Perturbative and non Perturbative calculations of holographic Renyi relative divergence

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Introduction

- In the first part of this talk we would like to consider perturbative calculations of Renyi type quantities , like $\mathrm{tr}\rho^{\gamma}$ and $\mathrm{tr}\left[\rho^{\gamma}\sigma^{1-\gamma}\right]$ involving power of DM.

- Conventionally Renyi type quantities are computed by replica trick. In this trick we first regard the Renyi index to be a positive integer, $\gamma = n$ and represent the quantity as a path integral on the n fold cover Σ_n .

- We then analytically continue the integer n to arbitrarily number γ to get the final result .

Introduction

However, replica trick has several disadvantages, when it is combined with perturbative expansion.

For example when we perturbatively expand $\mathrm{tr}\rho^n$ for $\rho = \rho_0 + \delta\rho$ naively, then at quadratic order we encounter following sum

$$\sum_{k,m} \mathrm{tr}\left[\rho_0^{k-1}\delta\rho\rho_0^{m-k-1}\delta\rho\rho_0^{n-m}\right]$$

To proceed, we first need to perform this sum to get a closed expression , then analytically continue the result in n. Both of them are usually difficult. Also, dealing with higher order terms is much more hard.

Introduction

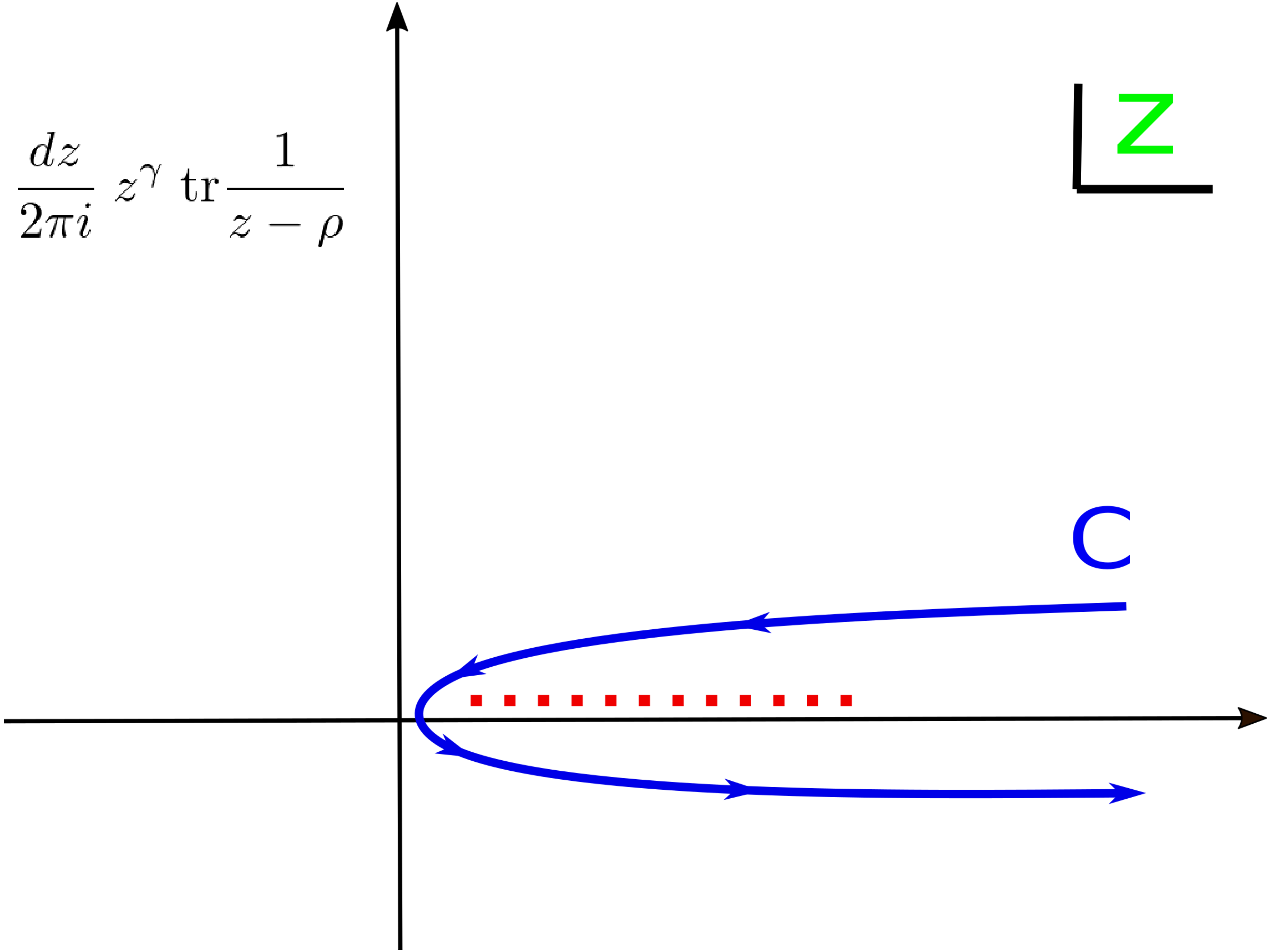
In order to overcome these difficulties, we developed a new way to perturbatively calculate the Renyi type quantities without using replica trick, as well as analytic continuation.

