Models of complexity growth and random quantum circuits

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June 25, 2019
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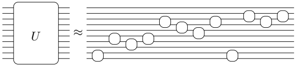
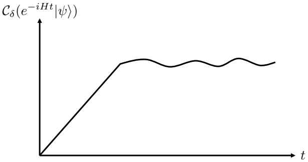
Based on:
[Kueng, NHJ, Chemissany, Brandão, Preskill], 1907.hopefully soon
[NHJ], 1905.12053

Based on:

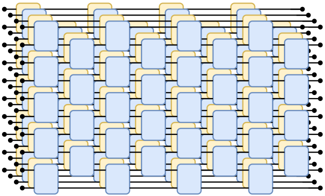
work in progress with Richard Kueng, Wissam Chemissany, Fernando Brandão, John Preskill

[Richard Kueng]

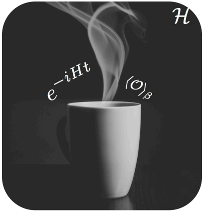




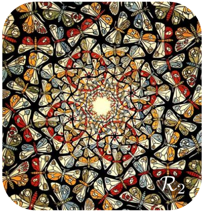
as well as [NHJ, "Unitary designs from statistical mechanics in random quantum circuits," arXiv:1905.12053]
(talk at the QI workshop 2 weeks ago)



We are interested in understanding universal aspects of strongly-interacting systems
$\rightarrow$ specifically in their real-time dynamics



Thermalization



Quantum chaos

Complexity

Transport

understanding these has implications in high-energy, condensed matter, and quantum information

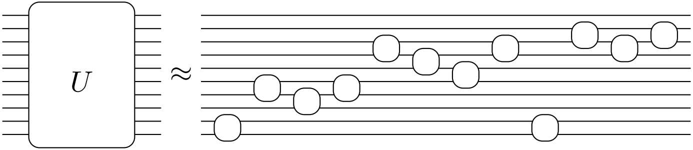
we'll focus on complexity in quantum mechanical systems

Complexity

some intuition

Complexity is a somewhat intuitive notion

The traditional definition involves building a circuit with gates drawn from a universal gate set, which implements the state or unitary to within some tolerance



We are interested in the minimal size of a circuit that achieves this

Complexity

a panoply of references

we've heard a lot about complexity growth already in this workshop

e.g. talks by Rob Myers, Vijay Balasubramanian, and Thom Bohdanowicz;
in talks later today/this week by Bartek Czech, Gabor Sarosi, Shira Chapman; and in many posters

and much progress has been made in studying complexity growth in holographic systems

[Susskind], [Stanford, Susskind], [Brown, Roberts, Susskind, Swingle, Zhao], [Susskind, Zhao], [Couch, Fischler, Nguyen], [Carmi, Myers, Rath], [Brown, Susskind], [Caputa, Magan], [Alishahiha], [Chapman, Marrochio, Myers], [Carmi, Chapman, Marrochio, Myers, Sugishita], [Caputa, Kundu, Miyaji, Takayanagi, Watanabe], [Brown, Susskind, Zhao], [Agón, Headrick, Swingle], ...

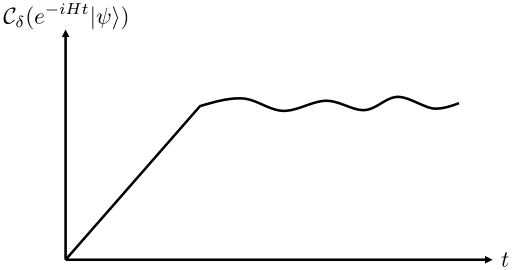
as well as extending definitions to understand a notion of complexity in QFT

[Chapman, Heller, Marrochio, Pastawski], [Jefferson, Myers], [Hackl, Myers], [Yang], [Chapman, Eisert, Hackl, Heller, Jefferson, Marrochio, Myers], [Guo, Hernandez, Myers, Ruan], ...

