The holographic marginal independence problem

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+ WIP with S. Cuenca, V. Hubeny and M. Rangamani

Current status on (holographic) entropy inequalities: a lightning review

N parties	RT	HRT	QM	
2	1 (SA)	1 (SA)	1 (SA)	Ryu-Takayanagi '06
3	2 (SA, MMI)	2 (SA, MMI)	2 (SA, SSA)	Headrick-Takayanagi '07, Hayden et al. '11, Wall '12
4	2 (SA, MMI)	2 (SA, MMI)	≥ 2	Bao et al. '15
5	7	≥ 2	≥ 2	Bao et al. '15 Cuenca '19
6	≥ 23	≥ 2	≥ 2	Cuenca '19

(number of non-redundant inequalities up to symmetries)

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Main open problems:

- how can we find good candidates for new inequalities efficiently ?
  (Hubeny-Rangamani-MR ’18)
- are all RT-inequalities provable by contraction maps (cutting and gluing) ?
- are all RT-inequalities also true for HRT ?
  (Cui et al. ’18, Hubeny ’18, Czech-Dong ‘19)
- how can we efficiently check if a set of proven inequalities is complete ?

A deeper conceptual issue

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More specifically it is interesting to focus on the following questions:

- what are the specific aspects of holography that imply the inequalities ?
- how do the holographic inequalities relate to restrictions which are purely quantum mechanical ?
- what kind of constraints do the inequalities impose on the structure of density matrices of spacial subregions for states which are dual to classical geometries ?
- is there a general physical principle from which all the inequalities can be derived ?

Main goal today: introduce a general problem which can be formulated both in quantum mechanics and holography and can shed light on the answers to all these questions.

Warm up: a simple example for 3 parties

Is it possible to construct a tripartite density matrix $\rho_{\mathcal{ABC}}$ which satisfies all of the following properties?

$$\rho_{\mathcal{ABC}} \neq \rho_{\mathcal{A}} \otimes \rho_{\mathcal{BC}} \qquad \rho_{\mathcal{ABC}} = \rho_{\mathcal{B}} \otimes \rho_{\mathcal{AC}} \qquad \rho_{\mathcal{ABC}} = \rho_{\mathcal{C}} \otimes \rho_{\mathcal{AB}}$$

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The answer is **no**, as can immediately be seen as follows

$$S_{\mathcal{ABC}} = S_{\mathcal{B}} + S_{\mathcal{AC}} = S_{\mathcal{B}} + S_{\mathcal{A}} + S_{\mathcal{C}} = S_{\mathcal{A}} + S_{\mathcal{BC}}$$

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We can think of this problem as a very simple example of a **quantum marginal problem.** The difference is that in this case the input is not a collection of specific density matrices, but only a **pattern of marginal independence**.

Given a specific pattern we just want to know if a global density matrix which respects this pattern exists or not.

Entropy vectors and entropy space

In the general case how can we classify all the possible patterns that we may consider ?
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Consider a N-partite density matrix $\rho_{\mathcal{A}_1\mathcal{A}_2\cdots\mathcal{A}_\mathsf{N}}$ acting on an arbitrary Hilbert space

$$\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_\mathsf{N}$$

We define the corresponding **entropy vector** as the following element of **entropy space**

$$\mathbf{S}(\rho_{\mathcal{A}_1\mathcal{A}_2\cdots\mathcal{A}_\mathsf{N}}) = (S_{\mathcal{A}_1}, S_{\mathcal{A}_2}, \ldots, S_{\mathcal{A}_1\mathcal{A}_2}, S_{\mathcal{A}_1\mathcal{A}_3}, \ldots, S_{\mathcal{A}_1\mathcal{A}_2\cdots\mathcal{A}_\mathsf{N}}) \in \mathbb{R}_+^{2^\mathsf{N}-1}$$

Two “non-overlapping” marginals $\rho_\mathcal{X}$ and $\rho_\mathcal{Y}$ are **independent if and only if**

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$$\mathbf{I}_2(\mathcal{X} : \mathcal{Y}) = 0$$

This equation defines a **codimension-one hyperplane** in entropy space. If the two marginals are independent the entropy vector for $\rho_{\mathcal{A}_1\mathcal{A}_2\cdots\mathcal{A}_\mathsf{N}}$ belongs to this hyperplane.

