Operator Growth in SYK model and Beyond

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Overview

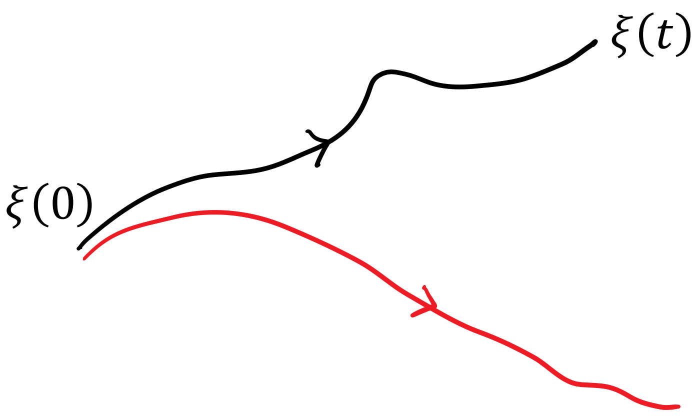
- Motivation
- Definition of operator size and size distribution
- General results: thermal state and single fermion excitations
- SYK model
  - Large q limit
  - Large βJ limit
- Dual theory: size distribution of bulk fermion operators

Ref.
1. Alexandre Streicher, XLQ, arxiv:1810.11958
2. Yingfei Gu, Yuri Lensky, XLQ, Pengfei Zhang, in progress

Part I: General results

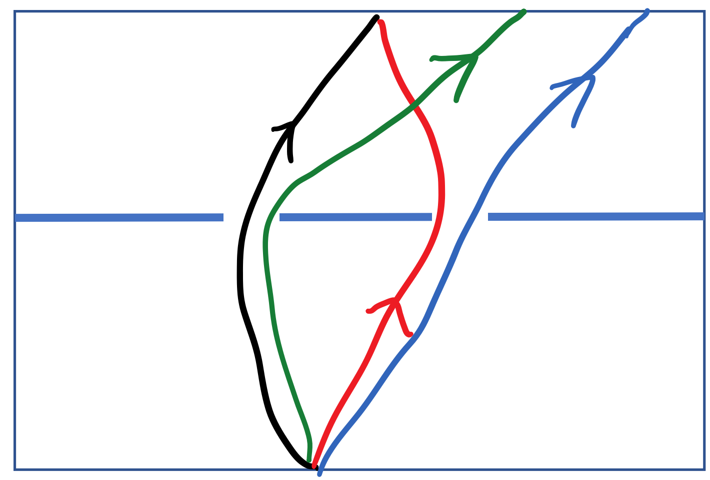
Motivation

- Classical chaos:
- $\xi(0) = \big(x(0), p(0)\big) \to \xi(t) = \big(x(t), p(t)\big)$
- Sensitivity to initial condition
- Lyapunov exponents
- $\xi(t)$ is a complicated function of $\xi(0)$
- Quantum analog: Heisenberg operator evolution $\hat{O}(t) = e^{itH}\hat{O}(0)e^{-itH}$
- $\hat{O}(t)$ is a complicated function of $\hat{O}$
- How to define **"complicated"**?

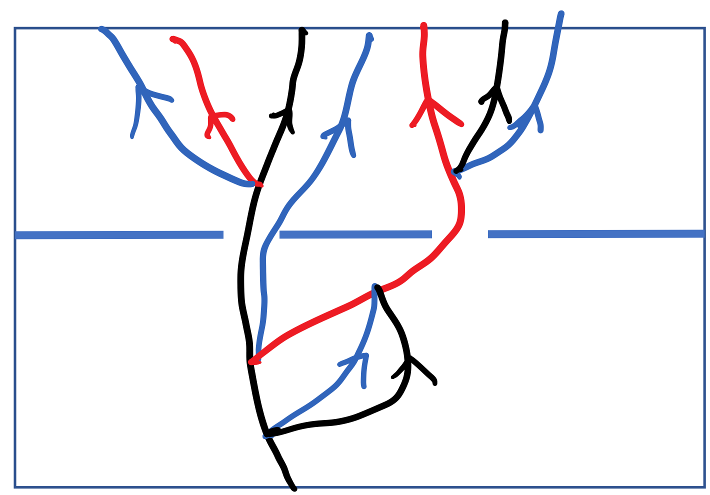


Many-body quantum chaos

- Non-interacting system:
  A particle has N possible positions.
  $\psi_x(t) = \sum_{y=1\ldots N} \phi_x(y)\hat{\psi}_y(0).$

