Complexity, Integrablity, & Chaos

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- arXiv: 1905.05765 VB, MATTHEW DE CROSS, ARJUN KAR, ONKAR PARRIKAR
- arXiv: 2101.0209 VB, MATTHEW DECROSS, ARJUN KAR, CATHY LI, ONKAR PARRIKAR

Complexity, Integrability, and Chaos

- In many branches of physics we talk about complex systems: many internal components, many-body interactions, multi-party correlations, emergent behaviors

e.g.

Strongly Interacting Materials | Gene Interaction Networks | Neural Networks | AdS/CFT & Emergent Spacetime | Black Hole Microstates & the Information Paradox

Complexity, black holes & AdS/CFT

1. complex microstates are hard to discriminate
2. Chaotic dynamics scramble information
3. Hawking radiation stores information in a highly complex (pseudorandom?) code
4. Interior volume growth may have something to do with dynamical complexity growth of the state, at least in holographic (AdS/CFT) contexts

4 + 2 leads to a conjecture:
Maximally chaotic processes such as black hole formation have linearly increasing complexity for an exponential amount of time.

- In fact in cases in string theory where we can construct microstates, they are clearly very complex.
- Here is an old argument (VB, de Boer, Jejjala, Simon, 2005)

[Microstates of black holes of mass M in AdS_5 universes with curvature scale l] $\Longleftrightarrow$ [States of $SU(N)$, $d=4$ super-Yang-Mills (SYM) theory with dimension $\Delta = Ml \sim N^2$]

$$|\text{microstate}\rangle \Leftrightarrow \mathcal{O}|0\rangle \quad \dim(\mathcal{O}) = N^2$$

- What do such operators look like?
- They are gauge-invariant polynomials in the fields $\{ \underset{\text{gauge field}}{A_\mu}, \underbrace{X, Y, Z}_{\text{adjoint complex scalars}}, \underset{\text{fermions}}{\psi} \}$

complex object made from many simple pieces

e.g., $\mathcal{O} = \text{Tr}[XXY\bar{X}\,ZZ\bar{X}\cdots]$ — $O(N^2)$ fields

Sprinkle traces, derivatives, other fields

$\boxed{\text{Almost all such long strings are random polynomials in the fields.}}$

$\Rightarrow$ Very difficult to tell apart using simple operations (e.g. statistically universal correlators of low dimension operators.)

$\hookrightarrow$ Apparent information loss in semiclassical black holes.

- Back to our conjecture:

  Maximally chaotic processes such as black hole formation have linearly increasing complexity for an exponential amount of time.

- In this statement, what do we want to mean by "complexity"?

- Intuitions from Computer Science – "complexity" counts the size of the algorithm needed to assemble the object starting with simple initial objects and elementary procedures acting on them.

EXAMPLE 1: KOLMOGOROV COMPLEXITY

The complexity of a sequence = length of the shortest program (let's say on a Turing Machine) required to produce it.

$\Longrightarrow$ Random sequences are complicated while π and $\sqrt{7}$ are simple

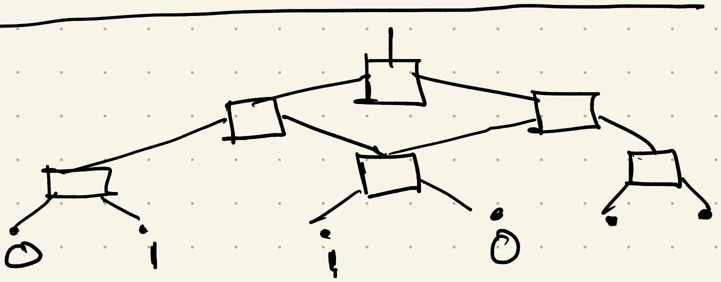
(Remember our black hole microstate operators!)

EXAMPLE 2: STOCHASTIC COMPLEXITY (Rissanen)

The complexity of a message is the size of the smallest code for compressing it.