If more than one instance of the mutual information is saturated, the vector belongs to the intersection of the corresponding hyperplanes.

The quantum marginal independence problem

To classify all possible patterns of marginal independence we just need to consider all hyperplanes corresponding to all instances of the mutual information for any given N.

This collection defines an **arrangement of hyperplanes** in entropy space (this is not the holographic entropy arrangement). The patterns of interest correspond to the linear subspaces in the **intersection poset** of this arrangement.

N-parties quantum marginal independence problem:
For which elements of the intersection poset of the mutual information arrangement there exists an N-partite density matrix with entropy vector belonging to that subspace?

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SA and SSA restrict the collection of subspaces which are “accessible” in quantum mechanics and other **unknown inequalities** could restrict this collection further. However, for N=4, even if there might exist infinitely many new quantum inequalities, SA and SSA are the only restrictions.

Conjecture:
any pattern which is allowed by SA and SSA is allowed in quantum mechanics.

The holographic marginal independence problem

In the holographic context nothing changes in the classification of the possible patterns of marginal independence. For an element of the intersection poset of the arrangement (a linear subspace of entropy space) we now want to know if there exists a **choice of geometric state** ψ **and configuration of boundary subsystems** $\mathscr{C}_{\mathsf{N}}$ such that the density matrix respects the corresponding pattern of marginal independence.

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Important aspects:

- this is a well defined question despite the fact that the entropy is divergent and scheme-dependent in QFT, since we only need to know if the mutual information for a particular partition (which is well defined) vanishes or not
- all we need to know to determine which subspace a pair of state and configuration belongs to is the collection of **proto-entropies** (connectivity of RT/HRT surfaces)

$$S_\epsilon(\rho_{\mathcal{X}}) = \frac{1}{4G}\sum_\mu \text{Area}_\epsilon(\omega^\mu) \longrightarrow S_{\mathcal{X}} = \sum_\mu \omega^\mu$$

(Hubeny-Rangamani-MR '18)

- this allows us to use only partial information from the area functional (we never need to evaluate any area) and nicely makes the entire formulation explicitly cut-off independent.

Back to the interpretation of inequalities

Comparison to quantum mechanics

- the marginal problem in QM is much simpler than the problem of finding general entropy inequalities and we might hope to be able to solve the problem in general
- in holography this allows us to distinguish entropic restrictions which are quantum mechanical from the ones which are intrinsically holographic
- the problem is more primitive than the analogous one for inequalities, in fact there is no difference at this level between holography and QM for N=3,4
- interestingly for N=5 holography starts to become more restrictive: there are pattern of marginal independence that can be realized in QM but not in holography !

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Systematic construction of the arrangement and the polyhedron (see talk by V. Hubeny)

- the solution to the holographic version of the problem gives a systematic derivation of the building blocks used to construct the holographic entropy arrangement and consequently the holographic entropy polyhedron
- if all holographic entropy inequalities are associated to primitive information quantities, the construction gives a candidate for the holographic entropy cone.

Role of the area functional

- remember that in order to define the holographic version of the marginal independence problem we only need the area functional to determine the connectivity of the RT/HRT surface, but we never need to use the value of the area
- is this enough information to derive all the inequalities ? This is tightly related to the previous point about primitivity of the corresponding information quantities
- if the answer is yes, then the information contained in the inequalities is more coarse-grained and we should only focus on the marginal independence problem. Otherwise the area functional contains additional information that we should extract.

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Structure of density matrix (WIP with K. Umemoto and J. Cresswell)

- the construction of the holographic entropy arrangement via the building blocks obtained from the solution to the holographic marginal independence problem automatically gives the structure of the density matrix of a large class of states which are ruled out by the holographic entropy inequalities (for any N).