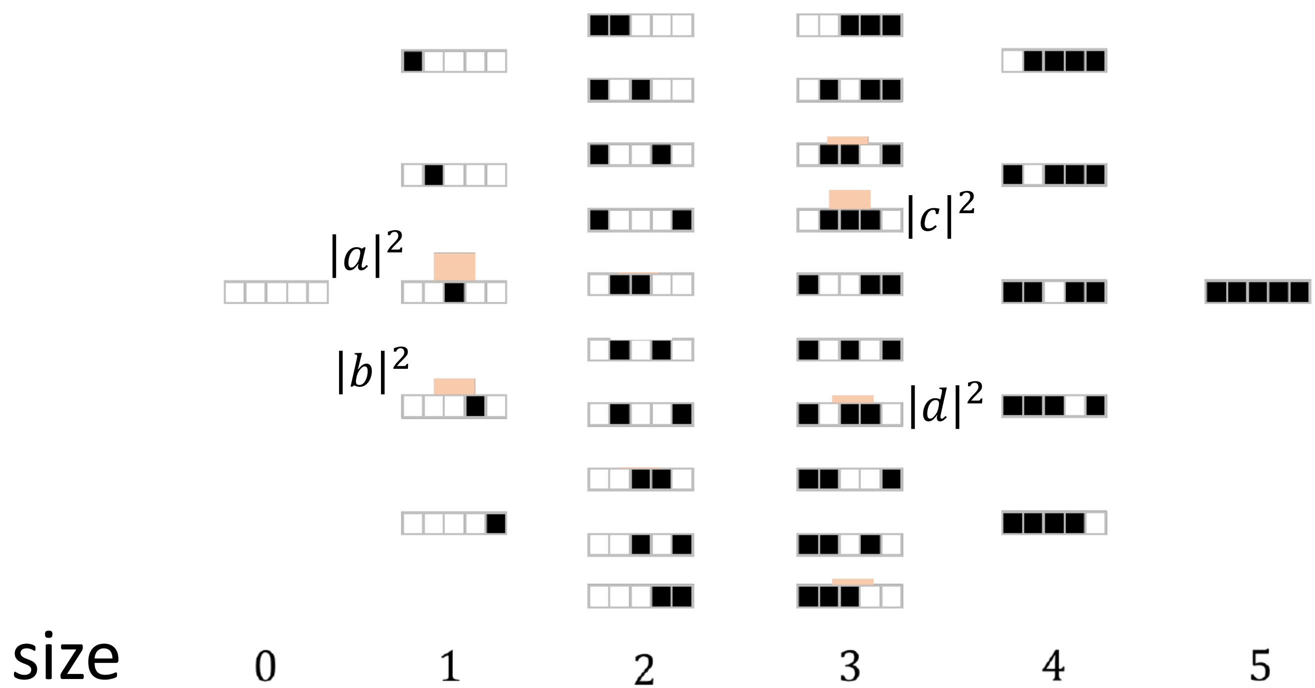


- Generic interacting system:
  A particle can decay into multi-particle states. Exponentially many final states in the Hilbert space.
- $\hat{\psi}_x(t) = \phi_x(y)\hat{\psi}_y(0) + \phi_x(y_1 y_2 y_3)\hat{\psi}_{y_1}\hat{\psi}^+_{y_2}\hat{\psi}_{y_3} + \cdots$



Operator size distribution

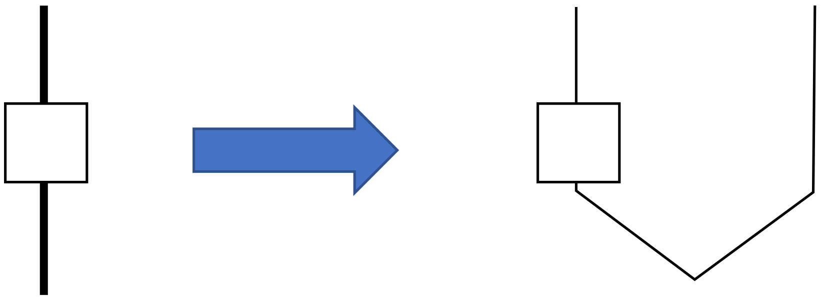
- p_l probability of size l (Roberts-Stanford-Susskind '14, Hosur XLQ '16, Roberts, Streicher, Stanford '18)
- Example: Majorana fermions
- $\hat{\psi}_1(t) = a\hat{\psi}_3 + b\hat{\psi}_4 + c\hat{\psi}_2\hat{\psi}_3\hat{\psi}_4 + d\hat{\psi}_1\hat{\psi}_3\hat{\psi}_4 + \cdots$



$p_1 = |a|^2 + |b|^2 \quad p_3 = |c|^2 + |d|^2$

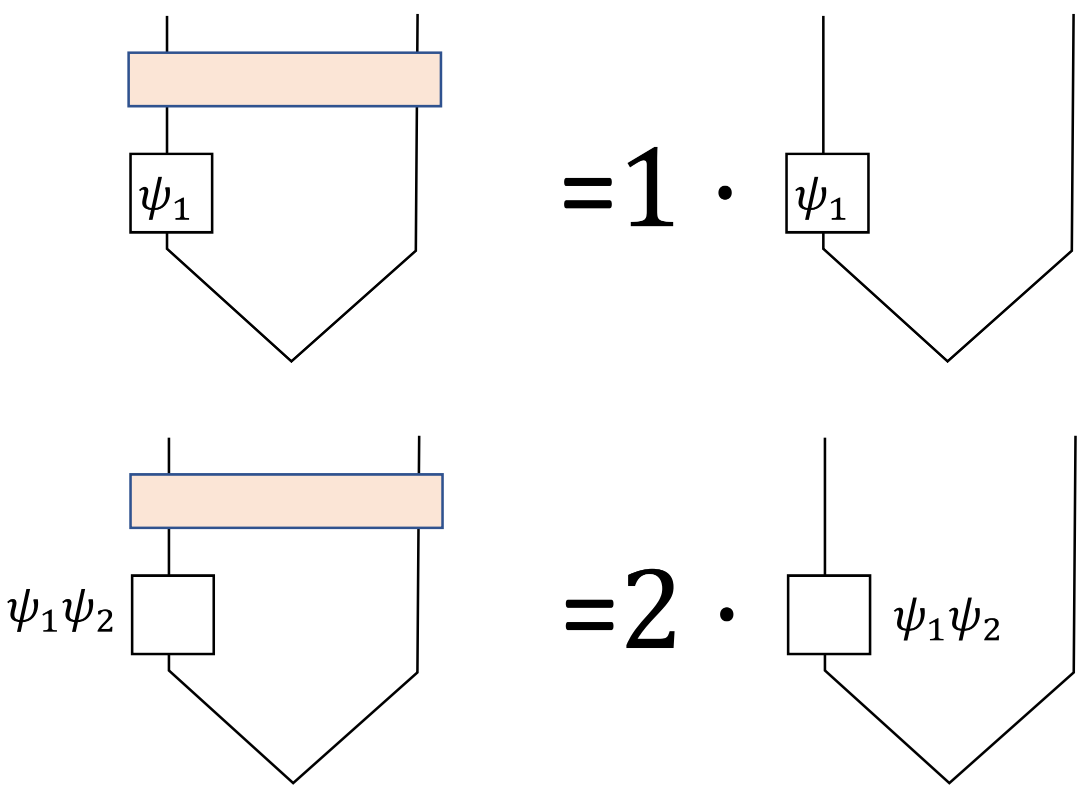
Operator size distribution

- Example: a random fermionic operator
- $p_l = 2^{1-N}\binom{N}{l}$, for odd l. Average size $\frac{N}{2}$
- Doubled formalism:
  - Take two systems L, R, each consisting of N Majoranas.
  - Prepare a maximally entangled state $|I\rangle$
- For every operator O, applying it to the left system, to obtain a state: $|O\rangle = O_L \otimes \mathbb{I}_R |I\rangle$
- $\langle A|B\rangle = \frac{1}{D} tr(A^{+}B)$



