Entanglement, free energy and C-theorem in defect CFT

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based on 1810.06995 with N. Kobayashi, Y. Sato and K. Watanabe
and a work in progress with K. Goto, L. Nagano and T. Okuda

Outline

1 Defect conformal field theories

2 Entanglement entropy and sphere free energy in DCFT

3 Towards a C-theorem in DCFT

Defects in quantum field theory

Defects = Non-local objects in QFTs

- Defined by boundary conditions around them
- Many examples:

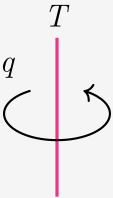
1-dim : Line operators (Wilson-'t Hooft loops)

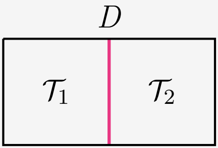
2-dim : Surface operators

Codim-1 : Domain walls, interfaces and boundaries

Codim-2 : Entangling surface for entanglement entropy

$$\int_{\mathbb{S}^2} F = 2\pi q$$





Defect as a probe of QFT phases

- Characterize phases of QFTs: ['t Hooft 78]

$$W\text{: Wilson loop}, \quad T\text{: 't Hooft loop}$$

$$\text{Confinement}: \ \langle W \rangle \sim e^{-\text{Area}}, \qquad \langle T \rangle \sim e^{-\text{Length}}$$

$$\text{Higgs}: \ \langle W \rangle \sim e^{-\text{Length}}, \quad \langle T \rangle \sim e^{-\text{Area}}$$

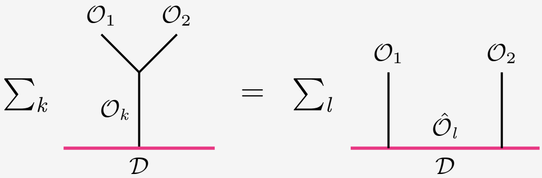
- Higher-dimensional generalization:
  - Wilson surface operators,

$$W_\Sigma = \exp\left(i \int_\Sigma A \right)$$

for a p-form gauge field A

Possible applications of defects

- Constrain bulk CFT data in defect CFT by conformal bootstrap
  [Liendo-Rastelli-van Rees 12]

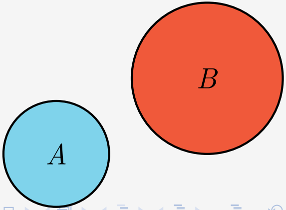


- Understand quantum entanglement in QFT:
  e.g. Mutual information
  [Cardy 13, Bianchi-Meineri-Myers-Smolkin 15]

$$I(A,B) \equiv S_A + S_B - S_{A\cup B}$$

as a correlator of two defects,

$$I(A,B) = \log \frac{\langle \mathcal{D}(\partial A)\,\mathcal{D}(\partial B)\rangle}{\langle \mathcal{D}(\partial A)\rangle\,\langle \mathcal{D}(\partial B)\rangle}$$

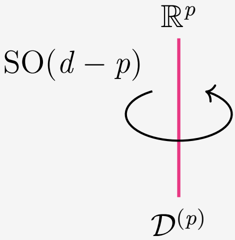


Conformal defects

- In Euclidean CFT$_d$, the conformal group is $\mathrm{SO}(d+1,1)$
- p-dimensional conformal defects $\mathcal{D}^{(p)}$ are either flat or spherical, preserving

$\mathrm{SO}(p+1,1)$: conformal symmetry on defects

$\mathrm{SO}(d-p)$: rotation in the transverse direction



- Defects allow for defect local operators $\hat{\mathcal{O}}_n$

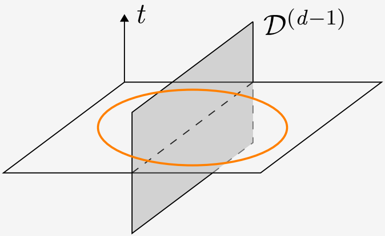
Main goal

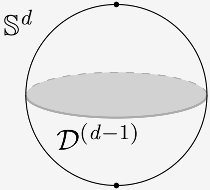
- How can we measure the degrees of freedom associated to defects?

  Possibilities:

  - Entanglement entropy across a sphere
  - Sphere free energy

- Is there a C-theorem in DCFT?