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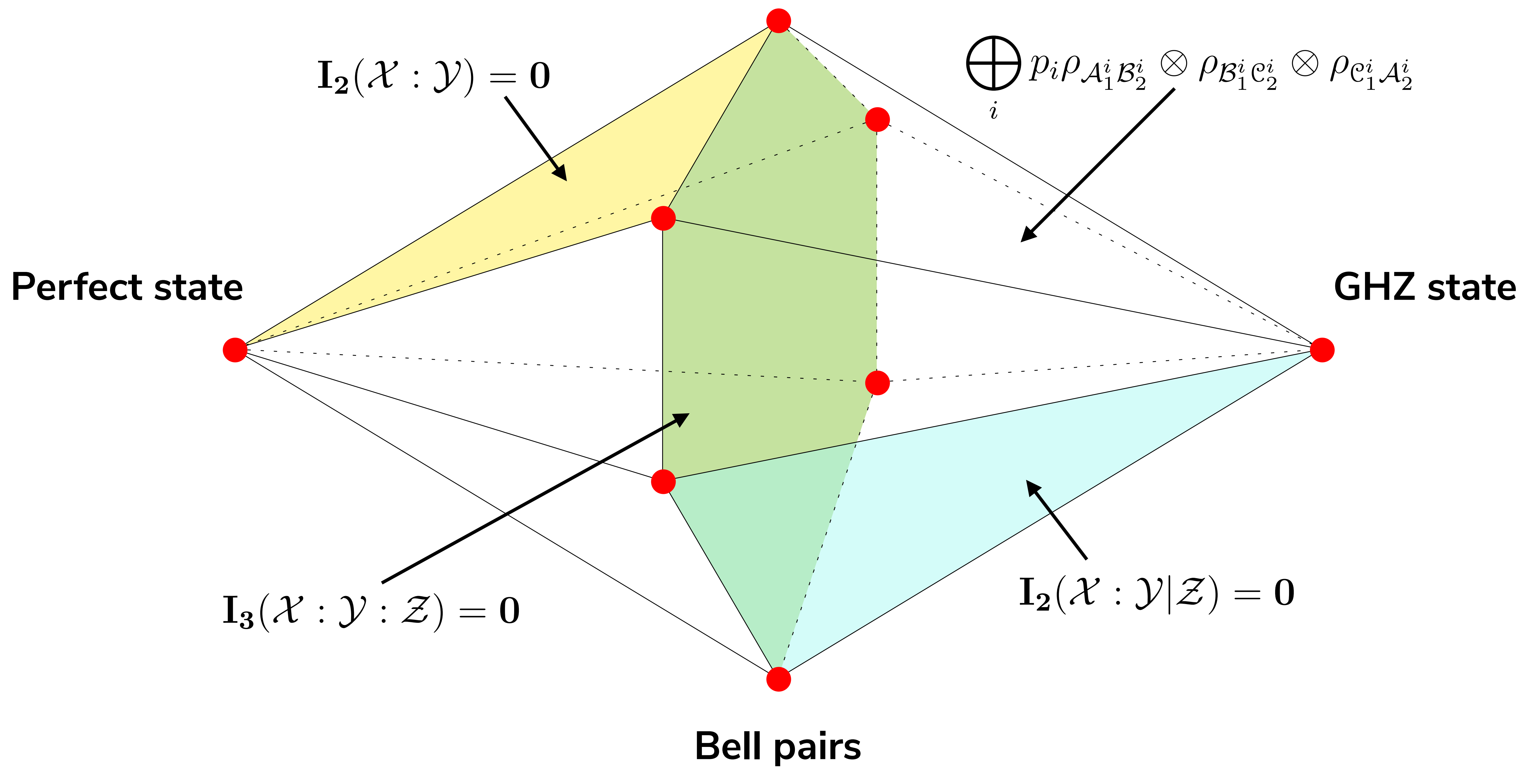
Identifying a general principle

- finding the general solution to both the quantum mechanical and holographic version of the problem can help identifying a general principle behind all the inequalities
- one possibility: is the holographic entropy cone the largest possible cone determined by marginal independence which fits inside the quantum mechanical cone (true for N=3)?

Three parties example

A cartoon of the 6d cross-section of the 7d quantum entropy cone for N=3:
The accessible subspaces of the marginal independence problem are represented by faces, edges and vertices to the left of the green plane (and on the plane itself).



Key questions for the future

- what is the solution to the quantum marginal independence problem for an arbitrary number of parties? Is the conjecture about SA and SSA true ?

- so far we have used the known holographic inequalities to solve the holographic version of the problem up to N=5. Can we solve the problem for an arbitrary number of parties more directly, without ever using the inequalities ?

- can we prove that for an arbitrary number of parties the holographic entropy inequalities (entropy cone) can always be reconstructed from solution of the holographic marginal independence problem ? Or do they contain more information ?

Thank you!