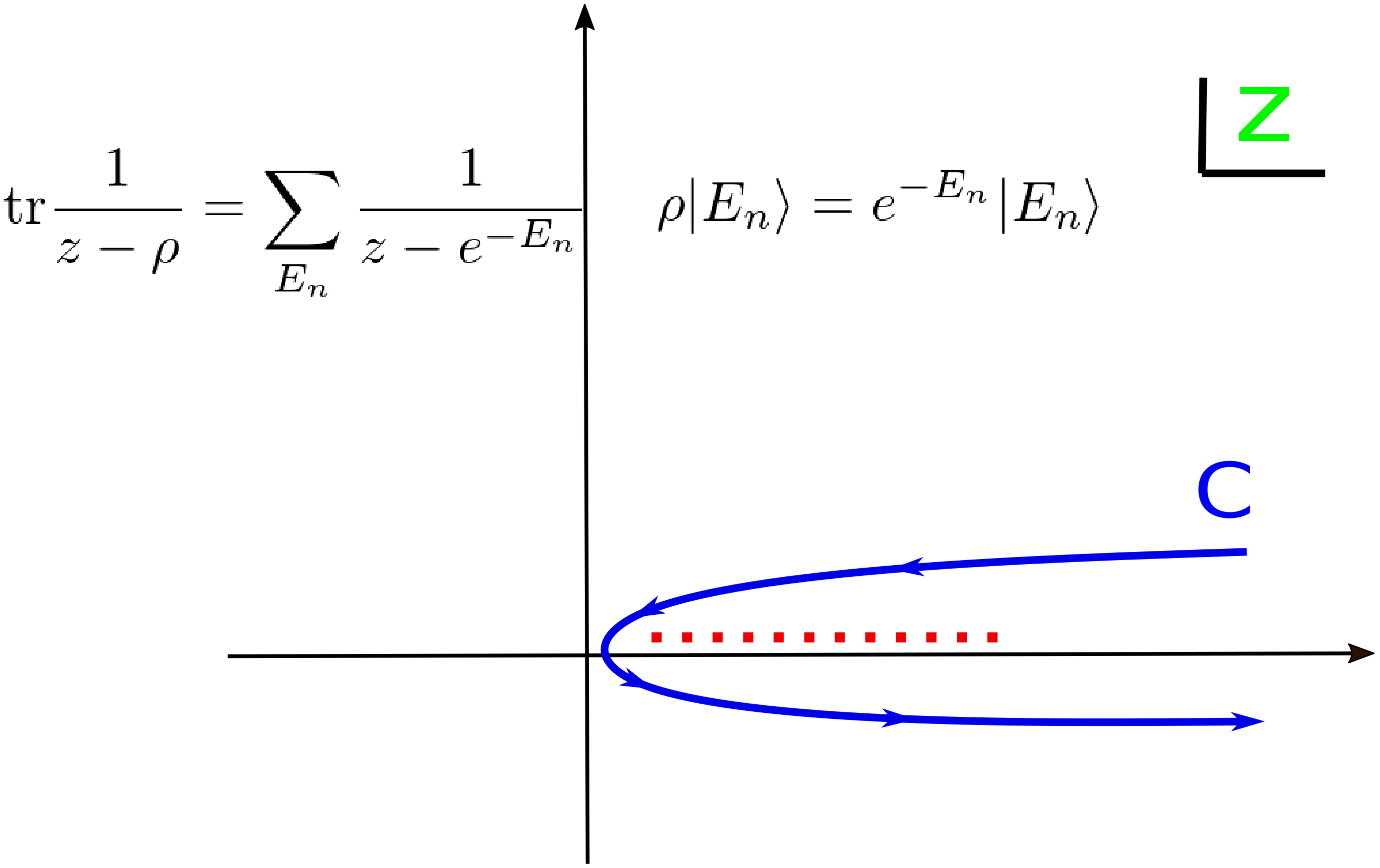
The idea we employ is simple, namely writing $\mathrm{tr}\rho^\gamma$ by a contour integral

