Dual of the bulk symplectic form and the volume of maximal slices in AdS/CFT

Gábor Sárosi

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with **Alex Belin** and **Aitor Lewkowycz**

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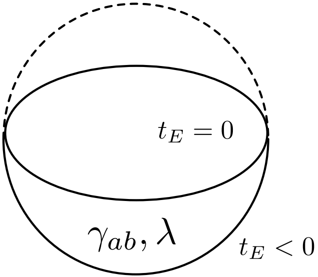
Motivation

Euclidean path integral states in AdS/CFT?

What are bulk coherent states?
Candidate: Euclidean path integral states

$$|\lambda\rangle = Pe^{-\int_{t_E<0}\lambda(t_E,x)O(t_E,x)}|0\rangle$$

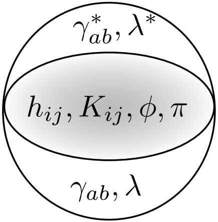


Overlaps like $\langle\lambda_1|\lambda_2\rangle$ should encode some properties of **bulk Cauchy slices**. What are these properties and how they are encoded in the overlap?

Motivation

Euclidean path integral states in AdS/CFT?

$\langle\lambda|\lambda\rangle =$



- Corresponding classical bulk: Euclidean saddle in the gravity calculation of $\langle\lambda|\lambda\rangle$ [Skenderis,van Rees]. Also leads to Lorentzian initial data [Marolf,Parrikar,Rabideau,Rad,Raamsdonk]
- Around the vacuum, $|\lambda\rangle$ states are perturbative coherent states [Botta-Cantcheff,Martinez,Silva]
- **Need complexified sources!** Otherwise all momenta vanish due to Z_2 symmetry of the Euclidean configuration. The complex λ is morally similar to $\alpha = q + ip$ of an oscillator.

Quantum Kähler structure

Coherent states have natural local geometry induced by their overlaps!

Parametrize states as $(\lambda, \lambda^*) \mapsto |\lambda\rangle$ such that the conjugation comes from the inner product on $\mathcal{H}$:

$$\partial_{\lambda^*}|\lambda\rangle = 0, \quad \partial_\lambda\langle\lambda| = 0.$$

- Pull-back of **Fubini-Study metric**: $ds^2 = \partial_\lambda\partial_{\lambda^*}\log\langle\lambda|\lambda\rangle d\lambda d\lambda^*$
- **Berry curvature** of states $|\Psi_{\lambda,\lambda^*}\rangle = \langle\lambda|\lambda\rangle^{-1/2}|\lambda\rangle$ parametrized by (λ, λ^*):

$$i\langle d\Psi_{\lambda,\lambda^*}| \wedge |d\Psi_{\lambda,\lambda^*}\rangle = i\partial_\lambda\partial_{\lambda^*}\log\langle\lambda|\lambda\rangle d\lambda \wedge d\lambda^*$$

$\Rightarrow$ Locally **Kähler geometry** with Kähler potential $\mathcal{K} = \log\langle\lambda|\lambda\rangle$
Running "backwards" the quantization procedure

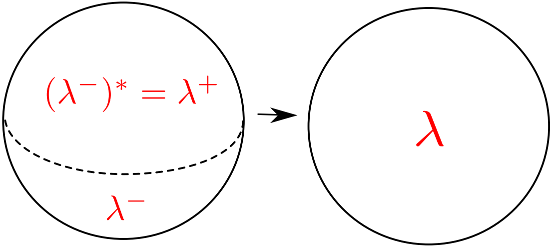
Quantum Kähler structure in QFT

Apply this to **path integral states in a QFT**

$$|\lambda\rangle = e^{-\int_{t_E<0} dx \lambda^-(t_E,x) O(t_E,x)}|0\rangle, \quad \langle\lambda| = \langle 0|e^{-\int_{t_E>0} dx \lambda^+(-t_E,x) O^\dagger(t_E,x)},$$

Kähler potential:

$$\mathcal{K} = \log\langle\lambda|\lambda\rangle = \log Z[\lambda], \quad \lambda(x) = \begin{cases} \lambda^-(t_E) & t_E < 0 \\ \lambda^+(-t_E) \equiv [\lambda^-(-t_E)]^* & t_E > 0, \end{cases}$$



Quantum Kähler structure in QFT

The metric and the Kähler form are obtained by second variations of the Kähler potential $\log Z[\lambda^+, \lambda^-]$ $\Rightarrow$ described by the **connected two point functions** in the state:

$$G^c_\lambda(x,y) = \langle O^\dagger(x) O(y)\rangle_\lambda - \langle O^\dagger(x)\rangle_\lambda \langle O(y)\rangle_\lambda .$$