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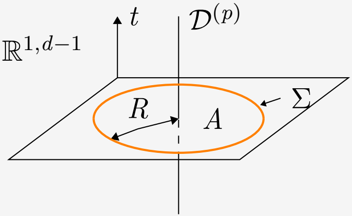
Entanglement entropy across a sphere



- A: a ball centered at the origin
- $\mathcal{D}^{(p)}$: a p-dim flat defect
- After a conformal transformation (CHM map [Casini-Huerta-Myers 11, Jensen-O'Bannon 13])

The n-th Rényi entropy

$$S_n^{(\mathrm{DCFT})} = \frac{1}{1-n} \log \frac{Z^{(\mathrm{DCFT})}[\mathbb{S}_n^d]}{\left(Z^{(\mathrm{DCFT})}[\mathbb{S}^d]\right)^n}$$

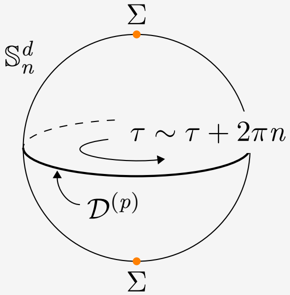
$\mathbb{S}_n^d$: n-fold cover of $\mathbb{S}^d$

Defect entropy

- The excess of EE is measured by

Defect entropy

$$S_{\text{defect}} \equiv \lim_{n\to 1}\left(S_n^{(\text{DCFT})} - S_n^{(\text{CFT})}\right)$$



Defect entropy

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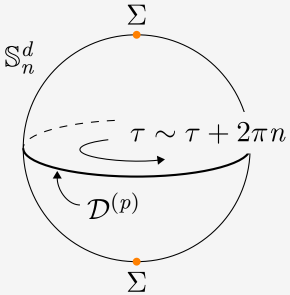
Defect entropy

$$S_{\text{defect}} \equiv \lim_{n\to 1}\left(S_n^{(\text{DCFT})} - S_n^{(\text{CFT})}\right)$$
$$= \lim_{n\to 1}\frac{1}{1-n}\log\frac{\langle\, \mathcal{D}^{(p)}\,\rangle_n}{\langle\, \mathcal{D}^{(p)}\,\rangle^n}$$



$$\langle\, \mathcal{D}^{(p)}\,\rangle_n \equiv \frac{Z^{(\text{DCFT})}[\mathbb{S}_n^d]}{Z^{(\text{CFT})}[\mathbb{S}_n^d]}$$

$$\langle\, \mathcal{D}^{(p)}\,\rangle \equiv \langle\, \mathcal{D}^{(p)}\,\rangle_1 \quad (\text{vev of } \mathcal{D}^{(p)})$$

$n \to 1$ limit

- Expansion around $n = 1$ ($\delta g_{\mu\nu} = O(n-1)$)

$$\log\, Z^{(\mathrm{DCFT})}[\mathbb{S}^d_n] = \log\, Z^{(\mathrm{DCFT})}[\mathbb{S}^d] - \frac{1}{2}\int_{\mathbb{S}^d} \delta g_{\mu\nu} \, \langle\, T^{\mu\nu} \,\rangle^{(\mathrm{DCFT})}_{\mathbb{S}^d} + O\left((n-1)^2\right)$$

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- In CFT [Casini-Huerta-Myers 11]

$$\langle T^{\mu\nu} \rangle^{(\mathrm{CFT})}_{\mathbb{S}^d} = 0$$

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- In DCFT

$$\langle T^{\mu\nu} \rangle^{(\mathrm{CFT})}_{\mathbb{S}^d} \neq 0 \quad (\propto a_T)$$

Universal relation

Defect entropy and sphere free energy [Kobayashi-TN-Sato-Watanabe 18]

$$S_{\text{defect}} = \log \langle \mathcal{D}^{(p)} \rangle - \frac{2(d-p-1)\,\pi^{d/2+1}}{\sin(\pi p/2)\; d\,\Gamma(p/2+1)\;\Gamma((d-p)/2)}\, a_T$$

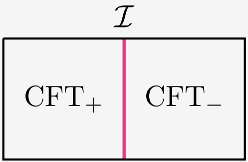
- Equality holds up to UV divergences
- When $p=1$ agrees with [Lewkowycz-Maldacena 13]
- When $p=d-1$ (codimension-one), the second term in rhs vanishes [McAvity-Osborn 95]

Interface entropy

- Interface CFT, $\mathcal{D}^{(d-1)} = \mathcal{I}$

Interface entropy

$$S_{\mathcal{I}} = S^{(\mathrm{ICFT})} - \frac{S^{(\mathrm{CFT}_+)} + S^{(\mathrm{CFT}_-)}}{2}$$