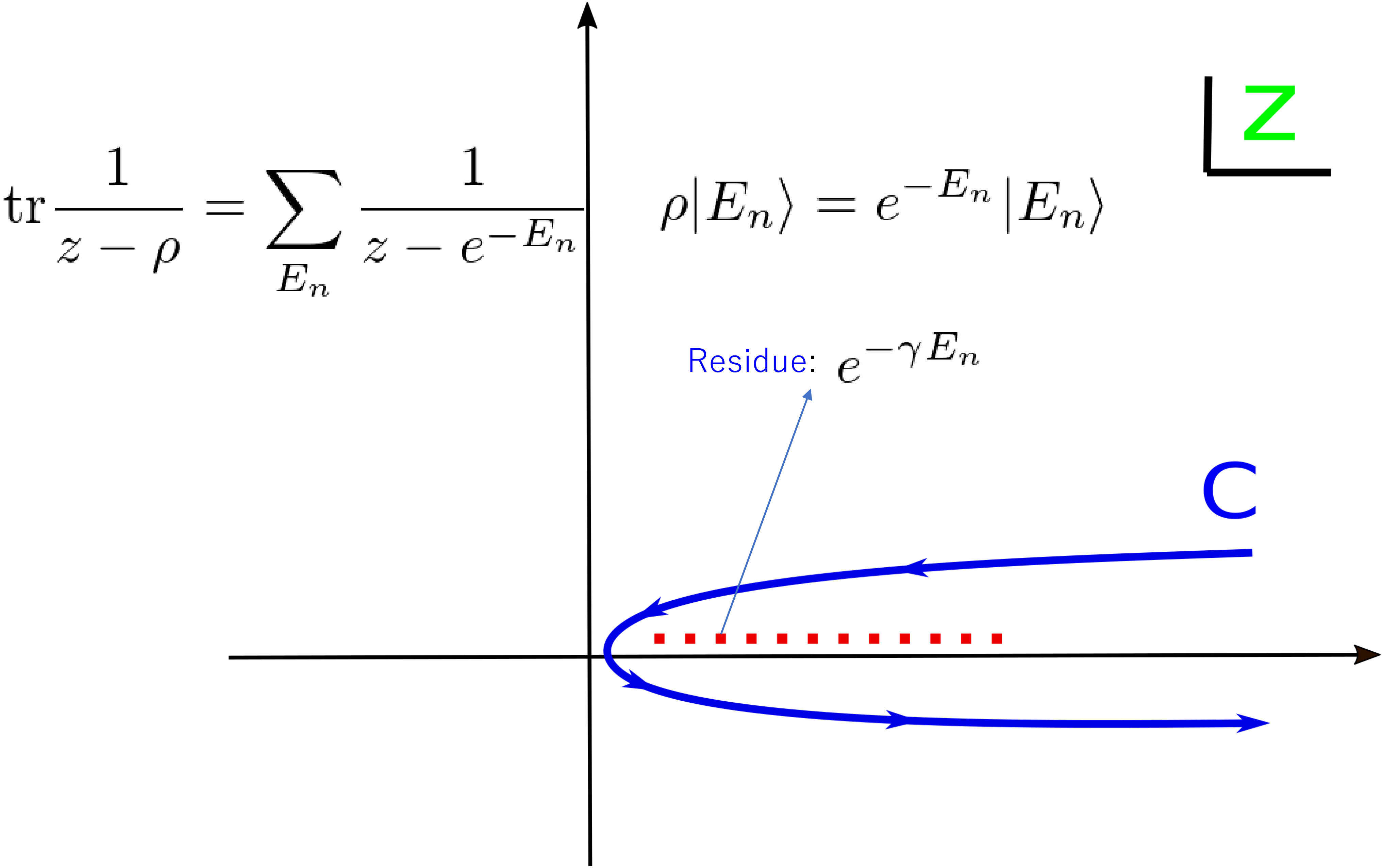
$$\mathrm{tr}\rho^\gamma = \int_C \frac{dz}{2\pi i}\, z^\gamma\, \mathrm{tr}\frac{1}{z-\rho}$$

Where the contour C is chosen so that it includes all the poles of the integrand but avoids the contribution of the branch cut coming from z^γ .

$$\mathrm{tr}\,\rho^{\gamma} = \int_C \frac{dz}{2\pi i}\; z^{\gamma}\; \mathrm{tr}\frac{1}{z-\rho}$$




$\mathrm{tr}\,\frac{1}{z-\rho} = \sum_{E_n} \frac{1}{z - e^{-E_n}}$
$\rho|E_n\rangle = e^{-E_n}|E_n\rangle$
z
C



$\mathrm{tr}\,\frac{1}{z-\rho} = \sum_{E_n} \frac{1}{z-e^{-E_n}}$
$\rho|E_n\rangle = e^{-E_n}|E_n\rangle$
z
Residue: $e^{-\gamma E_n}$
C

The result

$$\mathrm{tr}\rho^\gamma = \sum_n T_\gamma^{(n)}(\delta\rho)$$

$$T_\gamma^{(n)}(\delta\rho) = \int_C ds_1 \cdots ds_{n-1}\, \mathcal{K}_\gamma^{(n)}(s_1, \cdots s_{n-1})\ \mathrm{tr}\left[e^{-2\pi\gamma K} \prod_{k=1}^{n-1} e^{iKs_k} \tilde{\delta}\rho\, e^{-iKs_k}\ \tilde{\delta}\rho \right]$$

$$\mathcal{K}_\gamma^{(n)}(s_1, \cdots s_{n-1}) = \frac{i}{8\pi^2}\left(\frac{-i}{4\pi}\right)^{n-2} \frac{(s_1 + 2\pi i\gamma)\sin\pi\gamma}{\sinh\left(\frac{s_1+2\pi i\gamma}{2}\right)\prod_{k=2}^{n-1}\sinh\left(\frac{s_k - s_{k-1}}{2}\right)\sinh\left(\frac{s_{n-1}}{2}\right)}$$

$\{s_k\}$ parametrize the modular flow of the reference state, and K is the modular Hamiltonian.