The size superoperator

- In the doubled formalism, size is a linear operator



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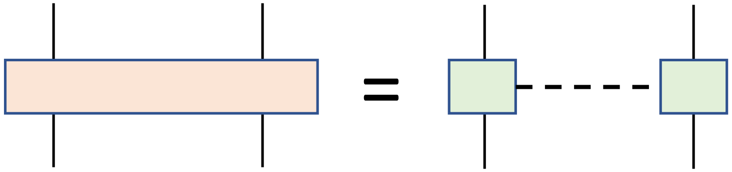
The size superoperator

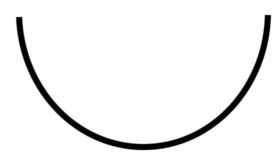
- Size superoperator $\hat{n}$ depends on reference state $|I\rangle$. $\hat{n}|I\rangle = 0$ is required
- A convenient choice of $|I\rangle$: $(\psi_{iL} - i\psi_{iR})|I\rangle = 0$
- $|I\rangle$ is the vacuum state of fermion with annihilation operator $f_i = \frac{1}{2}(\psi_{iL} - i\psi_{iR})$, $\{f_i, f_j^+\} = \delta_{ij}$
- In this basis $\hat{n} = \sum_i f_i^+ f_i$ is fermion number
- $\psi_{iL}\psi_{jL} \dots |I\rangle = f_i^+ f_j^+ \dots |I\rangle$
- In Majorana basis
- $\hat{n} = \frac{1}{2}(N - i\sum_i \psi_{iL}\psi_{iR})$

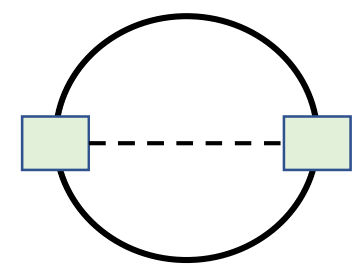


Generating function

- The size superoperator can be used to compute the full size distribution
- $p_l = \sum_\alpha \left|\langle l\alpha|\hat{O}|I\rangle\right|^2$
- $\mathcal{G}_\mu = \langle I|\hat{O}^+ e^{-\mu\hat{n}}\hat{O}|I\rangle = \sum_{l=0}^{N} e^{-\mu l} p_l[\hat{O}]$
- For random fermionic operator
- $\mathcal{G}_\mu^{rand} = \sum_{l \text{ odd}} e^{-\mu l} \binom{N}{l} 2^{1-N}$
  $= \left(\frac{1+e^{-\mu}}{2}\right)^N - \left(\frac{1-e^{-\mu}}{2}\right)^N.$

General results

- Independent from details, this formalism already tells us interesting information about operator size.
- Consider the TFD state:
- $\rho = \frac{1}{Z} e^{-\beta H}$
- $|TFD\rangle = \left|\rho^{\frac{1}{2}}\right\rangle = Z^{-\frac{1}{2}} e^{-\beta H_L} |I\rangle =$ 
- Size of $\rho^{\frac{1}{2}}$:
- $n\left(\rho^{1/2}\right) = \langle TFD|\hat{n}|TFD\rangle =$

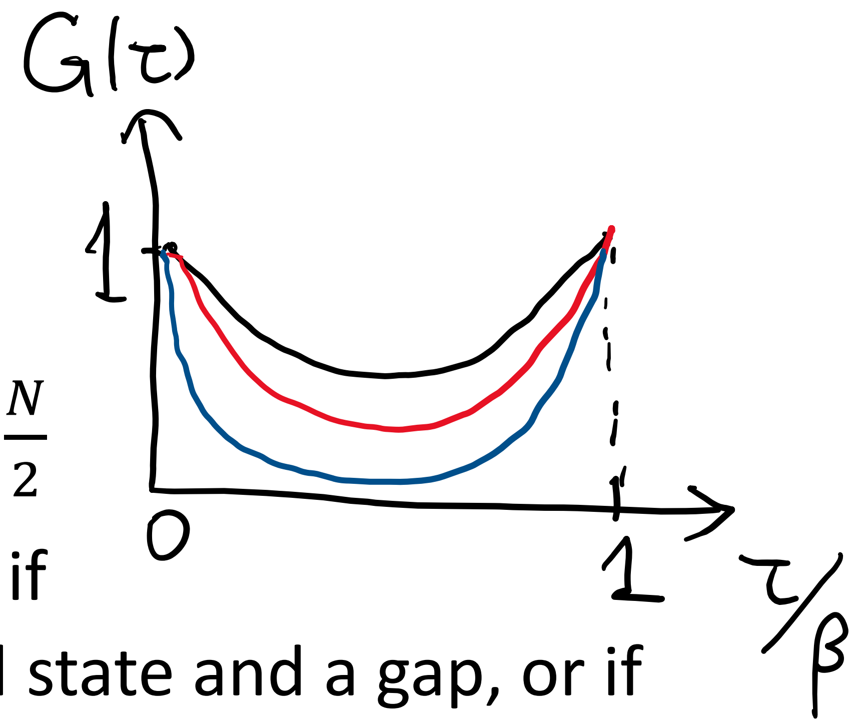


$$\frac{N}{2} - \frac{i}{2} \sum_i \langle TFD|\psi_{iL}\psi_{iR}|TFD\rangle$$

$$= \frac{N}{2} - \frac{1}{2} \sum_i \left\langle \psi_i\left(\frac{\beta}{2}\right) \psi_i(0) \right\rangle \equiv \frac{N}{2} - \frac{N}{2} G\left(\frac{\beta}{2}\right)$$

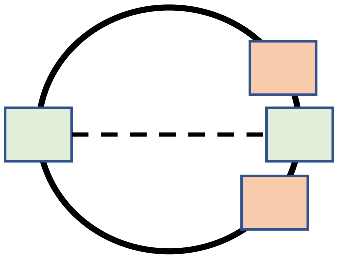
General results 1: thermal state

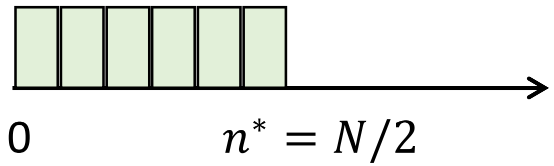
- Distance to scrambling (average size of random operator $n^* = N/2$):
- $\delta_\beta \equiv 1 - \frac{n}{n^*} = G\left(\frac{\beta}{2}\right)$
- Usually, $G\left(\frac{\beta}{2}\right)$ decays to zero at $\beta \to \infty$ ➔ Size approaches $\frac{N}{2}$
- For example, $\rho_{\beta\to\infty}$ has size $\frac{N}{2}$ if the system has unique ground state and a gap, or if the system is a CFT.

