Entanglement negativity in many-body quantum systems

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[Jonah Kudler-Flam and SR
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Partial transpose and entanglement negativity

• Partial transpose of a density operator $\rho$:

$$\langle e_i^{(A)} e_j^{(B)} | \rho^{T_B} | e_k^{(A)} e_l^{(B)} \rangle := \langle e_i^{(A)} e_l^{(B)} | \rho | e_k^{(A)} e_j^{(B)} \rangle$$

where $|e_i^{(A,B)}\rangle$ is the basis of $\mathcal{H}_{A,B}$.

• Entanglement negativity and logarithmic negativity:

$$\mathcal{N}(\rho) := (||\rho^{T_B}||_1 - 1)/2 = \sum_{\lambda_i < 0} |\lambda_i|,$$

$$\mathcal{E}(\rho) := \log ||\rho^{T_B}||_1 = \log(2 \mathcal{N}(\rho) + 1)$$
Entanglement in mixed states?

• How to quantify quantum entanglement between $A$ and $B$ when $\rho_{A\cup B}$ is mixed? E.g., finite temperature, $A, B$ are parts of a bigger system.

• The entanglement entropy is an entanglement measure only for pure states. For mixed states, it is not monotone under LOCC.

• Positive partial transpose (PPT) criterion.
  [Peres (96), Horodecki-Horodecki-Horodecki (96), Eisert-Plenio (99), Vidal-Werner (02), Plenio (05) ...]
Partial transpose and quantum entanglement

- EPR pair: $|\Psi\rangle = \frac{1}{\sqrt{2}} [ |01\rangle - |10\rangle ]$

  \[
  \rho = |\Psi\rangle\langle\Psi| = \frac{1}{2} [ |01\rangle\langle01| + |10\rangle\langle10| - |01\rangle\langle10| - |10\rangle\langle01| ]
  \]

- Partial transpose:

  \[
  \rho^{T_2} = \frac{1}{2} [ |01\rangle\langle01| + |10\rangle\langle10| - |00\rangle\langle11| - |11\rangle\langle00| ]
  \]

- Entangled states are badly affected by partial transpose: Negative eigenvalues:

  \[
  \{ \lambda_i \} = \text{Spec}(\rho^{T_B}) = \{ 1/2, 1/2, 1/2, -1/2 \}
  \]

- C.f. For a classical state: \( \rho = \frac{1}{2} [ |00\rangle\langle00| + |11\rangle\langle11| ] = \rho^{T_B} \)
Partial transpose and entanglement negativity

- (Logarithmic) Entanglement negativity:

\[ N(\rho) := \sum_{\lambda_i < 0} |\lambda_i| = (\|\rho^{TB}\|_1 - 1)/2, \]

\[ E(\rho) := \log(2N(\rho) + 1) = \log \|\rho^{TB}\|_1. \]

- Quantum entanglement measure for mixed state (monotone under LOCC).

- For mixed states, negativity extracts quantum correlations only.

- Computable.

- Negativity can zero even when the state is entangled (does not detect bound entanglement but only distillable one).
Applications of entanglement negativity to many-body physics?

- Entanglement negativity in CFTs, and TFTs
  [Calabrese-Cardy-Tonni(12-), Castelnovo (13), Lee-Vidal (13), Wen-Chang-SR, Wen-Matsuura-SR(16), and many others...]

- Partial transpose and topological invariants in SPT phases
  [Pollmann-Turner (12), Shapourian-Shiozaki-SR (16)]

- Experimental measurements?  [Elben et al (18-19); Cornfield et al (18)]
Negativity in 2d CFT

- The logarithmic negativity for two adjacent intervals of equal length $\ell$ in free fermion chain

$$\mathcal{E}(\ell) = \frac{c}{4} \ln \tan \frac{\pi \ell}{L}$$

- The numerical result (points) using the free fermion formula [Shapourian-Shiozaki-SR(17)] agrees with the CFT result (solid line) [Calabrese-Cardy-Tonni].

$$\mathcal{E}(\ell) = \frac{c}{4} \ln \tan \frac{\pi \ell}{L}$$
Partial transpose and topological invariant

- Topological invariant of 1d topological superconductor (the Kitaev chain) \( \text{Tr}(\rho_I \rho_I^T) \sim e^{2\pi i \nu/8} \).
Holographic description of entanglement negativity?
Basic proposal

Negativity = Minimal entanglement wedge cross section with back reaction


C.f. other proposal [Chaturvedi-Malvimat-Sengupta (16); Jain-Malvimat-Mondal-Sengupta (17)]
Entanglement wedge

- Entanglement wedge = the bulk region corresponding to the reduced density matrix on the boundary [Headrick et al (14), Jafferis-Suh (14), Jafferis-Lewkowycz-Maldacena-Suh (15), ...]

