The dual of non-extremal area: differential entropy in higher dimensions

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Differential entropy

Proposed in the context of AdS_3/CFT_2:
Balasubramanian, Chowdhury, Czech, de Boer and Heller '13

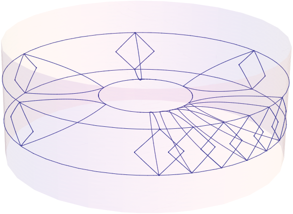
Headrick, Myers and Wien '14

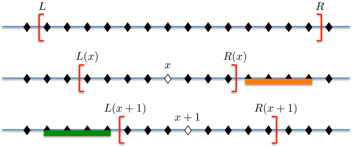
$$S_{diff}\left[R(x_0)\right] = \int dx_0\, \partial_R S\big(R(x_0), x_0\big)$$

Information theoretic quantity with multiple interpretations:

Observers making time limited measurements
Areas of non-minimal bulk surfaces
Cost of a constrained state merging protocol

Czech, Hayden, Lashkari and Swingle '14 (Figure credit)





Motivation for understanding the area of non-extremal surfaces:

bulk reconstruction, boundary rigidity

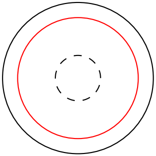
Ning Bao, ChunJun Cao, Sebastian Fischetti, Cynthia Keeler '19

area laws and RG-flows

Freedman, Gubser, Pilch and Warner '99, Myers and A. Sinha '10, Engelhardt and Fischetti '18

There is a long history of considering the area of bulk surfaces as probes of RG-flows:

$$ds^2 = f(r)dr^2 + r^2\left(-dt^2 + d\vec{x}^2\right)$$



Constructing a c-function requires a boundary quantity which computes this area

In this work we **propose an extention of differential entropy to higher dimensions**.

We provide an explicit construction of a well-defined quantity:

- correct divergence structure
- correct transformation properties
- no symmetry assumptions

How does a bulk area transform?

Consider a Riemannian manifold M and a co-dim 1 surface N:

$$Area(N) = \int_N d^d\sigma \sqrt{det\big(g_{\mu\nu}\partial_\alpha x^\mu(\sigma)\partial_\beta x^\nu(\sigma)\big)} = \int_N \iota_n \epsilon$$

Track this extra dependence: space of unit vectors on M

$$\mathbb{S}M = \{(x, V) \in TM | g_{ab}V^a V^b = 1\}$$

A co-dim 1 surface $N \subset M$ can be lifted to a section of $\mathbb{S}M$,

$$N \to \tilde{N}$$

$$Area(N) = \int_{\tilde{N}} \eta$$

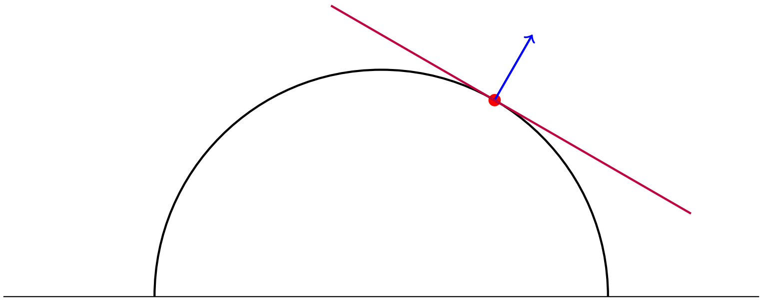
$$\eta = \iota_V \epsilon$$

Particle physicist's version:

Area has an index $d\Sigma^\mu$.

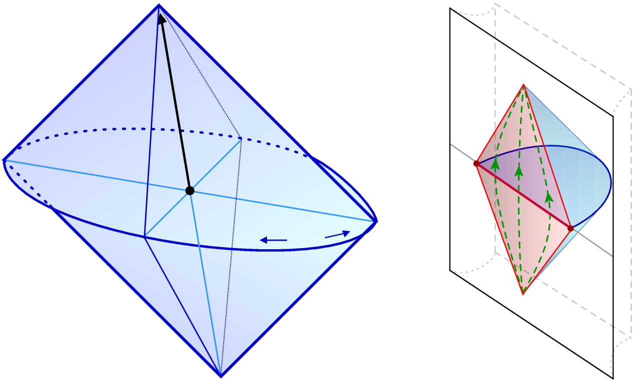
Natural boundary interpretation of $\mathbb{S}M$

Boundary anchored extremal surface with a particular point on the extremal surface picked out chooses a point in $\mathbb{S}M$.