Complexity

some expectations

it is believed(/expected/conjectured) that the complexity of a simple initial state grows (possibly linearly) under the time-evolution by a chaotic Hamiltonian



saturating after an exponential time

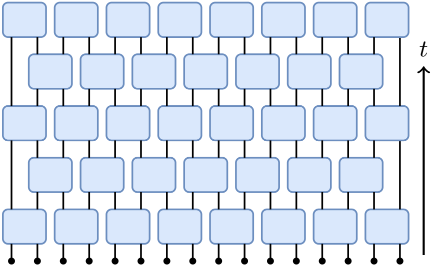
computing the quantum complexity analytically is very hard (especially for a fixed chaotic H and $|\psi\rangle$)

$\rightarrow$ we'll focus on ensembles of time-evolutions (RQCs)

Our goal

Consider random quantum circuits, a solvable model of chaotic dynamics

we take local RQCs on n qudits of local dimension q, with gates drawn randomly from a universal gate set G



and try to derive exact results for the growth of complexity

Overview

- Define complexity
- Complexity by design
- Complexity in local random circuits
- Solving random circuits
- (complexity from measurements)

State complexity

more serious version

Consider a system of n qudits with local dimension q, where $d = q^n$

Complexity of a state: the minimal size of a circuit that builds the state $|\psi\rangle$ from $|0\rangle$

We assume the circuits are built from elementary 2-local gates chosen from a universal gate set G. Let G_r denote the set of all circuits of size r.

Definition (δ-state complexity)

Fix $\delta \in [0,1]$, we say that a state $|\psi\rangle$ has δ-complexity of at most r if there exists a circuit $V \in G_r$ such that

$$\frac{1}{2}\left\| |\psi\rangle\langle\psi| - V|0\rangle\langle 0|V^\dagger \right\|_1 \leq \delta \,,$$

which we denote as $\mathcal{C}_\delta(|\psi\rangle) \leq r$.

Unitary complexity

more serious version

Consider a system of n qudits with local dimension q, where $d = q^n$

Complexity of a unitary: the minimal size of a circuit, built from a 2-local gates from G, that approximates the unitary U

Definition (δ-unitary complexity)

We say that a unitary $U \in U(d)$ has δ-complexity of at most r if there exists a circuit $V \in G_r$ such that

$$\frac{1}{2}\left\|\mathcal{U} - \mathcal{V}\right\|_\diamond \leq \delta\,,$$

where $\mathcal{U} = U(\rho)U^\dagger$ and $\mathcal{V} = V(\rho)V^\dagger$,

which we denote as $\mathcal{C}_\delta(U) \leq r$.

Complexity by design

We start with some general statements about the complexity of unitary k-designs

related ideas were presented in [Roberts, Yoshida] relating the frame potential to the average complexity of an ensemble

But first, we need to define the notion of a unitary design

Unitary k-designs

Haar: (unique L/R invariant) measure on the unitary group $U(d)$

The k-fold channel, with respect to the Haar measure, of an operator $\mathcal{O}$ acting on $\mathcal{H}^{\otimes k}$ is

$$\Phi^{(k)}_{\mathrm{Haar}}(\mathcal{O}) \equiv \int_{\mathrm{Haar}} dU\, U^{\otimes k}(\mathcal{O})U^{\dagger \otimes k}$$

For an ensemble of unitaries $\mathcal{E} = \{p_i, U_i\}$, the k-fold channel of an operator $\mathcal{O}$ acting on $\mathcal{H}^{\otimes k}$ is

$$\Phi^{(k)}_{\mathcal{E}}(\mathcal{O}) \equiv \sum_i p_i U_i^{\otimes k}(\mathcal{O})U_i^{\dagger \otimes k}$$

An ensemble of unitaries $\mathcal{E}$ is an exact k-design if

$$\Phi^{(k)}_{\mathcal{E}}(\mathcal{O}) = \Phi^{(k)}_{\mathrm{Haar}}(\mathcal{O})$$

e.g. $k = 1$ and Paulis, $k = 2, 3$ and the Clifford group

Unitary k-designs

Haar: (unique L/R invariant) measure on the unitary group $U(d)$

k-fold channel: $\Phi_{\mathcal{E}}^{(k)}(\mathcal{O}) \equiv \sum_i p_i U_i^{\otimes k}(\mathcal{O}) U_i^{\dagger \otimes k}$

exact k-design: $\Phi_{\mathcal{E}}^{(k)}(\mathcal{O}) = \Phi_{\text{Haar}}^{(k)}(\mathcal{O})$

but for general k, few exact constructions are known

Definition (Approximate k-design)