In a typical CFT set up the trace is a correlation function on the covering space Σ_γ

The quadratic term: Checks

$$T_\gamma^{(2)}(\delta\rho) = \left(\frac{i \sin \pi\gamma}{8\pi^2}\right) \int_C \frac{s + 2\pi i\gamma}{\sinh \frac{s}{2} \sinh \frac{s+2\pi i\gamma}{2}} \mathrm{tr}\left[\rho_0^\gamma \ \delta\rho(s)\delta\rho\right]$$

1. when the Renyi index is an integer $\gamma = n$, this reduces to the trivial sum.

$$T_n^{(2)}(\delta\rho) = \sum_{k,m} \mathrm{tr}\left[\rho_0^{k-1}\delta\rho\rho_0^{m-k-1}\delta\rho\rho_0^{n-m}\right]$$

2. In the $\gamma \to 1$ limit, it recovers the quadratic part of entanglement entropy.

The perturbative expansion of RRD

One can apply the same trick to the Renyi relative divergence $D_\gamma(\rho||\rho_0) = \mathrm{tr}\rho^\gamma \rho_0^{1-\gamma}$

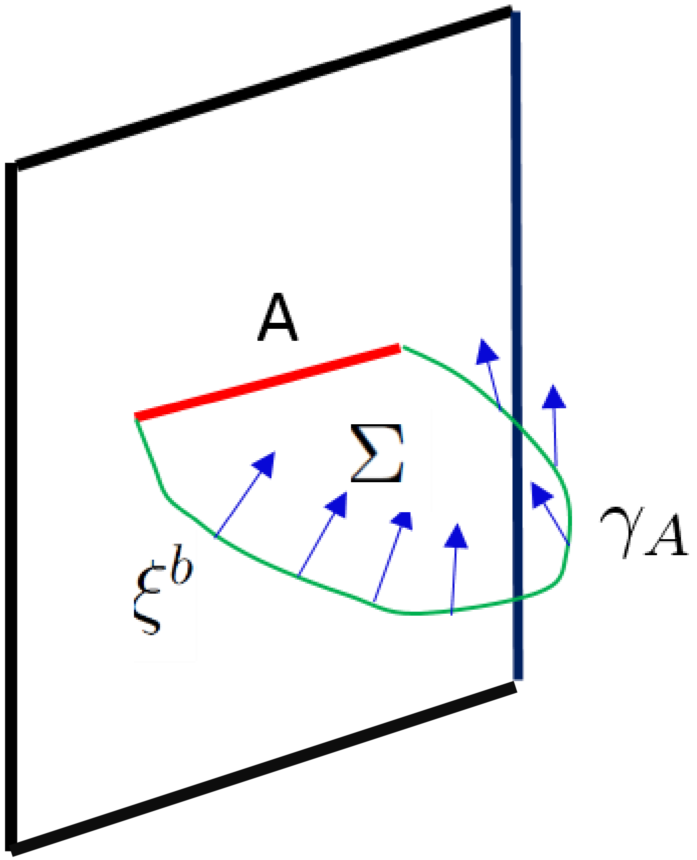
$$D_\gamma(\rho||\rho_0) = \sum_{n=2}^{\infty} D^{(n)}(\delta\rho)$$

$$D^{(n)}(\delta\rho) = \int \prod_{k=1}^{n-1} ds_k \, \mathcal{K}_\gamma^{(n)}(s_1, \cdots s_{n-1}) \, \mathrm{tr}\left[\rho_0 \prod_{k=1}^{n-1} \delta\rho(s_k)\delta\rho\right]$$

The same kernel function $\mathcal{K}_\gamma^{(n)}(s_1, \cdots s_{n-1})$ appears .

