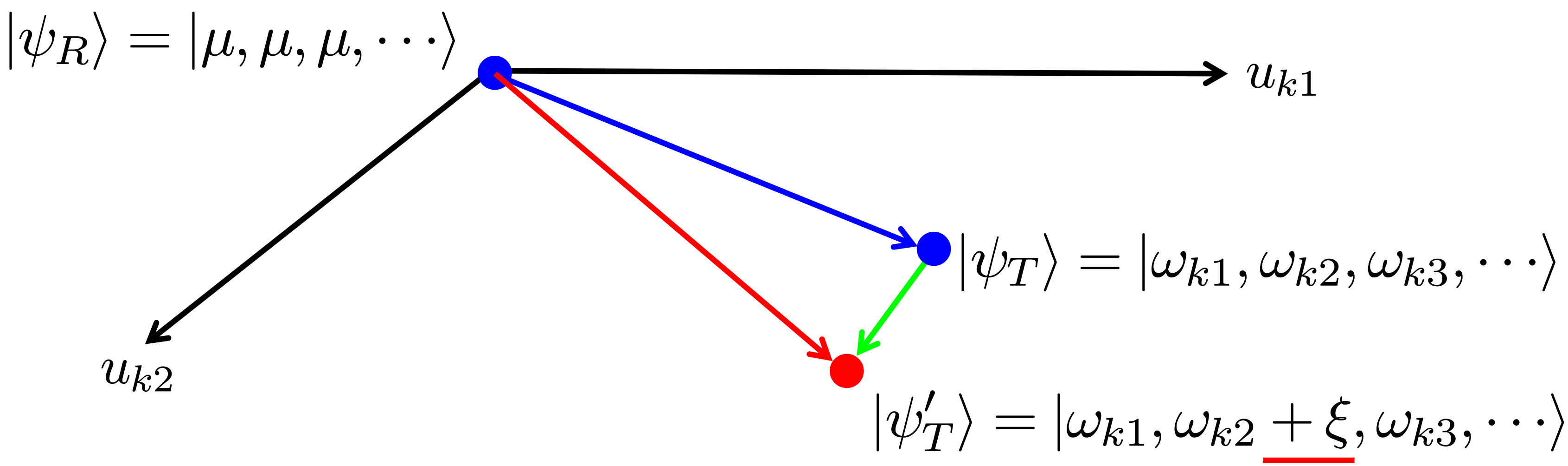
(build up spacetime with null cone layers)



First Law of Complexity:

- consider small variations of target state **in free scalar theory**:

→ **squeezed state:** apply stronger scaling for some modes



- using cost function:

$$\mathcal{D} = \int_0^1 ds \sum_{IJ} \delta_{IJ} Y^I(s) Y^J(s) \longrightarrow \delta\mathcal{C} = \underbrace{\xi}_{\delta x} \times \underbrace{\frac{\log[\omega_{k2}/\mu]}{\omega_{k2}}}_{P\big|_{s=1}}$$

First Law of Complexity:

Belin, Chen & RCM

- consider small variations of target state **in holography**:

squeezed state: introduce small "squeezing" of modes of bulk scalar

- quantum field operator: $\hat{\Phi} = \sum \left[e^{-i\omega_n t} u_n(r,\theta_i)\ a_n + e^{i\omega_n t} u_n^*(r,\theta_i)\ a_n^\dagger \right]$

$$\longrightarrow \quad a_n^\dagger = i \int_{t=const.} dr\, d^{d-1} \quad \sqrt{-g} g^{00} \left[e^{-i\omega_n t} u_n \overleftrightarrow{\partial_t}\, \hat{\Phi} \right] \ \ ; \ \ [a_n,\, a_m^\dagger] = \delta_{nm}$$

- squeezed states

$(\varepsilon \ll 1)$

$$|\varepsilon\, \xi_k\rangle \equiv e^{\varepsilon \sum S(\xi_n)} |0\rangle \quad \text{with} \qquad S(\xi_n) = \frac{1}{2}(\xi_n a_n^\dagger a_{\bar{n}}^\dagger - \xi_n^* a_k a_{\bar{n}})$$

$$a_n[\xi] = S^\dagger[\xi]\, a_n\, S[\xi] = a_n\ \cosh\left[\frac{\xi_n + \xi_{\bar{n}}}{2}\right] + a_{\bar{n}}^\dagger\, e^{i\phi}\ \sinh\left[\frac{\xi_n + \xi_{\bar{n}}}{2}\right]$$

$\longrightarrow$ ***S*[*ξ*]** squeezes correlatons in the scalar vacuum

$\longrightarrow$ **remains squeezed state in boundary theory**

First Law of Complexity:

Belin, Chen & RCM

- consider small variations of target state **in holography**:

squeezed state: introduce small "squeezing" of modes of ~~bulk scalar~~ boundary operator

- generalized free field: $\hat{\mathcal{O}}_\Delta = \sum \left[e^{-i\omega_n t}\tilde{u}_n(\theta_i)\ a_n + e^{i\omega_n t}\tilde{u}_n^*(\theta_i)\ a_n^\dagger\right]$

$$\longrightarrow \quad a_n^\dagger = \frac{i}{\mathcal{N}_n}\int_0^{2\pi R} dt \int d^{d-1} \quad \left[e^{-i\omega_n t}\tilde{u}_n \overleftrightarrow{\partial_t}\, \hat{\mathcal{O}}_\Delta\right] \quad ; \quad \left[a_n,\, a_m^\dagger\right] = \delta_{nm}$$

- squeezed states $\qquad (\varepsilon \ll 1)$

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Belin, Chen & RCM

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- working perturbatively in squeezing parameter (or G_N or $1/N^2$)

  ⟶ determine leading backreaction from semiclassical Einstein eq.

$$R_{ab} - \frac{1}{2} g_{ab} R = 8\pi G_N \langle \varepsilon \xi_n | :T_{ab}: | \varepsilon \xi_n \rangle$$

Belin, Chen & RCM

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- evaluate change in gravitational action including $\langle \varepsilon\xi_n | :I^\Phi: | \varepsilon\xi_n \rangle$

$$\longrightarrow \boxed{\delta\mathcal{C} \sim \xi_n}$$

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- to maintain semiclassical state (cf. double trace operators), restrict $\xi_n \sim O(1)$ in large *N* expansion producing $\delta C \sim O(1)$

  $\longrightarrow$ quantum corrections to C_A or C_V?

- justification to ignore quantum corrections for coherent state?

Conclusions/Questions/Outlook:

- complexity model for free scalar shows surprising similarities to holographic proposals for complexity of boundary CFT states
- first Law may provide avenue to concrete connection between "Neilsen's circuit complexity" and "holographic complexity"?
  - higher dimensions; other fields; other quantum states
  - insight into quantum corrections to C_A or C_V?
  - beyond spherical symmetry (tension with Fleury poster??)
  - similar extremization for Fubini-Study and path integral optimization procedures $\longrightarrow$ can apply analogous 1st law

- QFT/path integral description of "complexity" in boundary CFT?
- what is boundary dual of these gravitational observables?

$\longrightarrow$ preliminary suggestions:

Caputa et al (1703.00456; 1706.07056;1804.01999); Czech (1706.00965); Takayanagi (1808.09072); Camargo, Heller, Jefferson & Knaute (1904.02713)

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  - ➢ simila … l
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Lots to explore!

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