There is a nice subset of regions that is sufficient:

- Observers making time limited measurements
- Equivalent to a subregion of a timeslice that is the boost of a ball shaped region



The set of such observers is known as Kinematic Space, $\mathcal{K}$.

Czech, Lamprou, McCandlish, Mosk and Sully '16

de Boer, Haehl, Heller and Myers '16

Definition: E, the bundle of points on extremal surfaces.

Base: Kinematic space – the space of time-limited observers

Fibre: the extremal surface attached to this diamond extending into the bulk

$$\pi(E) = \mathcal{K} \qquad \pi^{-1}(k \in \mathcal{K}) = D_{d-1}$$

The normal to the extremal surface defines an embedding

$$E \to \mathbb{S}M$$

For empty AdS$_{d+1}$, this is an isomorphism.

A point on a surface can be picked out by giving a ray in Kinematic Space:

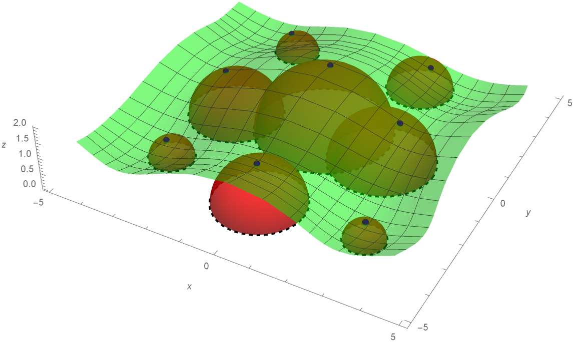
$$\mathbb{P}\mathcal{K} = T\mathcal{K}/\sim \qquad (k, V) \sim (k, \lambda V)$$

This plays nicely with the natural causal structure on Kinematic space from inclusions.

Embedding:

$$\mathbb{P}\mathcal{K} \to E \to \mathbb{S}M$$

The envelope of a surface $J \subset \mathcal{K}$ gives a surface $N \subset M$. Its lift to a section $\tilde{J} \subset \mathbb{P}\mathcal{K}$ commutes with the embeddings to give the lift $\tilde{N}$.



Moral:

Reconstructing area $\simeq$ pullback of the area form

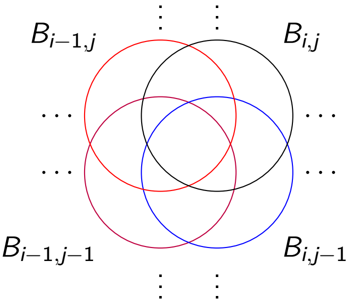
Differential Entropy

Now that we understand how to ask the question, we need to find the answer.

Principle that will guide us:

> The area of a bulk surface only has divergences where it approaches the asymptotic boundary.
>
> This can only occur at the boundaries of $N \subset M$ – which are also the boundaries of $J \subset \mathcal{K}$.
>
> Our proposal must be free of divergences (up to a potential boundary term).

Differential Entropy

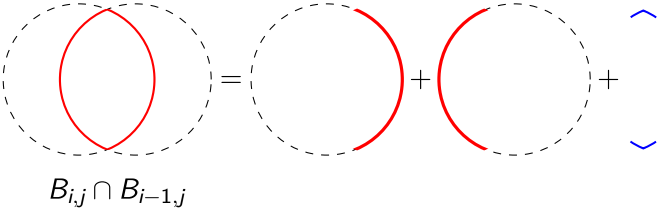


Discretised proposal for $2+1$ dimensional boundary:

$$S_{diff}[\{B_{i,j}\}] = \sum_{i,j=1}^{N} \Big[S(B_{i,j}) - S(B_{i,j} \cap B_{i-1,j}) - S(B_{i,j} \cap B_{i,j-1}) + S(B_{i,j} \cap B_{i,j-1} \cap B_{i-1,j} \cap B_{i-1,j-1}) \Big]$$