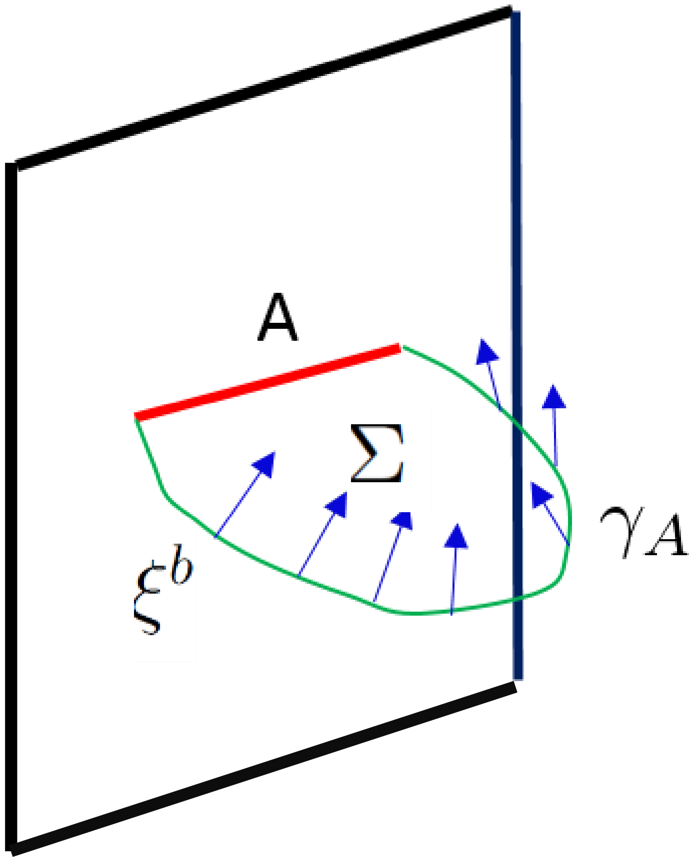
Difference: the Renyi index does not appear in the trace. The correlation function on Σ_1 (conformal to flat space.) -> Drastic simplification of the calculation.

The holographic expression of quadratic term



$$X_\gamma(\rho||\rho_0) = \frac{d}{d\gamma}\left[D_{-\gamma}(\rho||\rho_0) - D_\gamma(\rho||\rho_0)\right]$$

The holographic expression of quadratic term



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$$X_\gamma^{(2)}(\delta\rho) = -2\pi \int_\Sigma d\Sigma^a \xi^b \, T_{ab}(\phi, \phi_\gamma)$$

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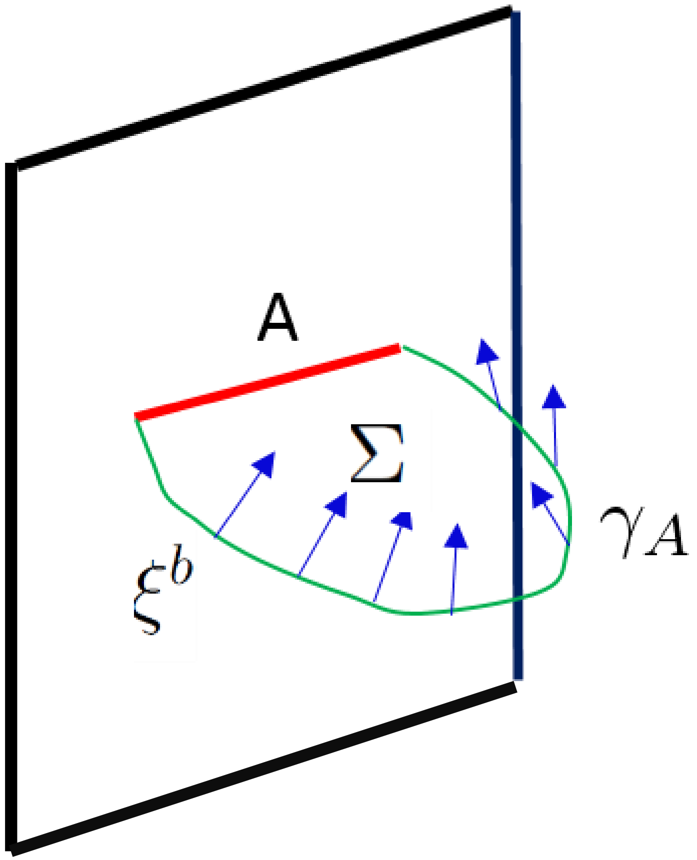