For $\epsilon > 0$, an ensemble $\mathcal{E}$ is an ϵ-approximate k-design if the k-fold channel obeys

$$\left\| \Phi_{\mathcal{E}}^{(k)} - \Phi_{\text{Haar}}^{(k)} \right\|_{\diamond} \leq \epsilon$$

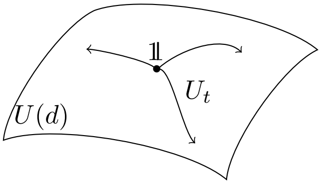
$\rightarrow$ designs are powerful

Intuition for k-designs

(eschewing rigor)

How random is the time-evolution of a system compared to the full unitary group $U(d)$?

Consider an ensemble of time-evolutions at a fixed time t: $\mathcal{E}_t = \{U_t\}$
e.g. RQCs, Brownian circuits, or $\{e^{-iHt},\ H \in \mathcal{E}_H\}$ generated by disordered Hamiltonians



quantify randomness:
when does $\mathcal{E}_t$ form a k-design?
(approximating moments of $U(d)$)

Complexity by design

an exercise in enumeration

Consider a discrete approximate unitary design $\mathcal{E} = \{p_i, U_i\}$.

Can we say anything about the complexity of U_i's?

The structure of a design is sufficiently restrictive, can count the number of unitaries of a specific complexity

Theorem (Complexity for unitary designs)

For $\delta > 0$, an ϵ-approximate unitary k-design contains at least

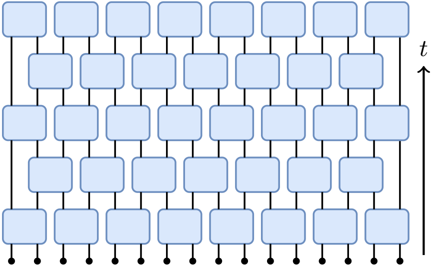
$$M \geq \frac{d^{2k}}{k!}\frac{1}{(1+\epsilon')} - \frac{n^r |G|^r}{(1-\delta^2)^k}$$

unitaries U with $\mathcal{C}_\delta(U) > r$.

This is essentially $\approx (d^2/k)^k$ for $r \lesssim kn$ (exp growth in design k)

Random quantum circuits

Consider G-local RQCs on n qudits of local dimension q, evolved with staggered layers of 2-site unitaries, each drawn randomly from a universal gate set G



where evolution to time t is given by $U_t = U^{(t)} \dots U^{(1)}$

RQCs and randomness

Now we need a powerful result from [Brandão, Harrow, Horodecki]

Theorem (G-local random circuits form approximate designs)

For $\epsilon > 0$, the set of all G-local random quantum circuits of size T forms an ϵ-approximate unitary k-design if

$$T \geq cn\lceil \log k \rceil^2 k^{10}(n + \log(1/\epsilon))$$

where c is a (potentially large) constant depending on the universal gate set G.

Less rigorous version: RQCs of size $T \sim n^2 k^{10}$ form k-designs

Complexity by design

curbing collisions

Now we can combine these two results to say something about the complexity of states generated by G-local random circuits

Fix some initial state $|\psi_0\rangle$, and consider the set of states generated by G-local RQCs: $\{U_i\,|\psi_0\rangle\,,\ U_i \in \mathcal{E}_{G\text{-local RQC}}\}$

Obviously, at early times: $\mathcal{C}_\delta(|\psi\rangle) \approx T$

but we must account for collisions: $U_1\,|\psi_0\rangle \approx U_2\,|\psi_0\rangle$

and collisions must dominate at exponential times as the complexity saturates

but the definition of an ϵ-approximate design restricts the number of potential collisions
$\rightarrow$ allows us to count the $\#$ of distinct states

Complexity by design

curbing collisions

Now we can combine these two results to say something about the complexity of states generated by G-local random circuits

Fix some initial state $|\psi_0\rangle$, and consider the set of states generated by G-local RQCs: $\{U_i\,|\psi_0\rangle\,,\ U_i \in \mathcal{E}_{G\text{-local RQC}}\}$

For $r \leq \sqrt{d}$, G-local RQCs of size T, where $T \geq c\,n^2(r/n)^{10}$, generate at least

$$M \gtrsim c' e^{r \log n}$$

distinct states with $\mathcal{C}_\delta(|\psi\rangle) > r$.

