Information propagation in long-range interacting systems

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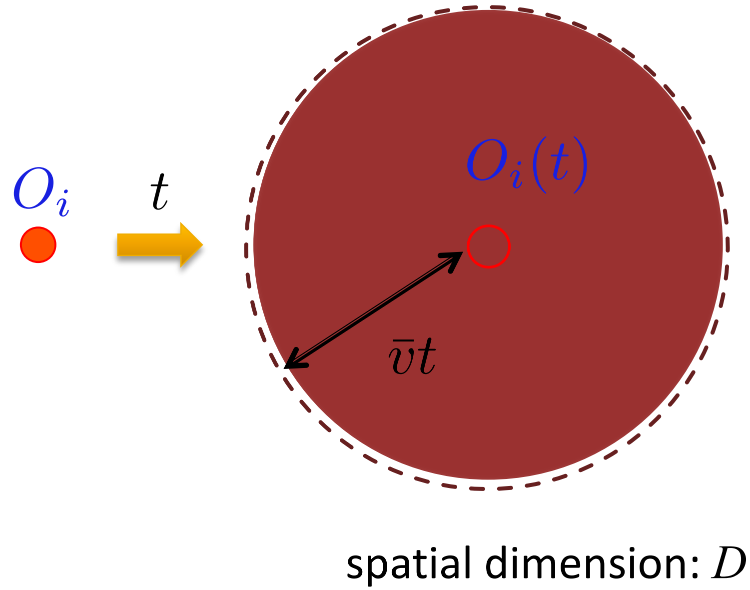
YITP Workshop, March 1st

Reference

T. Kuwahara and K. Saito, PRX 10, 031010 (2020), Featured in Physics
T. Kuwahara and K. Saito, PRL 126, 030604 (2021).

Contents

- Long-range interaction: $1/r^\alpha$



- Motivation
  - ➡ Lieb-Robinson bound
  - ➡ Linear-light-cone problem
- Results
  - ➡ For $\alpha > 2D+1$, the effective light cone is linear
  - ➡ The tight Lieb-Robinson bound is obtained

$$\|[O_i(t), O_j]\| \lesssim t^{2D+1}(d_{i,j} - \bar{v}t)^{-\alpha}$$

Set up

- Long-range Hamiltonian (system size : n , spatial dimension: D)

$$H = \sum_{i,j\in\Lambda} h_{i,j} + \sum_{i=1}^{n} h_i \quad \text{with} \quad \|h_{i,j}\| \le \frac{g_0}{d_{i,j}^{\alpha}}$$

$\alpha > 0, \quad g_0 = \mathcal{O}(1)$

$\|\cdots\|$: operator norm

$d_{i,j}$: distance between i and j

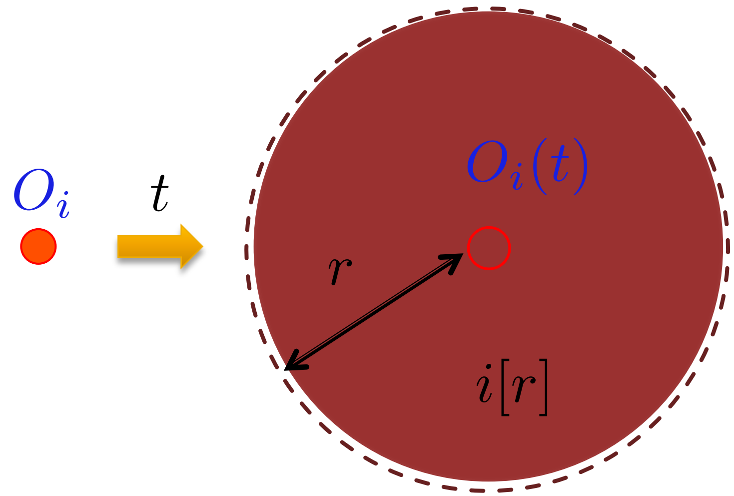
➡ Generalized to k-body interaction, time-dependent Hamiltonians

➡ α is experimentally controllable J. Zhang, et al., Nature **551**, 601 (2017).