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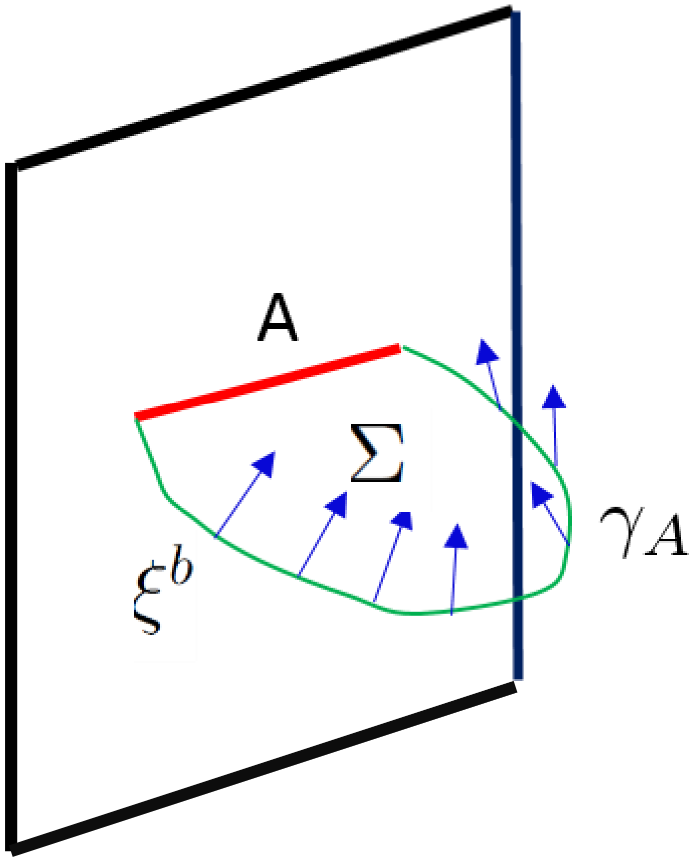
The holographic expression of quadratic term



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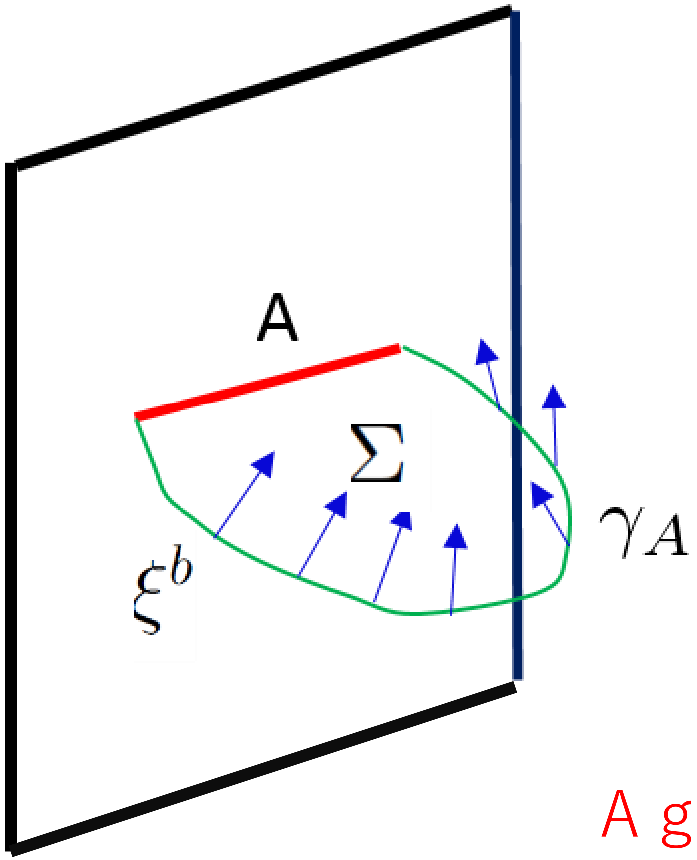
$$\phi_\gamma = e^{\gamma K}\phi e^{-\gamma K}$$

The holographic expression of quadratic term



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A generalization of Fisher information = Canonical energy

See also [Hijano, May]

Resumming the series

It turned out that the perturbative expansion we have developed does not converge in general, like usual QFT perturbations. [Sarosi, TU]

This is roughly speaking because $\delta\rho$ is not a bounded operator. [Lashkari, Liu, Rajagopal]

Resumming the perturbative series is important since it has to do with emergence of full dynamical gravity in the bulk from CFT point of view.

The gravity dual of the resummation

CFT side: Resumming the perturbative series of RRD $\mathrm{tr}\left[\rho^{\gamma}\sigma^{1-\gamma}\right]$

||

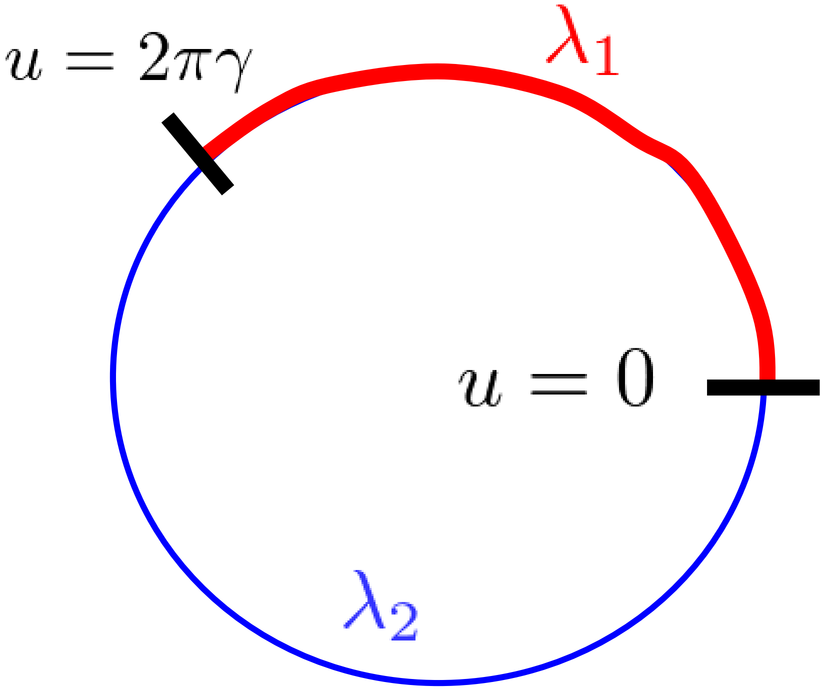
Gravity side: Finding the fully backreacted geometry dual to $\rho^{\gamma}\sigma^{1-\gamma}$
and evaluate its on shell action.