(This is straight out of information theory but is conceptually related to Kolmogorov: $\text{CODE} \rightarrow \boxed{\text{Decoder}} \rightarrow$ ORIGINAL SEQUENCE)

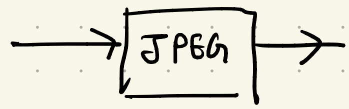
EXAMPLE 3: CIRCUIT COMPLEXITY



Given N input bits,
$C(N)$ = size (# of gates/width/depth) of the minimal circuit to compute the answer

An ambiguity

The size of the algorithm depends on the choice of computing architecture – e.g. the particular Turing machine design, the details of the decoding algorithm, or the circuit gate set



Avoiding the ambiguity

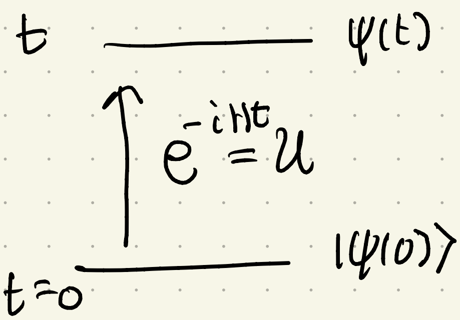
Ask how complexity of a problem class scales with input size N.

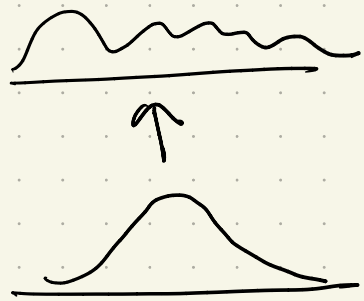
<u>Physical Systems</u>: Locality in spacetime suggests a more natural canonical definition of the architecture.

$\Rightarrow$ The elementary operators we use should be local in spacetime. This means they are defined at a point and have a bounded number of derivatives.

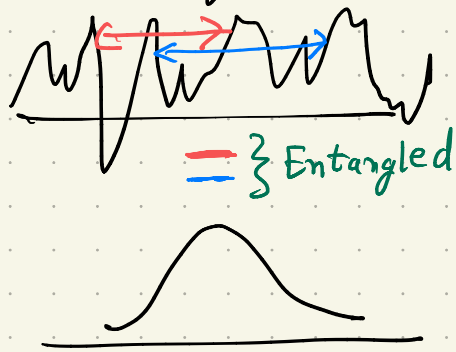
- If you have a discrete qubit system, define k-local = acting on no more than k bits.
- Let SIMPLE vs COMPLEX be defined by the degree of locality of the Hamiltonian, taken to be SIMPLE, since this generates the natural time evolution.

- Let's treat quantum time evolution as a computation.





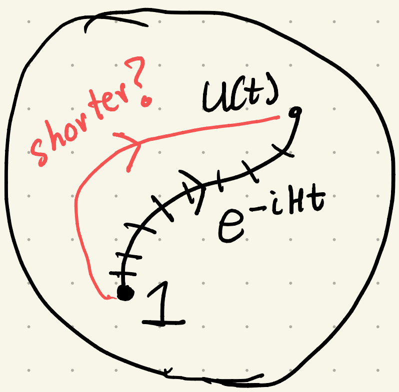
Seems like a simple evolution
~ some local changes in $|\psi\rangle$



Seems like a complex evolution
~ large local and global changes, substantial nonlocal correlation & entanglement

- How can we construct a measure of the complexity of this time evolution, and hopefully relate to other properties of physical interest?
- That is, we want to define $C(U(t)) = C(e^{-iHt})$

Let $\mathcal{H}$ = Hilbert space of dim N $\qquad SU(N)$ = unitary operations on $\mathcal{H}$

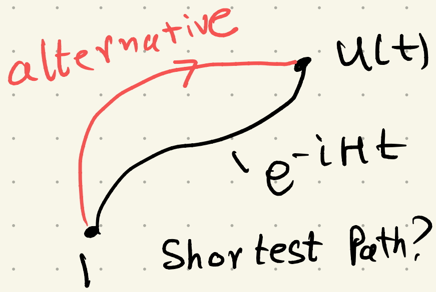


$SU(N)$

$N = 2^n$ $\quad n = \#$ of qubits

- How complicated is $U(t)$?
- The # of steps required ~ length of the curve on $SU(N)$ in some appropriate metric
- The COMPLEXITY should the length of the shortest geodesic on $SU(N)$ between 1 and $U(t)$
- This is a difficult problem, because $SU(N)$ is a complicated fiber bundle.

What metric on $SU(N)$ is useful for quantifying complexity?