- Time evolution: $O_i(t) = e^{iHt} O_i e^{-iHt}$

- Local approximation : $O_i(t, i[r])$

$$O_i(t, i[r]) := \frac{1}{\mathrm{tr}_{i[r]^c}(\hat{1})} \mathrm{tr}_{i[r]^c}\left[O_i(t)\right] \otimes \hat{1}_{i[r]^c},$$



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$\alpha > 0, \quad g_0 = \mathcal{O}(1)$

$\| \cdots \|$: operator norm

$d_{i,j}$: distance between i and j

Can we have

$$O_i(t) \overset{?}{\simeq} O_i(t, i[r])$$

Nature **551**, 601 (2017).

- Time evolution: $(t) = e^{iHt} O_i e^{-iHt}$

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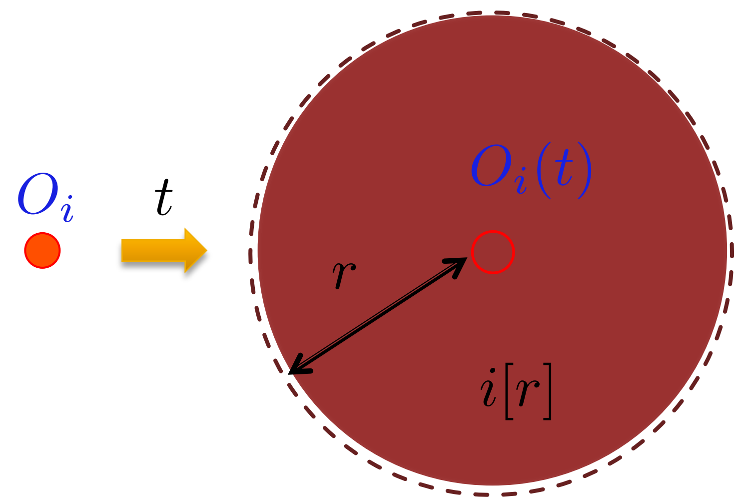


Lieb-Robinson bound

- Upper bound on $\|[O_i(t), O_j]\|$
- Short-range interacting systems

$$\|[O_i(t), O_j]\| \lesssim e^{-\mathrm{const.}(d_{i,j} - vt)}$$

Lieb and Robinson, Commun. Math. Phys. **28**, 251 (1972).

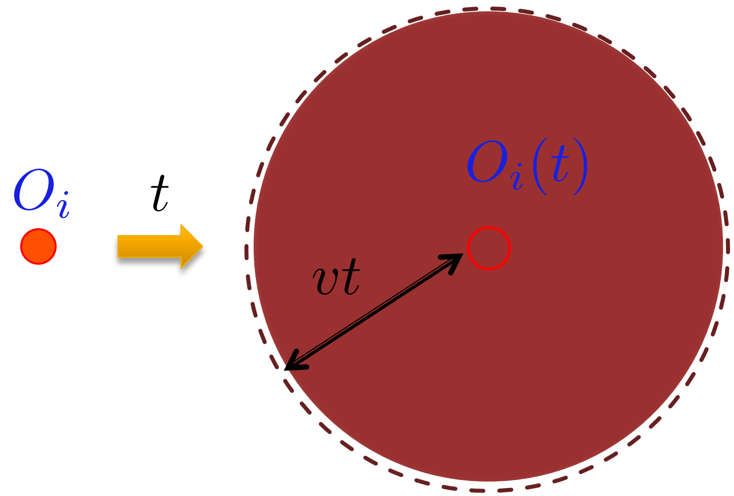
v: Lieb-Robinson velocity

S. Bravyi, et al., PRL **97**, 050401 (2006).

$$\|O_i(t) - O_i(t, i[r])\| \lesssim r^{D-1} e^{-\mathrm{const.}(r - vt)}$$

- Effective light cone: $R = vt$

$\Rightarrow O_i(t) \simeq O_i(t, i[r]) \quad \text{for} \quad r \gg R$



- Experimentally observable

M. Cheneau, et al., Nature 481, 484 (2012).
P. Richerme, et al., Nature 511, 198 (2014).

Linear-light cone problem in long-range interacting systems

- Short-range interaction

➡ Linear light cone: $R = vt$ $v = \text{finite}$

- Long-range interaction: $1/r^{\alpha}$

➡ Exponential light cone: $R = e^{vt}$

$$\|[O_i(t) - O_i(t, i[r])]\| \lesssim e^{vt} r^{-\alpha+D}$$

Hastings and Koma, Commun. Math. Phys. **265**, 781 (2006).