- Entanglement of purification [Takayanagi-Umemoto (17), Nguyen-Devakul-Halbasch-Zaletel-Swingle (17)]

- Odd entropy [Tamaoka (18)]

- Reflected entropy [Dutta-Faulkner (19)]
Perfect tensor holographic error correcting code

- Tensor network acts as an error correcting code encoding “bulk” logical qubits into “boundary” physical qubits
- Captures many aspects of holography; black holes, bulk reconstruction, subregion duality, holographic entanglement entropy, etc.

[Almheiri-Dong-Harlow(15), Harlow(17), Pastawski-Yoshida-Harlow-Preskill(15), Hayden et al (16)]
• Computed entanglement negativity in a tensor network model of holographic duality (using the language of QEC).

\[ S(\rho_A) = S(\chi_A) + S(\rho_a) \]

\[ E(\rho_{A\bar{A}}) = S_{1/2}(\chi_A) + E(\rho_{\text{bulk}}) \]

• With BH in bulk, negativity avoids horizon
Full-fledged holography

- No back reaction for the von Neumann entropy

- Back reaction for Rényi entropy

- For pure state, negativity = Rényi 1/2

- How well do we know about the back reaction? How much control do we have?
Rényi entropy and back reaction

• Rényi entropy $S_n$: [Dong (16)]

\[ n^2 \frac{\partial}{\partial n} \left( \frac{n-1}{n} S_n \right) = \frac{\text{Area(cosmic brane}_n)}{4G_N} \]

Cosmic brane has a tension $T_n = (n - 1)/(4nG_N)$

• Conjecture: For “spherical” configurations [C.f. Hung-Myers-Smolkin-Yale (11), Rangamani-Rota (14)]

\[ \mathcal{E} = \chi \frac{E_W}{4G_N} \]

• For (1+1)d CFT (with spherical entangling surface) $\chi = 3/2$,

\[ \mathcal{E} = \frac{3}{2} \frac{E_W}{4G_N} \]
Disjoint intervals at zero temperature

- Negativity for two disjoint intervals:
  \[ E_W = \begin{cases} 
  c \frac{1}{6} \ln \frac{1 + \sqrt{x}}{1 - \sqrt{x}}, & 1/2 \leq x \leq 1 \\
  0, & 0 \leq x \leq 1/2 
\end{cases} \]
  \[ x = \frac{\ell_1 \ell_2}{(\ell_1 + d)(\ell_2 + d)}. \]

- Minimal entanglement wedge cross section \( E_W \)

[Takayanagi-Umemoto(17), Nguyen-Devakul-Halbasch-Zaletel-Swingle (17)]
• $E_W$ should be compared with

$$\mathcal{E} = \lim_{n_e \to 1} \ln \text{Tr} (\rho^{T_2})^{n_e}$$

$$= \lim_{n_e \to 1} \ln \langle \sigma_{n_e} (w_1, \bar{w}_1) \bar{\sigma}_{n_e} (w_2, \bar{w}_2) \bar{\sigma}_{n_e} (w_3, \bar{w}_3) \sigma_{n_e} (w_4, \bar{w}_4) \rangle_{\mathbb{C}}.$$  

where $\sigma_n$ is the twist operator with dimension $h_n = (c/24)(n - 1/n)$. The replica limit from even integer to $n_e \to 1$. [Calabrese-Cardy-Tonni (12)]
**Adjacent limit**

- In the limit of adjacent intervals $d \to 0$:

\[
E_W \to \frac{c}{6} \ln \left[ \frac{4 \ell_1 \ell_2}{\epsilon (\ell_1 + \ell_2)} \right].
\]