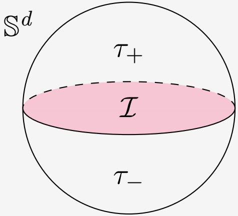
- Universal relation:

$$S_{\mathcal{I}} = \log \langle \mathcal{I} \rangle \,, \qquad \langle \mathcal{I} \rangle \equiv \frac{Z^{(\mathrm{ICFT})}[\mathbb{S}^d]}{\left(Z^{(\mathrm{CFT}_+)}[\mathbb{S}^d]\, Z^{(\mathrm{CFT}_-)}[\mathbb{S}^d]\right)^{1/2}}$$

Interface entropy of SUSY Janus interface

- Half-BPS Janus interface in $2d$ $\mathcal{N} = (2,2)$ and $4d$ $\mathcal{N} = 2$ SCFTs:

$$Z^{(\text{ICFT})}[\mathbb{S}^d] \propto \exp\left[c_0\, K(\tau_+, \bar{\tau}_-)\right]$$



$K(\tau_+, \bar{\tau}_-)$: Kähler potential on a conf mfd

[Jockers-Kumar-Lapan-Morrison-Romo 12, Gomis-Lee 12, Goto-Okuda 18, Drukker-Gaiotto-Gomis 10, Gerchkovitz-Gomis-Komargodski 14]

- Reproduces the relation (see Nagano's poster)
  ($2d$: [Bachas-BrunnerDouglas-Rastelli 13, Bachas-Plencner 16], $4d$: conjecture [Goto-Okuda 18])

Interface entropy as Calabi's diastasis [Goto-Nagano-TN-Okuda, WIP]

$$S_{\mathcal{I}} = \log |\langle \mathcal{I} \rangle|$$

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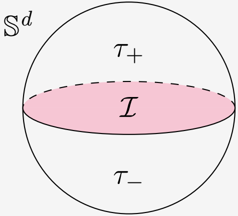

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Interface entropy as Calabi's diastasis [Goto-Nagano-TN-Okuda, WIP]

$$S_{\mathcal{I}} = -\frac{c_0}{2}\left[K(\tau_+, \bar{\tau}_+) + K(\tau_-, \bar{\tau}_-) - K(\tau_+, \bar{\tau}_-) - K(\tau_-, \bar{\tau}_+)\right]$$

(invariant under Kähler transformation)

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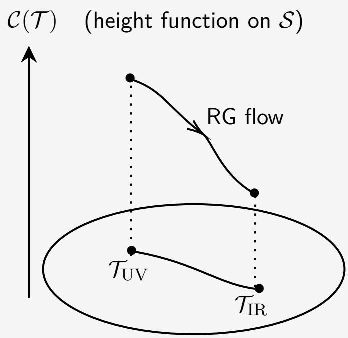
C-theorem

C-theorem (weak)

$\exists$ a function $\mathcal{C}(\mathcal{T})$ on a theory space s.t.

$$\mathcal{T}_{\rm UV} \xrightarrow[\text{RG flow}]{} \mathcal{T}_{\rm IR}$$

$$\Rightarrow \quad \mathcal{C}(\mathcal{T}_{\rm UV}) \geq \mathcal{C}(\mathcal{T}_{\rm IR})$$



$\mathcal{S}$ = space of QFTs

- $\mathcal{C}(\mathcal{T})$ called a $\mathcal{C}$-function ($\approx$ resource measure)
- Regarded as a measure of degrees of freedom in QFT
- Constrains the dynamics under RG if holds

Examples and conjectures

- $2d$: Zamolodchikov's c-theorem [Zamolodchikov 86]
- even d: A-theorem ($\langle T^{\mu}_{\mu} \rangle_{\mathbb{S}^d} \propto A$) [Cardy 88, Komargodski-Schwimmer 11]

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Generalized F-theorem conjecture [Giombi-Klebanov 14]

$$\tilde{F} \equiv \sin\left(\frac{\pi d}{2}\right) \log Z[\mathbb{S}^d], \qquad \tilde{F}_{\text{UV}} \geq \tilde{F}_{\text{IR}}$$