- What metric should we use to measure lengths of paths?
- Infinitesimal operation

$$\mathcal{O} = \sum_i V^i T^i \qquad T_i = \text{generators}$$

└ directions on tangent space

<u>Natural guess:</u>
Metric should be "small" in the simple (easy / local) directions and "large" in the (hard / nonlocal) directions

↳ This definition gives geodesic lengths that are (polynomially) equivalent to circuit complexity (for the time evolution treated as a quantum circuit) if you take (Nielsen et. al.)

Local direction metric $\sim \mathcal{O}(1)$

Non-Local direction metric $\sim \mathcal{O}(e^S)$

$S = \ln \dim \mathcal{H} = \ln 2^n \sim n$
$\Rightarrow e^S \sim N$

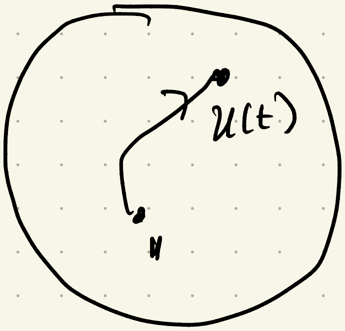
- We will adopt this definition and use it with SYK models with N fermions as examples.
- So if the tangent space of $SU(N)$ is split as $\{T_\alpha\}$ (easy) and $\{T_{\dot{\alpha}}\}$ hard, we take the complexity metric for $SU(N)$:

$$G_{ij} = \begin{pmatrix} \delta_{\alpha\beta} & \\ & (1+\mu)\,\delta_{\dot{\alpha}\dot{\beta}} \end{pmatrix} \qquad \mu \sim e^{\varepsilon S} \text{ for some } \varepsilon$$

(more generally, we can take, for any group:

$$G_{ij} = \frac{c_i + c_j}{2} K_{ij}, \quad K_{ij} = \text{Cartan-Killing form}$$

$$c_i = \begin{cases} O(1) & \text{for easy } i \\ e^{\varepsilon S} & \text{for hard } i \end{cases}$$
)



We want to find geodesics on SU(N) between $\mathbb{1}$ & e^{-iHt} in the metric $G_{ij} = \begin{bmatrix} \delta_{\alpha\beta} & \\ & (1+\mu)\,\delta_{\dot\alpha\dot\beta} \end{bmatrix}$

$$g(\underbrace{X, Y}_{\text{Tangents @ } U}) = G(XU^{-1}, YU^{-1})$$

Geodesic equation on SU(N) in Euler-Arnold form

$$G_{ij}\frac{dV^j}{ds} = f_{ij}^{\;\;k}\, V^j G_{kl} V^l$$

V^i = tangents to path
s = path parameter
$f_{ij}^{\;\;k}$ = structure consts. of group

Solve $\Downarrow$

$$U(s) = P\exp\left(\int_0^s ds'\, V(s')\right)$$

↳ Path ordered exponential

Complexity $\quad C(U(t)) = \min \int_0^1 ds \sqrt{G_{ij}V^i(s)V^j(s)}$

↳ over geodesics

This is a practical formalism for calculations!

Example: SU(2) $\quad T_1 = \gamma_1$; $T_2 = \gamma_2$; $T_3 = \gamma_1\gamma_2$ $\quad \gamma$ = GAMMA MATRIX

SYK with 2 fermions $\quad \Rightarrow G_{ij} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1+\mu \end{bmatrix} \quad \theta = \sum V^i T_i$

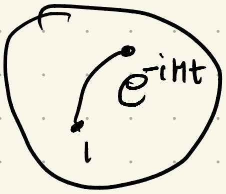
$$\Rightarrow \quad \frac{dV^1}{ds} = -2\mu V^2 V^3 \qquad \frac{dV^2}{ds} = 2\mu V^3 V^1 \qquad (1+\mu)\frac{dV^3}{ds} = 0$$

SOLVE:

$$V^1(s) = v^1\cos(V^3\mu s) - v^2\sin(V^3\mu s)$$
$$V^2(s) = v^2\cos(V^3\mu s) + v^1\sin(V^3\mu s)$$
$$V^3(s) = \frac{V^3}{2}$$

C = complexity = length $= \sqrt{(v^1)^2 + (v^2)^2 + (1+\mu)(V^3)^2}$

We are interested in the complexity of time evolution. So we want geodesics from $\mathcal{U}(0)=1$ to $\mathcal{U}(t)=e^{-iHt}$.