Although we have not solved the first problem, but found a toy model in which one can completely solve the second problem. [TU. Work in progress]

Set up

We consider the JT gravity + matter scalar field χ .

$$I = -\frac{\phi_0}{16\pi G}\left[\int dx^2\sqrt{g}R + 2\int K\right] - \frac{1}{16\pi G}\left[\int dx^2\sqrt{g}\phi(R+2) + 2\int \phi_b K\right] + I_M[g,\chi],$$



The class of density matrices

$$\rho_\lambda = \frac{1}{Z_\lambda}\exp\left[-H + \lambda\int_0^{2\pi} du\, \mathcal{O}(u)\right]$$

$\mathcal{O}$ is dual to the bulk scalar field χ

$D_\gamma(\rho_{\lambda_1}||\rho_{\lambda_2})$ has a path integral representation.

[Bernamonti,Galli,Myers Oppenheim]

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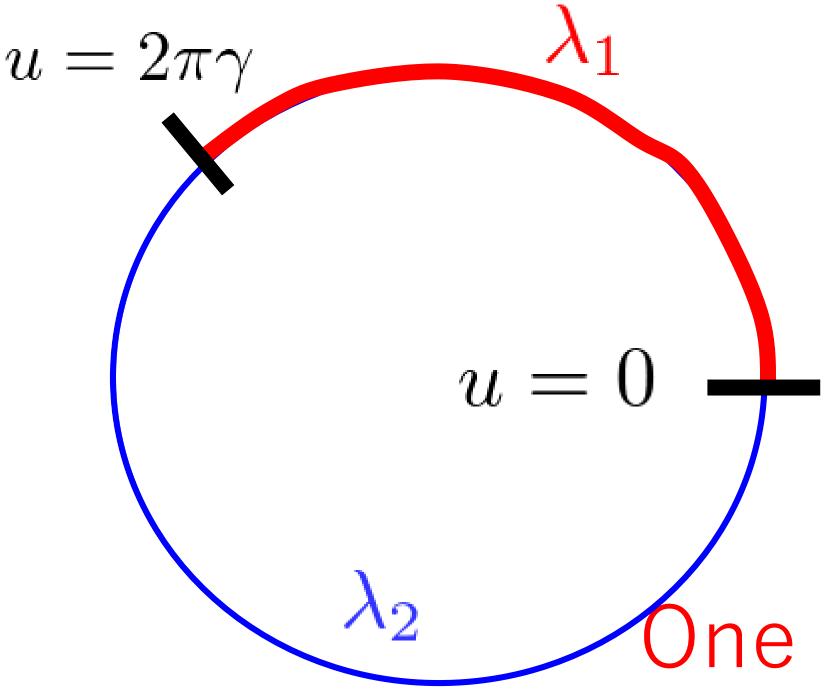