This establishes a polynomial relation between the growth of complexity and size of the circuit up to $r \leq \sqrt{d}$

$\rightarrow$ but what we really want is linear growth

RQCs and $T \sim k$

an appeal for linearity

To get a linear growth in complexity we need a linear growth in design

we had $T = O(n^2 k^{10})$, but would need $T = O(n^2 k)$

[Brandão, Harrow, Horodecki]: a lower bound on the k-design depth for RQCs is $O(nk)$

Can we prove that RQCs saturate this lower bound? (and are thus optimal implementations of k-designs)

k-designs from stat-mech in RQCs

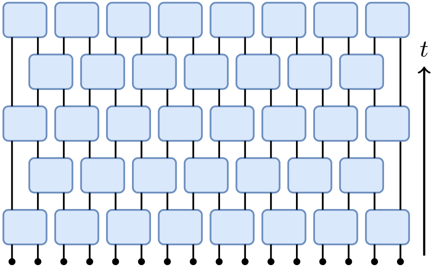
I'll now briefly summarize the result mentioned two weeks ago

using an exact stat-mech mapping, we can show that RQCs form k-designs in $O(nk)$ depth in the limit of large local dimension

this was for local Haar-random gates, but we believe it should extend to G-local circuits with any local dimension q

Random quantum circuits

Consider local RQCs on n qudits of local dimension q, evolved with staggered layers of 2-site unitaries, each drawn randomly from the Haar measure on $U(q^2)$



where evolution to time t is given by $U_t = U^{(t)} \dots U^{(1)}$

Study the convergence of random quantum circuits to **unitary k-designs**, i.e. depth where we start approximating moments of the unitary group

Our approach

- Focus on 2-norm and analytically compute the frame potential for random quantum circuits
- Making use of the ideas in [Nahum, Vijay, Haah], [Zhou, Nahum], we can write the frame potential as a lattice partition function
- We can compute the $k = 2$ frame potential exactly, but for general k we must sacrifice some precision
- We'll see that the decay to Haar-randomness can be understood in terms of domain walls in the lattice model

Frame potential

The frame potential is a tractable measure of Haar randomness, defined for an ensemble of unitaries $\mathcal{E}$ as [Gross, Audenaert, Eisert], [Scott]

k-th frame potential : $$\mathcal{F}_{\mathcal{E}}^{(k)} = \int_{U,V\in\mathcal{E}} dU\,dV \left|\mathrm{Tr}(U^\dagger V)\right|^{2k}$$

For any ensemble $\mathcal{E}$, the frame potential is lower bounded as

$$\mathcal{F}_{\mathcal{E}}^{(k)} \geq \mathcal{F}_{\mathrm{Haar}}^{(k)} \quad \text{and} \quad \mathcal{F}_{\mathrm{Haar}}^{(k)} = k! \;\; (\text{for } k \leq d)$$

with $=$ if and only if $\mathcal{E}$ is a k-design.

Related to ϵ-approximate k-design as

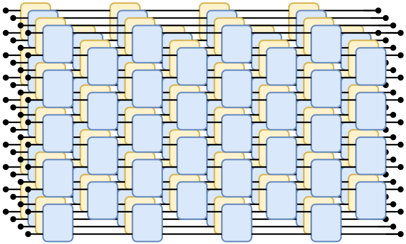
$$\boxed{\left\|\Phi_{\mathcal{E}}^{(k)} - \Phi_{\mathrm{Haar}}^{(k)}\right\|_{\diamond}^{2} \leq d^{2k}\left(\mathcal{F}_{\mathcal{E}}^{(k)} - \mathcal{F}_{\mathrm{Haar}}^{(k)}\right)}$$