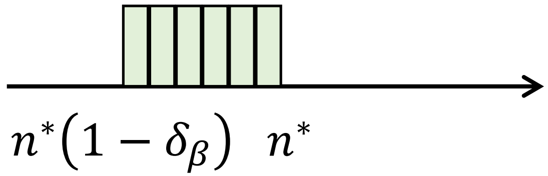


General results 2: single fermion excitation

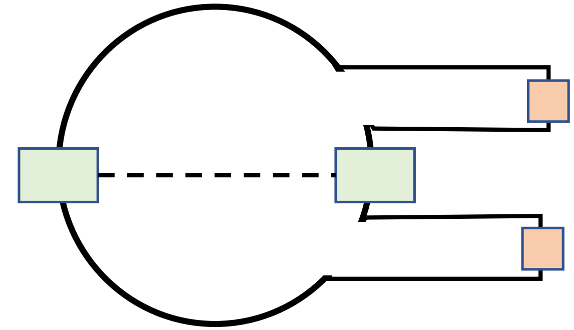
- At infinite temperature, a single fermion operator χ_i has size 1. In our doubled language, this means $\hat{n}\psi_{iL}|I\rangle = \psi_{iL}|I\rangle$
- As a generalization, we can consider $\psi_{iL}|TFD\rangle$
- $\delta n_\beta[\psi_i] = \langle TFD|\psi_{iL}\hat{n}\psi_{iL}|TFD\rangle - \langle TFD|\hat{n}|TFD\rangle = \langle TFD|[\psi_{iL},\hat{n}]\psi_{iL}|TFD\rangle.$
- Fermion anticommutation gives
- $\delta n_\beta[\psi_i] = -i\langle TFD|\psi_{iR}\psi_{iL}|TFD\rangle$
- $\frac{1}{N}\sum_i \delta n[\psi_i] = G\left(\frac{\beta}{2}\right) = \delta_\beta$







General results 3: Time evolution and OTOC

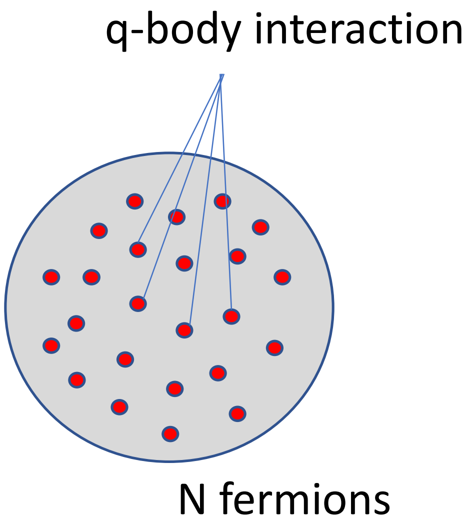


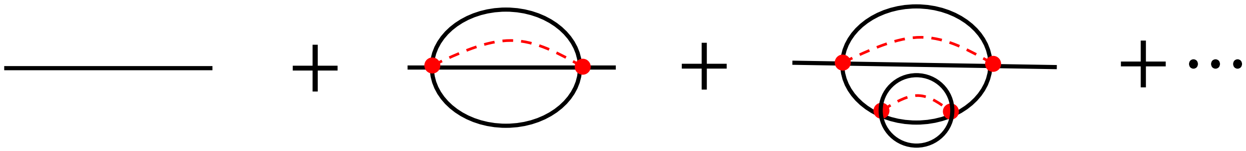
- The effect of (chaotic) dynamics is described by studying the size of Heisenberg operators.
- $\delta n_\beta[\psi_i(t)] = \langle TFD|[\psi_{iL}(t), \hat{n}]\psi_{iL}(t)|TFD\rangle.$
- This is related to the out-of-time-ordered correlation function (OTOC):
- $\delta n_\beta[\psi_i(t)] = \sum_j \left\langle \psi_j\left(\frac{\beta}{2}\right)\{\psi_i(\epsilon+it), \psi_j(0)\}\psi_i(-\epsilon+it)\right\rangle_\beta.$
- Infinite temperature case Roberts, Streicher, Stanford '17
- Finite temperature XLQ-Streicher '18

Part II: SYK model

Sachdev-Ye-Kitaev model

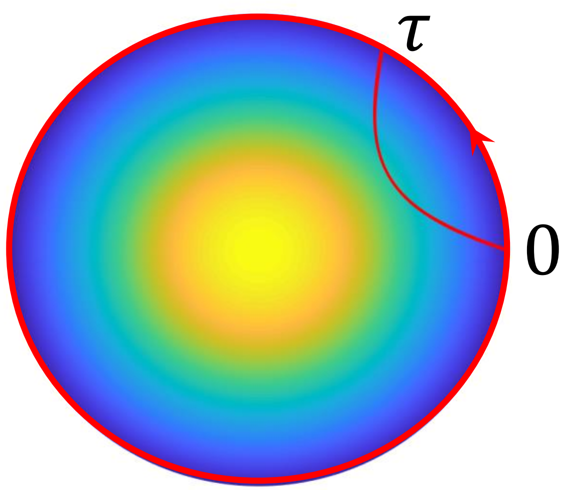
- $H = \sum_{ijkl} J_{ijkl}\chi_i\chi_j\chi_k\chi_l$ with Gaussian random coupling J_{ijkl}.
  (Sachdev-Ye, '93, Kitaev '15, Maldacena-Stanford '16)
- Or complex fermion model
  $H = \sum_{ij,kl} J_{ijkl} c_i^+ c_j^+ c_k c_l$.
- Generalization (Maldacena-Stanford '16) :
  $H = \sum_{i_1 i_2 \ldots i_q} J_{i_1 i_2 \ldots i_q}\chi_{i_1}\chi_{i_2}\cdots\chi_{i_q}$
- Averaging over disorder
- $\overline{Z^n} \simeq \overline{Z}^{\mathrm{n}}$ in large N limit.
- Large N order parameter
- $G(\tau_1,\tau_2) = \frac{1}{N}\sum_i \langle \chi_i(\tau_1)\chi_i(\tau_2)\rangle =$