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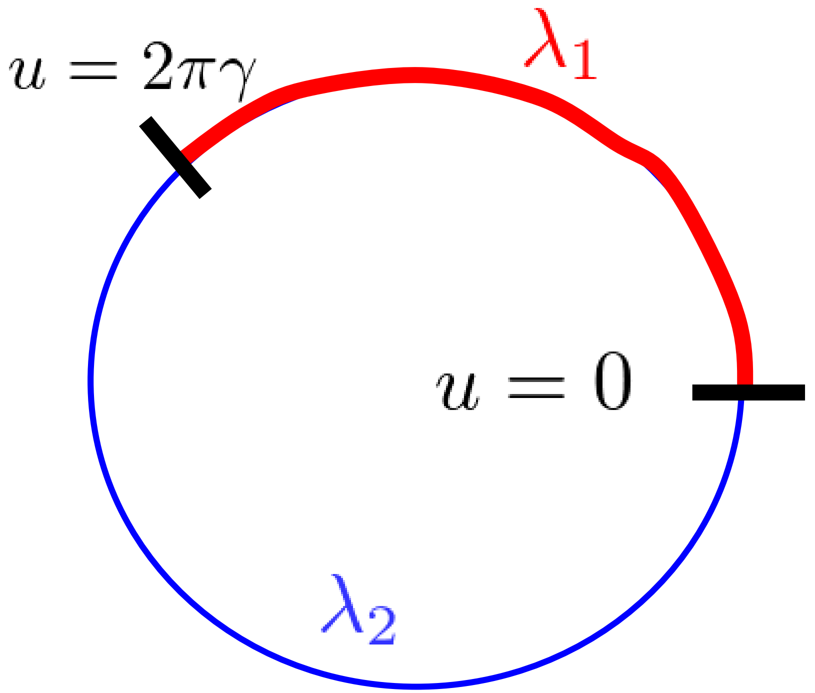
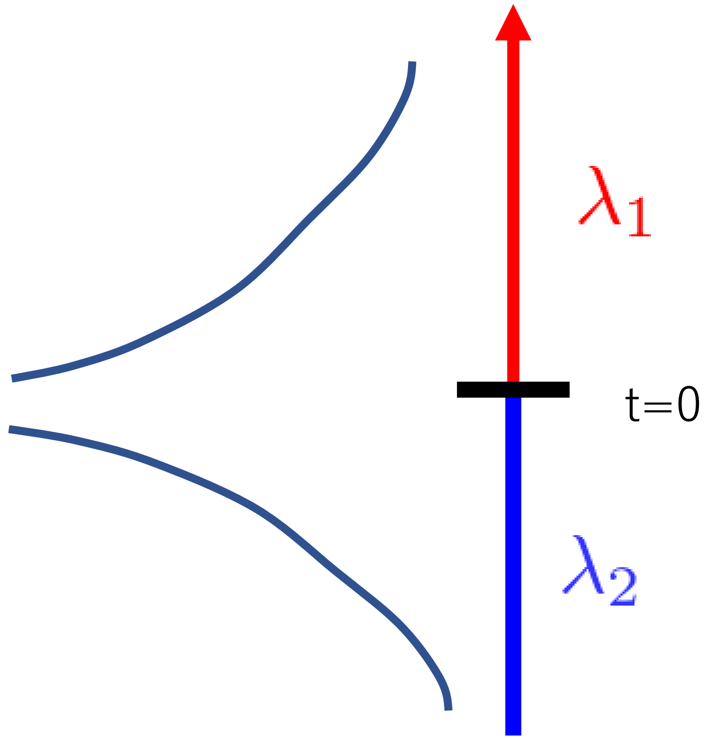
One can compute the RRD $D_\gamma(\rho_{\lambda_1}||\rho_{\lambda_2})$ exactly!

Sketch of the derivation



$$t(u) = \frac{\sin\frac{u}{2}}{\sin\left(\frac{u-2\pi\gamma}{2}\right)}$$



Janus solution in the Poincare coordinates
[Bak Kim Yi]

$$\chi(t,z) = \lambda_{12}\tan^{-1}\left(\frac{t}{z}\right)$$

$$\phi(t,z) = \phi_0(t,z) - 4\pi G\lambda_{12}^2\left[\frac{t}{z}\tan^{-1}\left(\frac{t}{z}\right)\right]$$

Sketch of the derivation (II)

From these data one can solve the EoM $\phi(t(u), \varepsilon t'(u)) = \frac{\bar{\phi}}{\varepsilon}$ to find the on shell reparametrization mode $t(u)$, and evaluate its action (Schwarzian+matter).

In the relative entropy limit, $\gamma \to 1$,we get

$$S(\rho_{\lambda_1}||\rho_{\lambda_2}) = \left(\frac{4\pi^2 G}{\bar{\phi}}\right)^2 (\lambda_{12})^4 + \frac{1}{\varepsilon} (\lambda_{12})^2$$

In the large source limit $\lambda_{12} \to \infty$, one can expand the resulting RRD by $\frac{1}{\lambda_{12}}$.
In this limit RRD become independent of the Renyi index γ ,

$$\lim_{\lambda_{12}\to\infty} D_\gamma(\rho_{\lambda_1}||\rho_{\lambda_2}) = S(\rho_{\lambda_1}||\rho_{\lambda_2})$$

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Coming from the matter part

$\sim \lambda_{12}^2 \int du du' \langle \mathcal{O}(u) \mathcal{O}(u') \rangle$

$$S(\rho_{\lambda_1} || \rho_{\lambda_2}) = \left(\frac{4\pi^2 G}{\bar{\phi}} \right)^2 (\lambda_{12})^4 + \frac{1}{\varepsilon} (\lambda_{12})^2$$

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Effect of full gravitational back reaction (Schwarzian)

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Conclusions

We developed ways to compute Renyi relative divergence.

Can we derive the kernel function $\mathcal{K}_{\gamma}^{(n)}(s_1, \cdots s_{n-1})$ from the gravity calculation ?

Higher dimensional generalizations?

By the Wick rotation $\gamma \to it$ we can study holographic relative modular flow => related to black hole interior?

Thank you