Frame potential for RQCs

The goal is to compute the FP for RQCs evolved to time t:

$$\mathcal{F}^{(k)}_{\mathrm{RQC}} = \int_{U_t, V_t \in \mathrm{RQC}} dU\, dV\, \left|\mathrm{Tr}(U_t^\dagger V_t)\right|^{2k}$$

Consider the k-th moments of RQCs, k copies of the circuit and its conjugate:

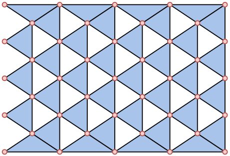


Lattice mappings for RQCs

Haar averaging the 2-site unitaries allows us to exactly write the frame potential as a partition function on a triangular lattice.

The result is then that we can write the k-th frame potential as

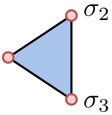
$$\mathcal{F}_{\mathrm{RQC}}^{(k)} = \sum_{\{\sigma\}} \prod_{\triangleleft} J_{\sigma_2\sigma_3}^{\sigma_1} = \sum_{\{\sigma\}}$$



with $\sigma \in S_k$, width $n_g = \lfloor n/2 \rfloor$, depth $2(t-1)$, and pbc in time.

The plaquettes are functions of three $\sigma \in S_k$, written explicitly as

$$J_{\sigma_2\sigma_3}^{\sigma_1} = \sigma_1$$



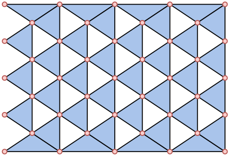
$$= \sum_{\tau \in S_k} \mathcal{W}g(\sigma_1^{-1}\tau, q^2) q^{\ell(\tau^{-1}\sigma_2)} q^{\ell(\tau^{-1}\sigma_3)} .$$

Lattice mappings for RQCs

Haar averaging the 2-site unitaries allows us to exactly write the frame potential as a partition function on a triangular lattice.

The result is then that we can write the k-th frame potential as

$$\mathcal{F}^{(k)}_{\mathrm{RQC}} = \sum_{\{\sigma\}} \prod_{\triangleleft} J^{\sigma_1}_{\sigma_2\sigma_3} = \sum_{\{\sigma\}}$$



with $\sigma \in S_k$, width $n_g = \lfloor n/2 \rfloor$, depth $2(t-1)$, and pbc in time.

We can show that $J^{\sigma}_{\sigma\sigma} = 1$, and thus the minimal Haar value of the frame potential comes from the $k!$ ground states of the lattice model

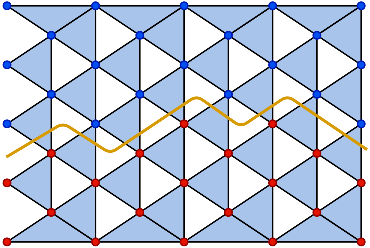
$$\mathcal{F}^{(k)}_{\mathrm{RQC}} = k! + \dots$$

RQC domain walls

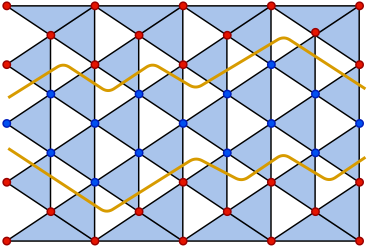
all non-zero contributions to $\mathcal{F}^{(k)}_{\mathrm{RQC}}$ are domain walls
(which must wrap the circuit)

e.g. for $k = 2$ we have

a single domain wall configuration:



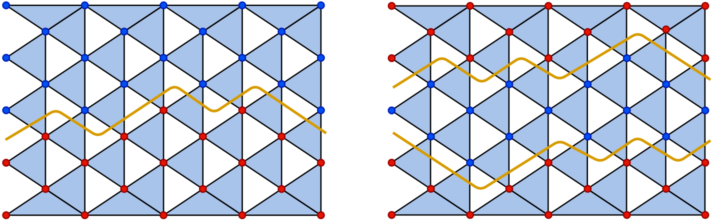
a double domain wall configuration:



k-designs from domain walls

To compute the k-design time, we simply need to count the domain wall configurations

$$\mathcal{F}_{\mathrm{RQC}}^{(k)} = k!\left(1 + \sum_{1\ \mathrm{dw}} wt(q,t) + \sum_{2\ \mathrm{dw}} wt(q,t) + \dots\right)$$