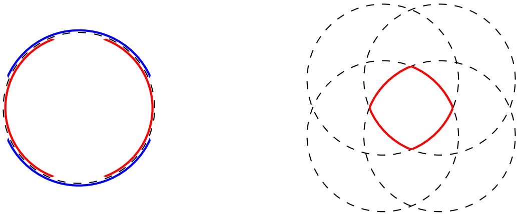
Intersections of balls

$$S_{diff}[\{B_{i,j}\}] = \sum_{i,j=1}^{N} \Big[S(B_{i,j}) - S(B_{i,j} \cap B_{i-1,j}) - S(B_{i,j} \cap B_{i,j-1}) \\ + S(B_{i,j} \cap B_{i,j-1} \cap B_{i-1,j} \cap B_{i-1,j-1}) \Big]$$

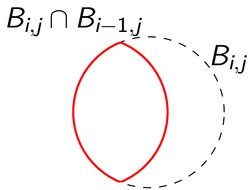


Divergences

$$S_{diff}[\{B_{i,j}\}] = \sum_{i,j=1}^{N} \Big[S(B_{i,j}) - S(B_{i,j} \cap B_{i-1,j}) - S(B_{i,j} \cap B_{i,j-1}) + S(B_{i,j} \cap B_{i,j-1} \cap B_{i-1,j} \cap B_{i-1,j-1}) \Big]$$



Continuum limit



$$S(B_{i,j} \cap B_{i-1,j}) = S(B_{i,j}) + \delta^{(1)} S\big[\delta B^{\leftarrow}\big] + \dots$$

S_{diff} has the form of a second shape derivative:

$$S_{diff}\Big[\{B_{i,j}\}\Big] = 2 \sum_{i,j} \delta^{(2)} S_{\mathrm{reg}}\Big[\rho^{\leftarrow}, \rho^{\downarrow}\Big] + O(a^3)\,,$$

Continuum limit

$$S_{diff}\Big[\{B(\sigma)\}\Big] = \int d\mu\, \delta^{(2)} S\Big[\rho_{\leftarrow}, \rho_{\downarrow}\Big]\,,$$

This limit is a bit subtle since $d\mu$ depends on how you take the continuum limit.

There is a choice such that

S_{diff} transforms correctly

S_{diff} computes the area of surfaces in empty AdS_4

Ingredients:

Natural causal structure on $\mathcal{K}$

No symmetry assumptions on the state

Constrained state merging protocol

Key step was to write the differential entropy in terms of conditional entropies

$$S_{diff} = \sum_{i,j} \Big[S(B_{i,j} - B_{i-1,j} | B_{i,j} \cap B_{i-1,j}) \\ - S(B_{i,j} \cap B_{i,j-1} - B_{i-1,j} \cap B_{i-1,j-1} | B_{i,j} \cap B_{i,j-1} \cap B_{i-1,j} \cap B_{i-1,j-1}) \Big]$$

Protocol: Construct ρ of whole system by only acting on spins in one ball at a time.
Understanding monotonicity of the cost of the protocol as the constraints are relaxed gives c-function.

Czech, Hayden, Lashkari and Swingle '14

Discussion

Higher dimensional and Lorentzian generalisations are discussed in our paper

Recent boundary rigidity results give tools for a general proof of the connection to area

$1/N$ corrections: How to define something that makes sense at finite N?

- Algebraic approaches: subsets of operator algebras
- Maybe the state merging protocol can give us new ideas?

Monotonicity of this quantity:

- From field theory
- From the cost of merging

We've introduced a new quantity that is interesting from a number of different points of view and which deserves further study.

Identified a structure for understanding bulk non-minimal areas from the boundary

Proposed an explicit formula that has correct structure:

- Divergence structure
- Transformation properties
- Works for arbitrary surfaces in AdS

Leads to a proposal for new boundary c-functions

Possible information theoretic interpretation in terms of the cost of a constrained state merging protocol.