The Kähler form can be rewritten as a **symplectic pairing**:

$$\Omega_\lambda(\delta\lambda_1, \delta\lambda_2) = i \int_{t_E>0} dx \Big(\delta\lambda_1^+ \delta_2^- \langle O\rangle - \delta\lambda_2^+ \delta_1^- \langle O\rangle \Big).$$

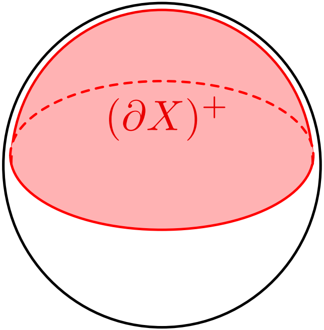
Sources and vevs are canonically conjugate.

Quantum Kähler structure for holographic CFTs

In **AdS/CFT** we can equate the boundary Kähler form with **Wald's symplectic flux** through **half of the Euclidean boundary**

$$i\int_{(\partial X)^+} dx \Big(\delta\lambda_1^+\delta_2^-\langle O\rangle - \delta\lambda_2^+\delta_1^-\langle O\rangle\Big) = i\int_{(\partial X)^+}\omega_{\mathrm{bulk}}(\phi,\delta_1\phi,\delta_2\phi).$$

with $\phi|_{\partial X} = \lambda$. (This follows directly from GPKW)

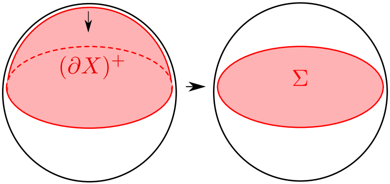


Quantum Kähler structure for holographic CFTs

We can push the flux into the bulk, using that the flux ω_{bulk} is on-shell conserved.

Therefore,

$$i\int_{(\partial X)^+} dx \Big(\delta\lambda_1^+ \delta_2^- \langle O \rangle - \delta\lambda_2^+ \delta_1^- \langle O \rangle \Big) = i\int_\Sigma (\delta_1 \pi \delta_2 \varphi - \delta_2 \pi \delta_1 \varphi).$$



On a slice Σ that continues "nicely" to a Lorentzian initial data surface, $\delta\lambda^+ = (\delta\lambda^-)^*$ is equivalent with

- $\delta\varphi|_\Sigma$ real
- $\delta\pi^{\text{Lor}}|_\Sigma = i\delta\pi|_\Sigma$ real

Complex structure and Kähler metric

Complex structure comes from the boundary inner product, acts on sources as

$$J: \quad \delta\lambda^-(x) \mapsto i\delta\lambda^-(x), \quad \delta\lambda^+(x) \mapsto -i\delta\lambda^+(x)$$

Or for the complete boundary condition: $J[\delta\lambda(x)] = i\text{sign}(t_E)\delta\lambda(x)$

Boundary complex structure $\Leftrightarrow$ Bulk quantum-polarization (separation into positive and negative energy modes)

Kähler (or FS) metric $\Leftrightarrow$ Klein-Gordon product of positive energy part

Application 1: Volume of the maximal slice

Symplectic form of Einstein gravity:

$$\Omega_{\text{bulk}}(\delta_1, \delta_2) = \int_\Sigma (\delta_1 \pi^{ab} \delta_2 h_{ab} - \delta_2 \pi^{ab} \delta_1 h_{ab}),$$

where $\pi^{ab} = \sqrt{h}(K^{ab} - h^{ab} K)$. Put in one of the slots:

$$\delta_Y \pi^{ab} = \frac{1}{2}\sqrt{h} h^{ab}, \quad \delta_Y h_{ab} = 0 \quad \Rightarrow \Omega_{\text{bulk}}(\delta_Y, \delta_2) = \delta_2 V$$

- δ_Y solves the constraints of GR only on the maximal slice
- It has an interpretation as moving initial data inside the causal diamond (WdW patch). Not a diffeo in general!

Central question: What's the interpretation of the deformation of the CFT background metric $\delta_Y \gamma_{ab}$, that gives rise to this?

$\Rightarrow$ see Aitor's talk tomorrow for some progress towards answering this.

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Complexity=volume?

There is a possible connection to the complexity=volume conjecture.

We have seen that the pull-back of the **Fubini-Study** metric to the $|\lambda\rangle$ states is

$$G_{ab} = (\partial_{\lambda_a^+}\partial_{\lambda_b^-} + \partial_{\lambda_b^+}\partial_{\lambda_a^-}) \log Z[\lambda^+, \lambda^-], \quad a = (x, i)$$

- What are the geodesics of G_{ab}?
- Do functionals on them (like length, energy) have nice holographic duals?