- Agrees with CFT result (universal) [Calabrese-Cardy-Tonni (12)]:

\[
\mathcal{E} = \lim_{n_e=1} \ln \langle \sigma_{n_e} (w_1, \bar{w}_1) \bar{\sigma}_{n_e}^2 (w_2, \bar{w}_2) \sigma_{n_e} (w_4, \bar{w}_4) \rangle_{\mathbb{C}}
= \frac{c}{4} \ln \left[ \frac{\ell_1 \ell_2}{\ell_1 + \ell_2} \right] + \text{const.}
\]

if the constant is properly chosen, $\mathcal{E} = (3/2) E_W$. 
Disjoint intervals

- We need four pt correlation function for the disjoint case:
  \[
  \mathcal{E} = \lim_{n_e \to 1} \ln \langle \sigma_{n_e}(w_1, \bar{w}_1)\bar{\sigma}_{n_e}(w_2, \bar{w}_2)\bar{\sigma}_{n_e}(w_3, \bar{w}_3)\sigma_{n_e}(w_4, \bar{w}_4) \rangle_C.
  \]

- Will focus on the dominant conformal block
  \[
  \langle \sigma_{n_e}(\infty)\bar{\sigma}_{n_e}(1)\bar{\sigma}_{n_e}(x, \bar{x})\sigma_{n_e}(0) \rangle \sim \mathcal{F}(h_p, h_i, x)\bar{\mathcal{F}}(h_p, h_i, \bar{x})
  \]

- Intermediate state is either identity operator of double twist operator \(\sigma_n^2\) with dimension
  \[
  \hat{h}_{\sigma_n^2} = \begin{cases} 
  \frac{c}{24} \left( \frac{n}{1} - \frac{1}{n} \right) & \rightarrow 0 \quad n : \text{odd} \\
  \frac{c}{12} \left( \frac{n}{2} - \frac{2}{n} \right) & \rightarrow -\frac{c}{8} \quad n : \text{even}
  \end{cases}
  \]
Series expansion

- Dominant conformal block with double twist operator \((y = 1 - x)\): \([\text{Headrick (10), Kulaxizi-Parnachev-Policastro (14)}]\)

\[
\mathcal{F}(h_p, y) = y^{h_p} \left[ 1 + \frac{h_p}{2} y + \frac{h_p(h_p + 1)^2}{4(2h_p + 1)} y^2 
\right.
\]
\[
+ \frac{h_p^2(1 - h_p)^2}{2(2h_p + 1)[c(2h_p + 1) + 2h_p(8h_p - 5)]} y^2 + \cdots \right].
\]

- Taking the large \(c\) limit and then replica limit \(h_p \to -c/8\),

\[
\mathcal{F}(h_p, y) = y^{-\frac{c}{8}} \left[ 1 - \frac{cy}{16} + \frac{c^2y^2}{512} - \frac{c^3y^3}{24576} + \cdots \right].
\]

Matches with the cross-ratio expansion of \((3/2)E_W\) in large \(c\).
Monodromy method

[Hartman (13), Faulkner (13), Kulaxizi-Parnachev-Policastro (14)]
Geodesic Witten diagram calculation

[Hirai-Tamaoka-Yokoya (18), Prudenzia (19)]
\[ \langle \sigma_n(x_1) \bar{\sigma}_n(x_2) \bar{\sigma}_n(x_3) \sigma_n(x_4) \rangle \]
\[ \sim \int_{\gamma_{ij}} \int_{\gamma_{kl}} G_{b\partial} G_{b\partial} \Delta \sigma_n^2 G_{bb\partial} G_{b\partial} \]
\[ \sim (|x_{12}| |x_{34}|)^{-2\Delta_n} \int_{\gamma_{12}} \int_{\gamma_{34}} d\lambda d\lambda' e^{-\Delta \sigma_n^2 \sigma(y,y')} \]

where \( \sigma(y,y') \) is the distance between bulk points \( y \) and \( y' \).

- In the large \( c \) limit, the integral localizes at the minimal entanglement wedge:
  \[ \langle \sigma_n(x_1) \bar{\sigma}_n(x_2) \bar{\sigma}_n(x_3) \sigma_n(x_4) \rangle \sim e^{-\Delta \sigma_n^2 \sigma_{\text{min}}} \]

We then take the replica limit \( \Delta \sigma_n^2 \rightarrow -c/4 \).
A few more comments

- Check in other configurations; e.g., bipartite at finite temperatures, thermofield double, etc.

- Back reaction for other configurations

- Bit thread picture; cross section = maximal number of distillable Bell pairs? [Agón-de Boer-Pedraza(18)] Negativity gives upper bound of distillable entanglement.

- Applications; operator negativity.