SU(N)

- We want a basis for SU(N). We can take a representation based on the gamma matrices, for example.

$$\gamma_a \quad a\in\{0,\dots N-1\} \quad \{\gamma_a,\gamma_b\}=2\delta_{ab}$$

and let $T_{a_1\cdots a_m}=\gamma_{a_1}\cdots\gamma_{a_m}$ $\quad a_p<a_q$ if $p<q$

$$\equiv T_i \qquad i\equiv(a_1\cdots a_m)$$

- In the SYK model (which we will discuss later) $\gamma_a\sim\psi_a$ (elementary fermion)

$\Longrightarrow$ k-local $\Longleftrightarrow$ no more than k γ_s in $T_i\equiv T_\alpha$

k-non local $\Longleftrightarrow$ more than k γ_s in $T_i\equiv T_{\tilde\alpha}$

- We want to solve the Euler-Arnold equation

$$G_{ij}\frac{dV^j}{ds}=f_{ij}^{\ \ k}V^j G_{kl}V^l \qquad G_{ij}=\begin{bmatrix}\delta_{\alpha\beta} & \\ & (1+\mu)\delta_{\tilde\alpha\tilde\beta}\end{bmatrix}$$

$$\mathcal{U}(s)=P\exp\left(\int_0^s ds'\,V(s')\right) \qquad \text{B.C. } \mathcal{U}(0)=1\,;\ \mathcal{U}(t)=e^{-iHt}$$

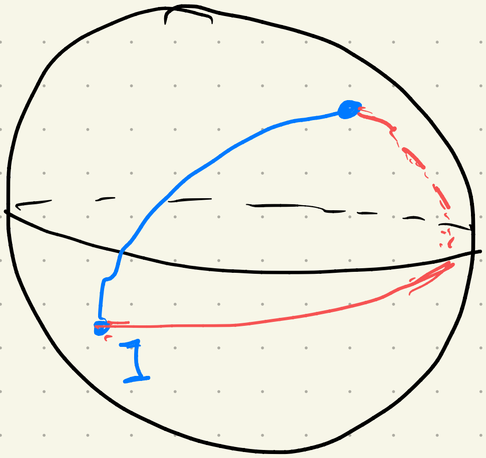
- Suppose the Hamiltonian is local: $H=\sum_\alpha J^\alpha T^\alpha$
- Then you can show that there is ALWAYS a geodesic along the easy directions $V^\alpha(s)=v^\alpha$; $V^{\tilde\alpha}(s)=0$ with $v^\alpha=J^\alpha t$ $\Longrightarrow$ $V=Ht$

$\Longrightarrow$ Time evolution trajectory is geodesic.

IF this is the shortest geodesic $\boxed{C=t\sqrt{\sum_\alpha (J^\alpha)^2}}$ LINEAR GROWTH OF COMPLEXITY

How the linear complexity growth can end.

Geodesic loops



These are hard to study in general, but we will make progress in free and integrable theories

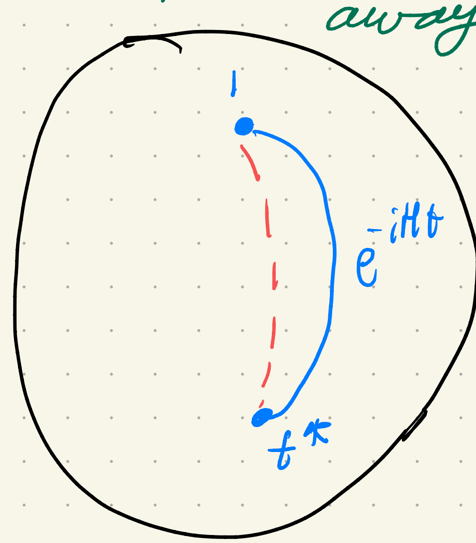
Conjugate Points

Given a geodesic
$U[s] : [0,1] \to U(\mathcal{H})$
with $U(0) = P$ and $U(1) = Q$,
a perturbation produces a new curve that satisfies the geodesic equation with the same b.c.s to first order.