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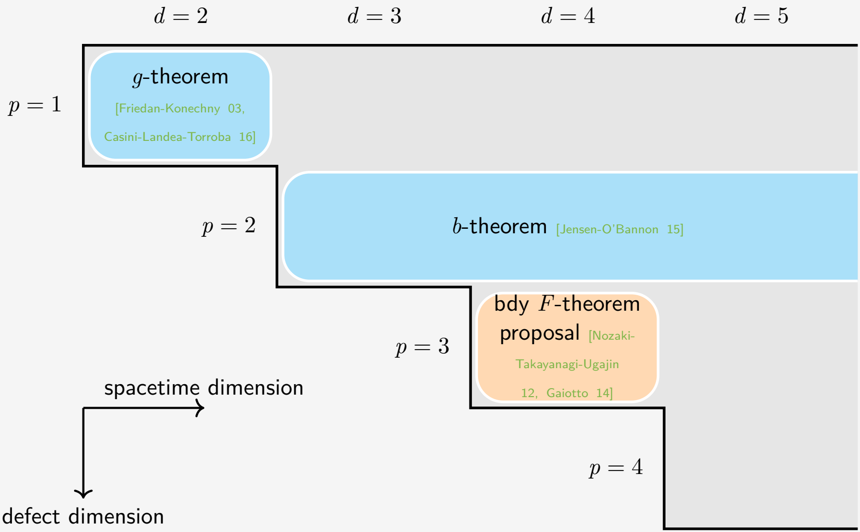
- Reduces to the F- and A-theorems

$$\tilde{F} = \begin{cases} F & d : \text{odd} \\ \frac{\pi}{2} A \quad (\text{conformal anomaly}) & d : \text{even} \end{cases}$$

Status of C-theorems (+ conjectures) in DCFT



Possible $\mathcal{C}$-function in DCFT

- Candidate $\mathcal{C}$-functions
  - entanglement entropy: holographic model $(p = d - 1)$
    [Estes-Jensen-O'Bannon-Tsatis-Wrase 14]
  - sphere free energy: bdy F-thm, b-thm $(p = 2)$,
    Wilson loop RG flow $(d = 4, p = 1)$ [Beccaria-Giombi-Tseytlin 17]
- These two agree when $p = d - 1$ due to the universal relation

Are both $\mathcal{C}$-functions for any d and p?

$\mathcal{C}$-theorem in DCFT: conjecture

- Defect RG flow triggered by a relevant defect operator:

$$I = I_{\mathrm{DCFT}} + \hat{\lambda} \int \mathrm{d}^p \hat{x} \sqrt{\hat{g}}\, \hat{\mathcal{O}}(\hat{x})$$

Conjecture [Kobayashi-TN-Sato-Watanabe 18]

The universal part of the sphere free energy

$$\tilde{D} \equiv \sin\left(\frac{\pi p}{2}\right) \log |\langle \mathcal{D}^{(p)} \rangle|$$

does not increase along any defect RG flow

$$\tilde{D}_{\mathrm{UV}} \geq \tilde{D}_{\mathrm{IR}}$$

- Same form as the generalized F-thm: $\tilde{F} = \sin\left(\frac{\pi d}{2}\right) \log Z[\mathbb{S}^d]$

Checks

- Sphere free energy decreases under defect RG flows in
  - Conformal perturbation theory
  - Wilson loops $(p = 1)$ in $3d$ and $4d$
  - Holographic models (a proof assuming null energy condition)
- Counterexamples for the monotonicity of entanglement entropy
  - Wilson loop RG flows, surface operators $(p = 2)$
    [Jensen-O'Bannon-Robinson-Rodgers 18]
  - Holographic Wilson loop [Kumar-Silvani 16, 17] and surface operators [Rodgers 18]

$$S_{\text{defect}} = \log \langle \mathcal{D}^{(p)} \rangle - \frac{2(d-p-1)\,\pi^{d/2+1}}{\sin(\pi p/2)\;d\,\Gamma(p/2+1)\;\Gamma((d-p)/2)}\,a_T$$

Summary and future work

- Summary:
  - Find the universal relation between EE and sphere free energy
  - Derive the interface entropy as Calabi's diastasis
  - Propose a $\mathcal{C}$-theorem in DCFT
- Future work:
  - Proof in SUSY theories? (cf. F- and a-maximizations [Jafferis 10, Closset-Dumitrescu-Festuccia-Komargodskia-Seiberg 12, Intriligator-Wecht 03])
  - Proof using entropic inequalities as in g-thm [Casini-Landea-Torroba 16]?
  - Constrains on the dynamics of defect RG flows?