$\rightarrow$ decay to Haar-randomness from dws

RQC 2-design time

We have the $k = 2$ frame potential for random circuits

$$\mathcal{F}_{\mathrm{RQC}}^{(2)} \leq 2\left(1 + \left(\frac{2q}{q^2+1}\right)^{2(t-1)}\right)^{n_g - 1}$$

and recalling that $\left\|\Phi_{\mathrm{RQC}}^{(2)} - \Phi_{\mathrm{Haar}}^{(2)}\right\|_\diamond^2 \leq d^4\big(\mathcal{F}_{\mathrm{RQC}}^{(2)} - \mathcal{F}_{\mathrm{Haar}}^{(2)}\big)$,

the circuit depth at which we form an ϵ-approximate 2-design is then

$$t_2 \geq C\big(2n\log q + \log n + \log 1/\epsilon\big) \quad \text{with} \quad C = \left(\log\frac{q^2+1}{2q}\right)^{-1}$$

and where for $q = 2$ we have $t_2 \approx 6.2n$, and in the limit $q \to \infty$ we find $t_2 \approx 2n$

k-designs in RQCs

For general k, we then have the contribution from the ground states and single domain wall sector, plus higher order contributions

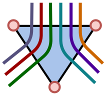
$$\mathcal{F}_{\mathrm{RQC}}^{(k)} \leq k!\left(1 + (n_g - 1)\binom{k}{2}\binom{2(t-1)}{t-1}\left(\frac{q}{q^2+1}\right)^{2(t-1)} + \dots\right)$$

k-designs in RQCs

For general k, we then have the contribution from the ground states and single domain wall sector, plus higher order contributions

$$\mathcal{F}_{\mathrm{RQC}}^{(k)} \leq k! \left(1 + (n_g - 1) \binom{k}{2} \binom{2(t-1)}{t-1} \left(\frac{q}{q^2+1}\right)^{2(t-1)} + \dots \right)$$

Moreover, the multi-domain wall terms are heavily suppressed and higher order interactions are subleading in $1/q$ as



$$\sim \frac{1}{q^p}$$

In the large q limit, the single domain wall sector gives the ϵ-approximate k-design time: $t_k \geq C(2nk \log q + k \log k + \log(1/\epsilon))$, which is

$$t_k = O(nk)$$

k-designs from stat-mech

RQCs form k-designs in $O(nk)$ depth

we showed this in the large q limit, but this limit is likely not necessary

Conjecture: *The single domain wall sector of the lattice partition function dominates the multi-domain wall sectors for higher moments k and any local dimension q.*

As the lower bound on the design depth is $O(nk)$, RQCs are then **optimal implementations of randomness**

Back to complexity

We'll now end on a much more speculative note

If this result holds for G-local random circuits, and for any local dimension q, then the circuits of size $T = O(n^2 k)$ form approx unitary k-designs

Therefore, G-local RQCs of size T generate at least $M \geq (d/k)^k$ distinct states with complexity $\mathcal{C}_\delta(|\psi\rangle) \approx T$. For $k \leq \sqrt{d}$, we have

$$M \gtrsim e^{T \log n}$$

This would then realize a conjecture by [Brown, Susskind] in an explicit example:

the $\#$ of states with $\mathcal{C}_\delta(U_T |\psi_0\rangle) \approx T$, generated by time-evolution to time T (in this case RQCs of size T), scales exponentially in T

Future science

- Can we prove anything about $\mathcal{C}_\delta(e^{-iHt}\left|\psi\right\rangle)$ for a fixed Hamiltonian?
- Can we rigorously bound the higher order terms in $\mathcal{F}^{(k)}_{\mathrm{RQC}}$ at small q? and then extend the result to G-local RQCs
- Explore the implications of an operational definition of complexity (in terms of a distinguishing measurement). More suited for holography?

Thanks!

(ご清聴ありがとうございました)