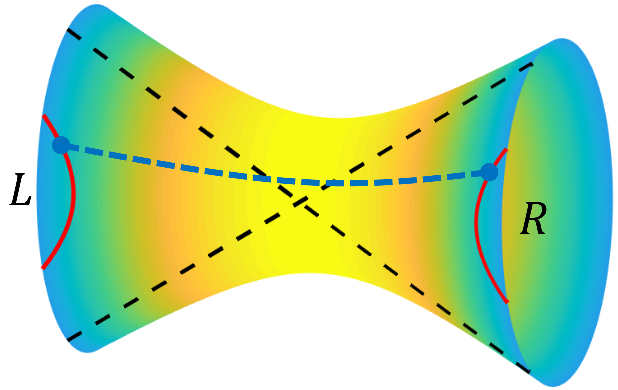




Sachdev-Ye-Kitaev model

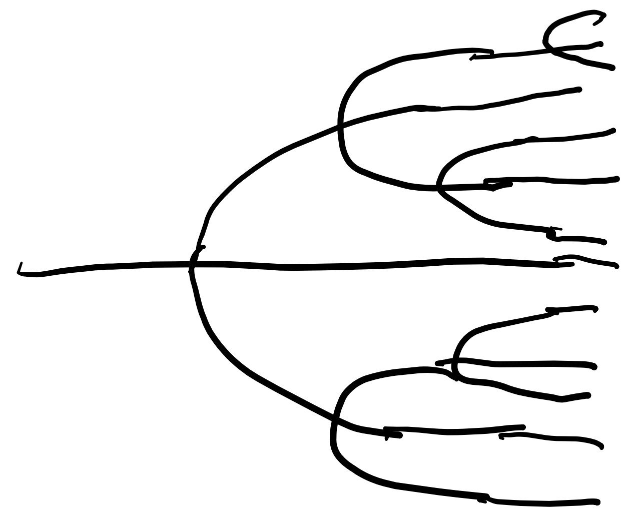
- Critical correlation function ($\Delta = 1/q$)
  $$\langle \chi_i(\tau)\chi_i(0)\rangle \propto \left|\sin\left(\frac{\tau}{\beta}\pi\right)\right|^{-2\Delta} \mathrm{sgn}(\tau).$$
- Real time: thermal double state
  $$|TFD(t)\rangle = Z^{-\frac{1}{2}}\sum_n e^{-E_n\left(\frac{\beta}{2}+it\right)}|n\rangle_L|n\rangle_R.$$
- $\langle \chi_{iL}(t)\chi_{iR}(t)\rangle \propto \left|\cosh\left(\frac{\pi t}{\beta}\right)\right|^{-2\Delta}$
- Chaos. Maximal Lyapunov exponent $\lambda = 2\pi T$ (Kitaev '15, Maldacena-Stanford '16, Maldacena-Shenker-Stanford '15)
- (Approximately) dual to Jackiw-Teitelboim gravity



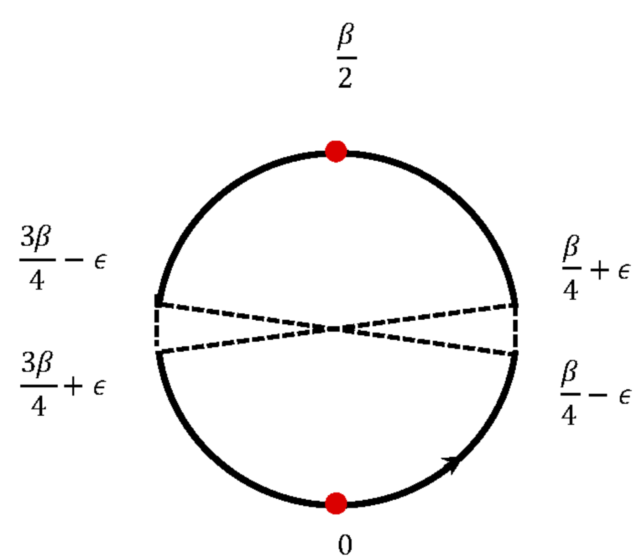


Operator growth in the SYK model

- Infinite temperature case (Roberts-Stanford-Streicher '18)
- For finite temperature, we study the size distribution $\langle TFD|e^{-\mu\hat{n}}|TFD\rangle$ and $\langle TFD|\psi_{iL}e^{-\mu\hat{n}}\psi_{iL}|TFD\rangle$
- $\langle TFD|e^{-\mu\hat{n}}|TFD\rangle \equiv Z_\mu$ is a partition function of SYK with twisted boundary condition

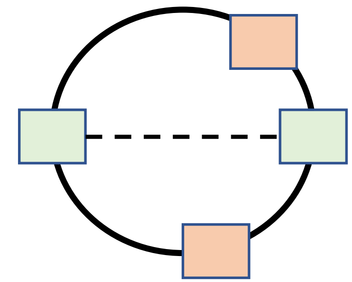


$$\psi\left(\frac{\beta}{4}+\epsilon\right)=\cosh\mu\,\psi\left(\frac{\beta}{4}-\epsilon\right)-\sinh\mu\,\psi\left(\frac{3\beta}{4}-\epsilon\right)$$



Operator growth in the SYK model

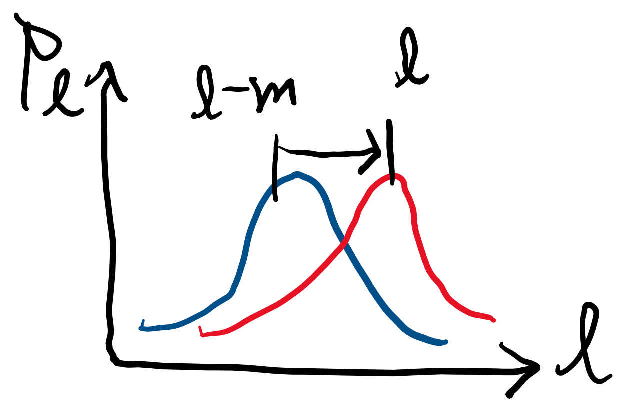
- Two-point function with μ term can be obtained by Schwinger-Dyson equation