$\Longrightarrow$ the geodesic we started with is a saddlepoint, not a minimum of the length. So, we can find a shorter path

Relevance of conjugate Points

minimizing geodesic can be a finite distance away



- MISLEADING INTUITION: Most sectional curvatures of Lie groups are negative (Milnor), so we can take as a toy model a high genus Riemann surface with a metric induced from its covering space, the hyperbolic disk $\mathbb{H}^2$. There are no conjugate points in this toy model, so they are irrelevant.
- This is a bad idea because
  (a) The main result of Milnor is that any right-invariant metric on $SU(n)$ with $n>2$ MUST have *some* positive sectional curvature, or else it is flat
  (b) Conjugate points are some of the earliest obstructions in free and integrable theories.

How to find conjugate points truncating complexity growth

- Start with the "linear" geodesic along the time evolution trajectory $V(s) = Ht$, which means that the path of unitaries is $\mathcal{U}(s) = e^{-istH}$.

- We want to perturb the trajectory

$$V(s) \longrightarrow V(s) + \delta V(s)$$

to first order. and still solve the Euler Arnold equation

$$G_{ij} \frac{dV^j}{ds} = f_{ij}^{\ \ k} V^j G_{kl} V^l \qquad G_{ij} = \begin{bmatrix} \delta_{\alpha\beta} & \\ & (1+\mu)\delta_{\bar{\alpha}\bar{\beta}} \end{bmatrix}$$

with the same boundary condition.

- In first-order perturbation theory this gives the Jacobi equation. Let $\delta V = \delta V_{Local} + \delta V_{Non\,Local}$

$$\Longrightarrow \quad i \frac{d\delta V_L}{ds} = \mu t \, [H, \delta V_{NL}]_L$$

Projections to the easy/hard subspaces

This is the Raychaudhuri Equation in 1st order formalism

$$i \frac{d\delta V_{NL}}{ds} = \frac{\mu t}{1+\mu} [H, \delta V_{NL}]_{NL}$$

- To locate conjugate points define the super-operator Υ_μ acting on the operator perturbation δV such that:

$$P\left(e^{-i\int_0^1 ds\,(Ht + \delta V(s))}\right) = e^{-itH}\left(1 - i\,\Upsilon_\mu(\delta V(0)) + O(\delta V^2)\right)$$

- Υ_μ maps the perturbation at the identity $\delta V(0)$ to the deviation in the endpoint.
- Expanding the LHS in a Dyson series, we get: $\Upsilon_\mu(\delta V(0)) = \int_0^1 ds\, e^{itsH}\, \delta V(s)\, e^{-istH}$

- Finally, using the Jacobi to solve for the perturbed trajectory, we get:

Projection to local subspace

$$Y_\mu(\delta V(0)) = \int_0^1 ds\, e^{iHts}\Big[\delta V_L(0) - i\mu t \sum_{\dot\alpha} \frac{e^{\frac{-i\mu t \lambda_{\dot\alpha} s}{1+\mu}} - 1}{\left(\frac{-i\mu t \lambda_{\dot\alpha}}{1+\mu}\right)} \delta\tilde{V}^{\dot\alpha}(0)\,[H, \tilde{T}_{\dot\alpha}]_L + \sum_{\dot\alpha} e^{\left(\frac{-i\mu t \lambda_{\dot\alpha} s}{1+\mu}\right)} \delta\tilde{V}^{\dot\alpha}(0)\,\tilde{T}_{\dot\alpha}\Big] e^{-iHts}$$

where: $\{\tilde{T}_{\dot\alpha}\}$ is a basis for the generators of the nonlocal subspace which diagonalizes the adjoint action of H: $[H, \tilde{T}_{\dot\alpha}] = \lambda_{\dot\alpha}\tilde{T}_{\dot\alpha}$

Conjugate points occur when $Y_\mu(\delta V(0)) = 0$ (i.e., no endpoint deviation.

So: A CONJUGATE POINT $\Longleftrightarrow$ Y_μ HAS A ZERO MODE EXISTS

This looks a bit complicated, but it is simpler than it looks, and we can use it to prove several general results.