A functional that is additive under $Z = Z_1 Z_2$ has to be linear in G_{ab}:

$$\mathcal{C} = \frac{1}{2}\int_{\lambda_i}^{\lambda_f} G_{ab}\dot{\lambda}^a\dot{\lambda}^b$$

It is a complexity-like quantity in the sense that it measures a **distance between two states, restricted to a path on which we can source only simple(=single trace) operators**. It is not gate counting though.

Complexity=volume?

The on-shell variation of $\mathcal{C} = \frac{1}{2}\int_{\lambda_i}^{\lambda_f} G_{ab}\dot{\lambda}^a\dot{\lambda}^b$ with respect to the endpoint is a boundary term:

$$\delta\mathcal{C} = G_{ab}\dot{\lambda}^a\delta\lambda_f^b \quad = \Omega_{ab}(J[\dot{\lambda}])^a\delta\lambda_f^b$$

In the second equality, we have used that the symplectic form and the metric are related by the complex structure $J[\lambda] = i\text{sign}(t_E)\lambda$.

We see that if

$$J[\dot{\lambda}] = \delta_Y\lambda \quad \text{then} \quad \delta\mathcal{C} = \delta V.$$

- This is hard to check in general, but notice that it is a conjecture **purely about classical gravity** and not field theory.
- We have checked that this holds to leading order in perturbation theory for Virasoro coherent state in a 2d CFT (Bañados geometries).
- Gives reasonable time dependence for the TFD in a (very-) mini superspace approximation, with early time quadratic and late time linear growth.

Application 2: Quantum information metric

Defined as: $|\langle\Psi(J)|\Psi(J+\delta J)\rangle| = 1 - G_{JJ}\delta J^2 + \cdots$, where J is some parameter. Also a second order change in the overlap, so we can access it.

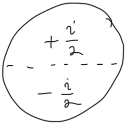
It was noticed by [Miyaji,Numasawa,Shiba,Takayanagi,Watanabe] that when J is a source for a marginal scalar deformation, G_{JJ} behaves like the volume of the maximal Cauchy slice.

For states with vanishing one point functions, we can write

$$G_{JJ} = \Omega(\delta_c\lambda, \delta_s\lambda),$$

i.e. the symplectic pairing between

$$\delta_c\lambda = 1, \qquad \delta_s\lambda = \frac{i}{2}\text{sign}t_E$$



Application 2: Quantum information metric

Equating bulk and boundary symplectic forms gives

$$G_{JJ} = -\frac{1}{2}\int_{\Sigma} \delta_s \pi,$$

where we used that $\delta_c \lambda = 1$ implies $\delta_c \phi = 1$ everywhere in the bulk for a massless field.

- Integral of sign deformed momenta over *any* Cauchy slice
- For vacuum AdS, $\delta_s \pi$ is constant on the Z_2 symmetric slice
- For the time evolved thermofield double, $\delta_s \pi$ is constant over the final maximal slice
- In general, $\delta_s \pi$ is *not* constant on the maximal slice
  $\Rightarrow$ **It is not the volume but a nice toy model for it.**

Application 3: Relative entropy

Consider a state deformed by *modular flow*

$$|\lambda, n\rangle = e^{-n(K_0 - K_\lambda)}|\lambda\rangle, \quad n = is,$$

where $K_0 = -\log \rho_0$ and $K_\lambda = -\log \rho_\lambda$ are modular Hamiltonians over some subregion.

Think of n as another complex source. It can in principle be turned on by sourcing a complex conical deficit in the background metric (replica trick). Evaluate the boundary symplectic form with a shift in *modular time* $\delta_s = i\Delta s \partial_n$

$$i[\delta^+_\lambda \delta_s - \delta^-_\lambda (\delta_s)^*] \log\langle\lambda, n|\lambda, n\rangle = \delta S_{\rm rel}(\rho_\lambda || \rho_0)$$

This is then equal to the bulk symplectic form on a Cauchy slice

$$\delta S(\rho_\lambda || \rho_0) = \int_\Sigma \omega_{\rm bulk}(\phi, \delta\phi, \partial_s \phi),$$

When $|\lambda\rangle$ is close to the vacuum one recovers Fisher information = canonical energy [Lashkari,Raamsdonk,15]

Summary

- Some integrals over bulk Cauchy slices, such as the symplectic flux and the Klein-Gordon inner product have natural and simple boundary expressions in terms of overlaps of nearby path integral states.
- This is useful for understanding the duals of some quantities in a new light, like relative entropy and quantum information metric.
- Steps towards identifying the precise dual of the volume of the maximal Cauchy slice.

Questions?