- $G = \left[\partial_\tau - \Sigma_\mu\right]^{-1}$
  $\Sigma_\mu(\tau_1, \tau_2) = J^2 \mathcal{G}_\mu^{q-1}(\tau_1, \tau_2).$
- Note that

$$\mathcal{G}_\mu(\tau_1, \tau_2) = \frac{\langle TFD | \psi_{iL}(\tau_1) e^{-\mu \hat{n}} \psi_{iL}(\tau_2) | TFD \rangle}{\langle TFD | e^{-\mu \hat{n}} | TFD \rangle}$$

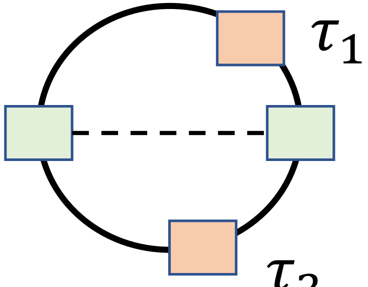
- This is a ratio of two generating functions. Expansion of $\mathcal{G}_\mu = \sum_m e^{-\mu m} K_m$ Coefficient K_m satisfies
- $p_l \left[\psi_i(t) \rho^{\frac{1}{2}}\right] = \sum_m K_m \, p_{l-m} \left[\rho^{\frac{1}{2}}\right].$
- K_m is the "transition probability" that the fermion operator increases size of $\rho^{\frac{1}{2}}$ by m



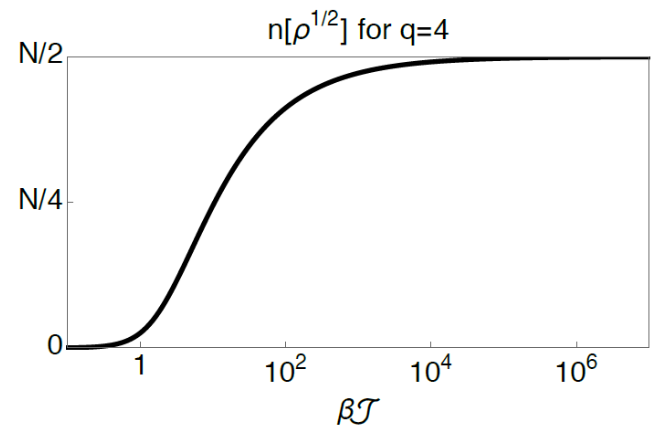
Large-q solution

- In large q, the Schwinger-Dyson equation simplifies $\left[\partial_\tau - \frac{J^2}{q}\mathrm{G}^{\mathrm{q}-1}\right] * \mathrm{G} = \mathbb{I}$,
- $G = G_0 e^{\frac{g}{q}}$, ➔ Liouville equation $\partial_{\tau_1}\partial_{\tau_2} g = -2J^2 e^g$
- Define $\mu = \frac{\hat{\mu}}{q}$, $\hat{\mu}$ changes the boundary condition of g

$$\mathcal{G}_\mu(\tau_1, \tau_2) = \frac{e^{-\hat{\mu}/q} G_\mu(\tau_1 - \tau_2)}{\left(1 - \frac{(1-e^{-\hat{\mu}})}{\sin^2\gamma_\mu}(G_\mu(\tau_1-\tau_2))^{q/2} \sin\left(\alpha_\mu\left(\tau_1 - \frac{\beta}{4}\right)\right)\sin\left(\alpha_\mu\left(\tau_2 - \frac{\beta}{4}\right)\right)\right)^{2/q}}$$



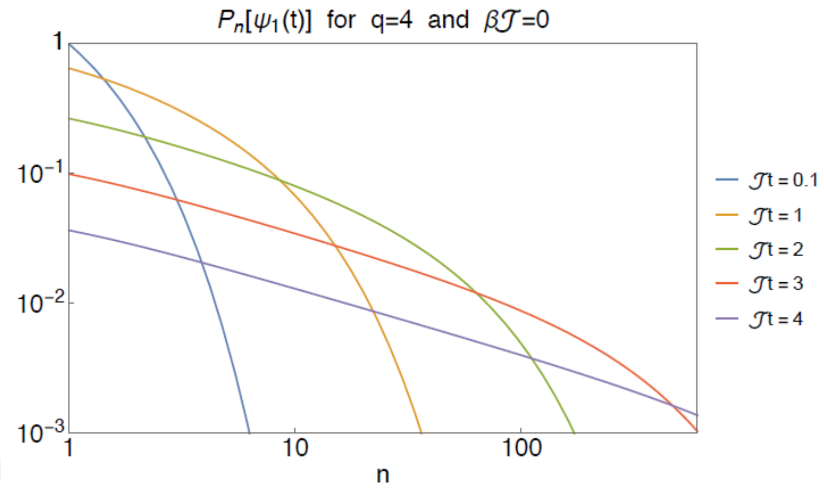
Large-q solution



- Size of thermal state
- $n\left[\rho^{\frac{1}{2}}\right] = \frac{N}{2}(1-\delta_\beta),$
- $\delta_\beta = \left(\frac{\alpha}{\mathcal{J}}\right)^{\frac{2}{q}}$
- At low temperature $\alpha \simeq \frac{\pi}{\beta}$.
- Fermion size $\delta n_\beta[\psi_1(t)] = 1 + 2\left(\frac{\mathcal{J}}{\alpha}\sinh \alpha t\right)^2$
- Lyapunov exponent 2α
- Consistency at $t \to 0$: conjecture $\delta n_\beta[\psi_1(t)] = \delta_\beta\left[1 + 2\left(\frac{\mathcal{J}}{\alpha}\sinh \alpha t\right)^2\right].$

Large-q solution

- Full size distribution can be obtained by expanding $\mathcal{G}_\mu$
- The entire distribution is the same as infinite temperature case except a renormalization of parameter $\sinh \mathcal{J}t \to \frac{\mathcal{J}}{\alpha}\sinh \alpha t$