CLAIM 1

(i) If the Hamiltonian has an adjoint eigenoperator $ad_H(\mathcal{O}) = [H, \mathcal{O}] = \lambda \mathcal{O}$, with $\lambda \in \mathbb{R}$ and $\mathcal{O} \in \{\text{local operators}\}$ then a conjugate point occurs at $t_* = \frac{2\pi}{\lambda}\mathbb{Z}$

(ii) If $\exists\ \mathcal{O}' \in \{\text{nonlocal operators}\}$ such that $[H, \mathcal{O}'] = \lambda' \mathcal{O}'$ with $\lambda' \in \mathbb{R}$ then a conjugate point occurs as $t_* = \frac{2\pi(1+\mu)}{\lambda'}\mathbb{Z}$

PROOF

Evaluate the super-operator Y_μ on such adjoint eigenoperators of H

ONLY THIS CONTRIBUTES IF $\mathcal{O}$ IS PURELY LOCAL

$$Y_\mu(\delta V(0)) = \int_0^1 ds\, e^{iHts}\Big[\delta V_L(0) - i\mu t \sum_{\dot\alpha} \frac{e^{\frac{-i\mu t + \lambda_{\dot\alpha} s}{1+\mu}} - 1}{\left(\frac{-i\mu t + \lambda_{\dot\alpha}}{1+\mu}\right)}\, \delta\tilde V^{\dot\alpha}(0)\,[H, \tilde T_{\dot\alpha}]_L$$

$$+ \sum_{\dot\alpha} e^{\left(\frac{-i\mu t + \lambda_{\dot\alpha} s}{1+\mu}\right)} \delta\tilde V^{\dot\alpha}(0)\, \tilde T_{\dot\alpha}\Big] e^{-iHts}$$

ONLY THIS CONTRIBUTES IF $\mathcal{O}$ IS PURELY NONLOCAL

Local Projections

NonLocal Projections

CLAIM 2

Let $M_{\alpha\beta}(t) = \int_0^1 ds \int_0^1 ds' \, \mathrm{Tr}\left[e^{i(s-s')tH} T_\alpha e^{-i(s-s')tH} T_\beta\right]$ where T_α and T_β are any two simple (local) operators. If $M_{\alpha\beta}(t)$ has a zero mode at time t^*, then a conjugate point occurs at t^*.

$M_{\alpha\beta} \sim$ temporally smeared, infinite temperature 2-point function of local operators.

PROOF: Let $\delta V(0) = \sum_\alpha X^\alpha T^\alpha$ where X^α is the zero mode of $M_{\alpha\beta}(t)$

The Frobenius norm of Y_μ is

$$\| Y_\mu(\delta V(0)) \|_F^2 = \mathrm{Tr}\left[(Y_\mu(\delta V(0)))^\dagger \; Y_\mu(\delta V(0)) \right]$$

By explicit computation, using that $\delta V(0)$ is entirely in the easy directions

$$= \sum_{\alpha\beta} (X^\alpha)^* M_{\alpha\beta}(t) X^\beta$$

Then $\sum_\beta M_{\alpha\beta} X^\beta = 0 \Rightarrow \| Y_\mu \|_F = 0$

$\Rightarrow Y_\mu(\delta V(0)) = 0$

$\Rightarrow$ there is a conjugate point

CLAIM 3 (EIGENSTATE COMPLEXITY HYPOTHESIS)

Let $|m\rangle, |n\rangle$ be any energy eigenstates and let T_α = local generators and $T_{\dot\alpha}$ = nonlocal generators. Then linear growth in complexity persists for a time of $\mathcal{O}(e^{\epsilon S})$ if:

$$R_{mn} = \frac{\sum_\alpha |\langle m|T_\alpha|n\rangle|^2}{\sum_\alpha |\langle m|T_\alpha|n\rangle|^2 + \sum_{\dot\alpha} |\langle m|T_{\dot\alpha}|n\rangle|^2}$$

$$= e^{-2S}\,\mathrm{poly}(S)\, r_{mn}$$

Says that energy eigenstates cannot be mapped onto each other by easy operations

PROOF Make the assumption and calculate the zero modes of Y_μ

$$\Longrightarrow \boxed{t^* = \frac{2\pi}{\Delta_{max}}\left(1+e^{\epsilon S}\right)}$$

Δ_{max} = largest gap

Applications

① Free theory.

SYK with $H = i \sum_{ij} J_{ij} \psi^i \psi^j$

$C(t) \lesssim \sqrt{N}$ and we can find an explicit way of fast-forwarding time evolution.
We improve a bound of Aharonov & Atiq by a constant factor

② INTEGRABLE THEORY

$H = H_0 + \varepsilon H_1$

$C(t) \leq \text{poly}(N)$

③ CHAOTIC SYK

$C(t) \leq e^N$

Using the above results for conjugate points
+
direct computation of geodesic loops