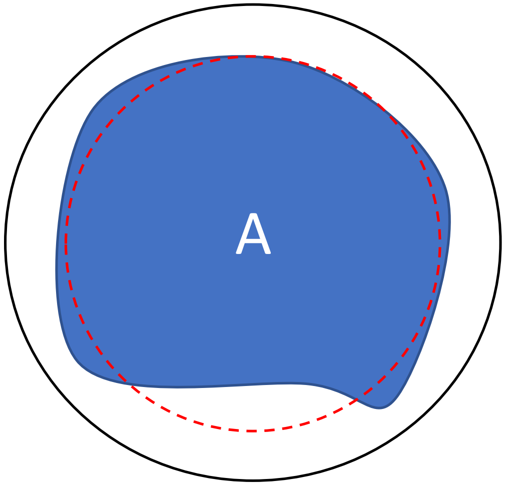


$$K^{\beta}_{\delta_\beta(1+qn)}\left[\psi_1(t)\right] = (-1)^n \binom{-2/q}{n} \frac{\left(\frac{\mathcal{J}}{\alpha}\sinh(\alpha t)\right)^{2n}}{\left(1+\left(\frac{\mathcal{J}}{\alpha}\sinh(\alpha t)\right)^2\right)^{n+\frac{2}{q}}}$$

$$\propto e^{-\frac{4}{q}\alpha t} \exp\left[-\left(\frac{2\alpha}{\mathcal{J}}\right)^2 e^{-2\alpha t} n\right]$$

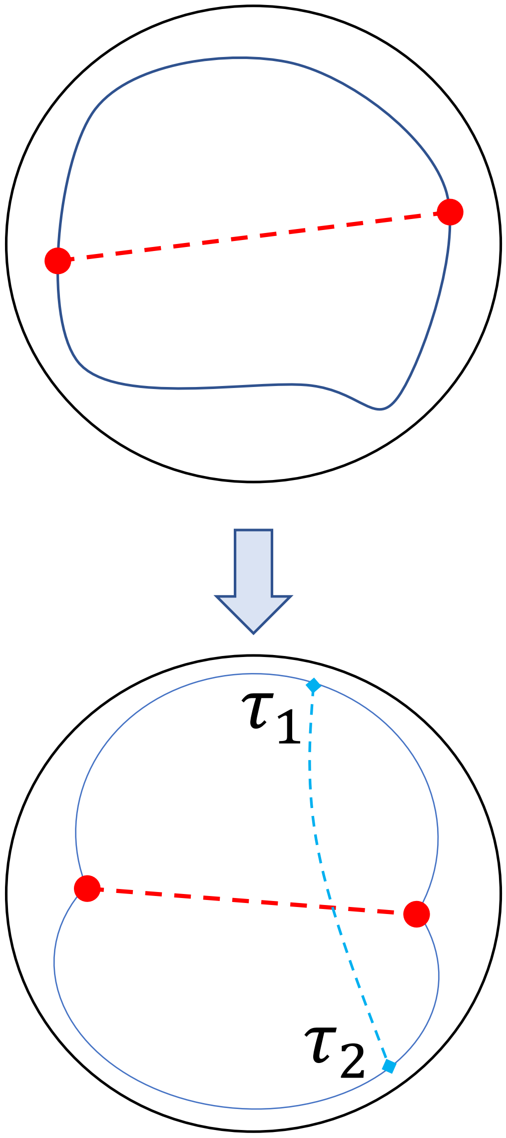
Low temperature solution

- At large βJ, the dynamics of the SYK model can be described by reparameterization modes $f(\tau)$, defined by
$$G_f(\tau_1,\tau_2) = |f'(\tau_1)f'(\tau_2)|^{\Delta} G_S\big(f(\tau_1), f(\tau_2)\big)$$
- Effective theory describes the breaking of reparameterization symmetry.
- Schwarzian action $S = \frac{N}{J}\alpha \int_0^{\beta} d\tau\, \mathrm{Sch}\left[\tan\left(\frac{\pi}{\beta}f(\tau)\right), \tau\right]$
- Preserves SL(2,R) gauge symmetry
- Bulk picture $S = N\alpha(L - A)$. $L = \beta J$, maximize A.



Low temperature solution

- $e^{-\mu\hat{n}}$ term adds an interaction between the time $\tau = 0$ and $\tau = \frac{\beta}{2}$.
- $\delta S \propto N \left\langle \psi\left(\frac{\beta}{2}\right)\psi(0)\right\rangle_f = N \cosh D^{\Delta}$
- Solution: two arcs (similar to Gu, Lucas XLQ '17)
- Analytic continuation $\tau_1 \to \epsilon + it_1$, $\tau_2 \to -\epsilon + it_2$ to determine $\mathcal{G}_\mu(t_1, t_2)$.



(Yingfei Gu, Yuri Lensky, XLQ, Pengfei Zhang, in progress)

Low temperature solution

- For $t_1 = t_2$,

$$g(\mu) = b_\Delta \left(\frac{\alpha \sin\left(\frac{\alpha}{2}\right)}{2\pi \cos\left(\alpha\left(\frac{1}{2} - \frac{\epsilon}{\pi}\right)\right) - 2\pi \cos\left(\frac{\alpha}{2}\right) \cosh\left(\frac{\alpha t}{\pi}\right)} \right)^{2\Delta}$$

$(\alpha \simeq \pi + c\mu)$

- Schwarzian

$$g(\mu) = b_\Delta \left(\frac{1}{\alpha' \mu \cosh t + 2\epsilon} \right)^{2\Delta}.$$

- Large q

$$g_q(\mu) = \frac{1}{2} \frac{e^{-\mu}}{\left(1 + (1 - e^{-q\mu}) \left(\frac{J}{\alpha_\mu}\right)^2 \sinh^2(\alpha_\mu t)\right)^{2\Delta}}.$$

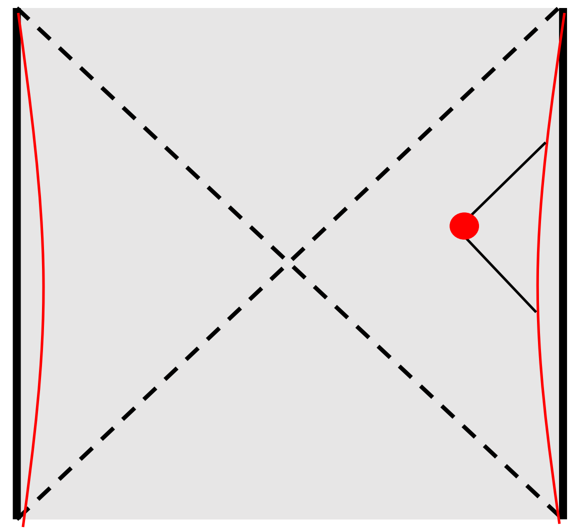
- Agreement for long time requires to take the UV cutoff $\epsilon = \frac{\pi}{\beta J}$, and small $\hat{\mu} = q\mu \ll 1$

Part III: Holographic dual theory

The dual theory: bulk operator size

- The SYK is approximately dual to Jackiw-Teitelboim gravity coupled with matter fields.
- Gravitational dynamics $\Longleftrightarrow$ Repara. Modes
- Matter field $\Longleftrightarrow$ Conformal fields
- In particular, fermion

$$\chi_{i\alpha}(\rho,t) \Longleftrightarrow \psi_i(t)$$



- Bulk fermion mass

$$m=\Delta-\frac{1}{2}=\frac{1}{q}-\frac{1}{2}.$$

- HKLL construction

$$\chi_{i\alpha}(\rho,t)=\int_{t_-}^{t_+}dt' K_\alpha(\rho,t-t')\psi_i(t').$$

- Size of $\chi_{i\alpha}(\rho,t)$ can be computed.

Yingfei Gu, Yuri Lensky, XLQ, Pengfei Zhang, in progress

The dual theory: bulk operator size

- Generating function
- $\mathcal{G}_\mu^B(\rho,t) \equiv \frac{\langle TFD|\chi_{i\alpha}(\rho,t)e^{-\mu\hat{n}}\chi_{i\alpha}(\rho,t)|TFD\rangle}{\langle TFD|e^{-\mu\hat{n}}|TFD\rangle}$

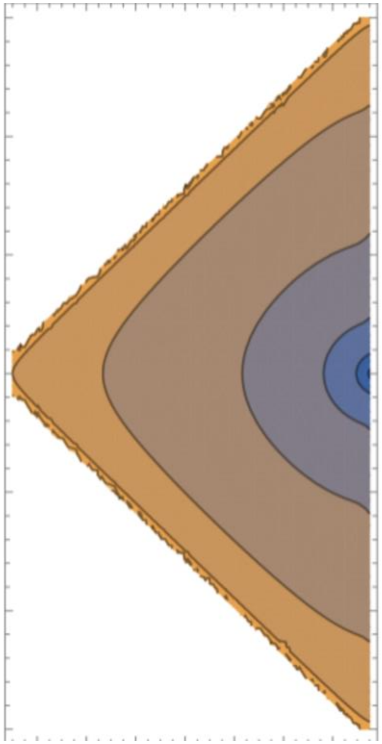
$$= \int dt_1 dt_2 K_\alpha(\rho,t-t_1)K_\alpha(\rho,t-t_2) \cdot \frac{\langle TFD|\psi_i(t_1)e^{-\mu\hat{n}}\psi_i(t_2)|TFD\rangle}{\langle TFD|e^{-\mu\hat{n}}|TFD\rangle}$$

$$= \int dt_1 dt_2 K_\alpha(\rho,t-t_1)K_\alpha(\rho,t-t_2)\mathcal{G}_\mu(t_1,t_2)$$

- Simplification: contribution is dominated by the light cone $t = t_\pm$
- $\mathcal{G}_\mu^B(\rho,t) = \frac{1}{2}\Big(\mathcal{G}_\mu(t_+,t_+) + \mathcal{G}_\mu(t_-,t_-)\Big) \pm Re\left(\mathcal{G}_\mu(t_+,t_-)\right)$

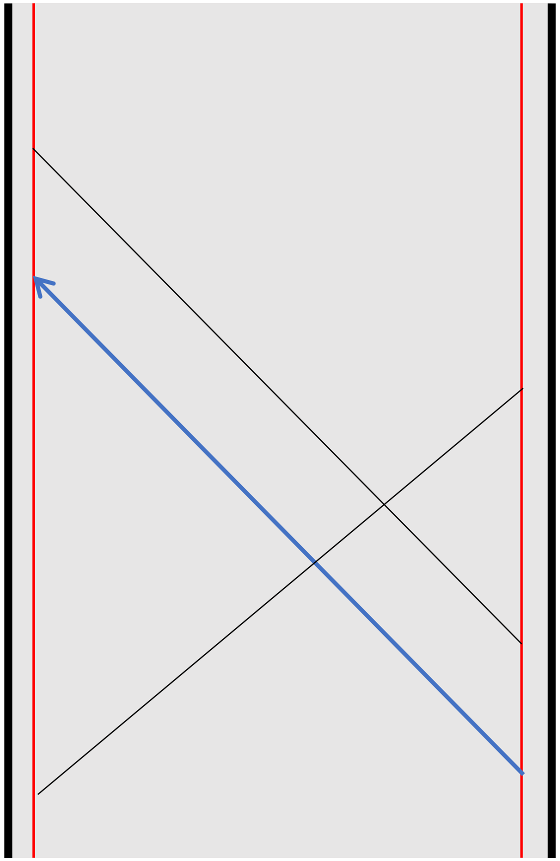
The dual theory: bulk operator size

- Bulk fermion size is almost determined by the size of boundary fermions at $t_\pm$.
- Approximately,
- $\langle n \rangle = \frac{(\beta J)^2}{2\pi^2}(\cosh t \coth \rho - 1) + \frac{\beta J}{2\pi}(\cosh \rho - \sinh \rho \cosh t) + O\big((\beta J)^0\big).$
- Size grows exponentially along radial and temporal direction with proper distance.
- Operator size diverges near the horizon
- The calculation only applies to $|t_\pm| < t^*$ (before scrambling time)



Conclusion and further discussion

- Operator size growth characterizes chaos (at early time)
- The same general results apply to qubit systems.
- SYK model operator size growth can be computed for low temperature or large q
- Size growth of bulk operator provides an interpretation of emergent bulk direction ρ
- Generalization to global AdS2?
- Does the size of bulk operator depend on the representation?
- Relation to other works, such as
  A. Brown et al 1804.04156



Thanks!