- Various numerical simulations observe linear-light cone even under long-range interactions

P. Hauke and L. Tagliacozzo, PRL **111**, 207202 (2013).

L. Cevolani, et al., New J. Phys. **18**, 093002 (2016).

Linear-light cone problem in long-range interacting systems

- Short-range interaction

➡ Linear light cone: $R = vt$ $v = \text{finite}$

- Long-range interaction: $1/r^{\alpha}$

[Linear light cone Problem]

"What is the critical exponent α_c above which the linear light cone is ensured in long-range interacting systems?"

1 (2006).

- Various numerical simulations observe linear-light cone even under long-range interactions

P. Hauke and L. Tagliacozzo, PRL **111**, 207202 (2013).
L. Cevolani, et al., New J. Phys. **18**, 093002 (2016).

Previous studies

- Nearly linear light cone

exponential light cone → polynomial light cone

$$R \propto e^{vt}$$

Hastings and Koma, CMP **265**, 781 (2006).

$$R \propto t^{\frac{\alpha - D + 1}{\alpha - 2D}} \to R \propto t \quad (\alpha \to \infty)$$

M. Foss-Feig, et al., PRL 114, 157201 (2015).

↓ improve M. C. Tran, et al., PRX **9**, 031006 (2019).

$$R \propto t^{\frac{\alpha - D}{\alpha - 2D}} \to R \propto t \quad (\alpha \to \infty)$$

- Improved Lieb-Robinson bound (1D, $\alpha > 3$)

Chen and Lucas, PRL 123, 250605 (2019).

$$\|[O_i(t), O_j]\| \lesssim \frac{t}{d_{i,j}} \to \|[O_i(t) - O_i(t, i[r])]\| \lesssim t \cdot r^{-1+D}$$

Useful up to $t = \mathcal{O}(d_{i,j})$

But, cannot achieve $O_i(t) \simeq O_i(t, i[vt])$

Our result

- Lieb-Robinson bound: for $\alpha > 2D+1$

$$\|[O_i(t), O_j]\| \le C_H |t|^{2D+1} (d_{i,j} - \bar{v}|t|)^{-\alpha}$$

$$\|[O_i(t) - O_i(t, i[r])]\| \le C'_H |t|^{D+1} (r - \bar{v}|t|)^{-\alpha+D}$$

D : spatial dimension

$C_H, C'_H, \bar{v}$: constants

➡ The bound gives $O_i(t) \simeq O_i(t, i[R]) \quad \text{for} \quad R \gtrsim \bar{v}t$

- Linear light cone is obtained for $\alpha > 2D+1$

➡ The bound is generalized to non-local operators $O_i \to O_X, \quad O_j \to O_Y$

➡ Cannot be extended to $\alpha \le 2D+1$

- For $\alpha \le D$, even the polynomial light cone can break down

Z. Eldredge, et al., PRL 119, 170503 (2017).

Our result

- Lieb-Robinson bound: for ***α* > 2*D*+1**

$$\|[O_i(t), O_j]\| \le C_H |t|^{2D+1} (d_{i,j} - \bar{v}|t|)^{-\alpha}$$

$$\|[O_i(t) - O_i(t, i[r])]\| \le C'_H |t|^{D+1} (r - \bar{v}|t|)^{-\alpha+D}$$

D : spatial dimension

$C_H, C'_H, \bar{v}$: constants

➡ The bound gives $O_i(t)$ … $(t, i[R])$ for $R \gtrsim \bar{v}t$

➡ … O_Y

➡

Can we improve the current Lieb-Robinson bound?

➡ NO! The linear light cone can break down for $\alpha < 2D + 1$

- For $\alpha \le D$, even the polynomial light cone can break down

Z. Eldredge, et al., PRL 119, 170503 (2017).

Optimality of the bound

- Quantum state transfer S. Bravyi, et al., PRL **97**, 050401 (2006).

➡ Initial state ρ_0 : product state of $|0\rangle$ with R qubits

➡ Unitary operation to qubit A, $u_0 = \hat{1}$ or $u_1 = |1\rangle\langle 0| + |0\rangle\langle 1|$

➡ Time evolution $\rho(t) = U_t^\dagger (u_s^\dagger \rho_0 u_s) U_t$

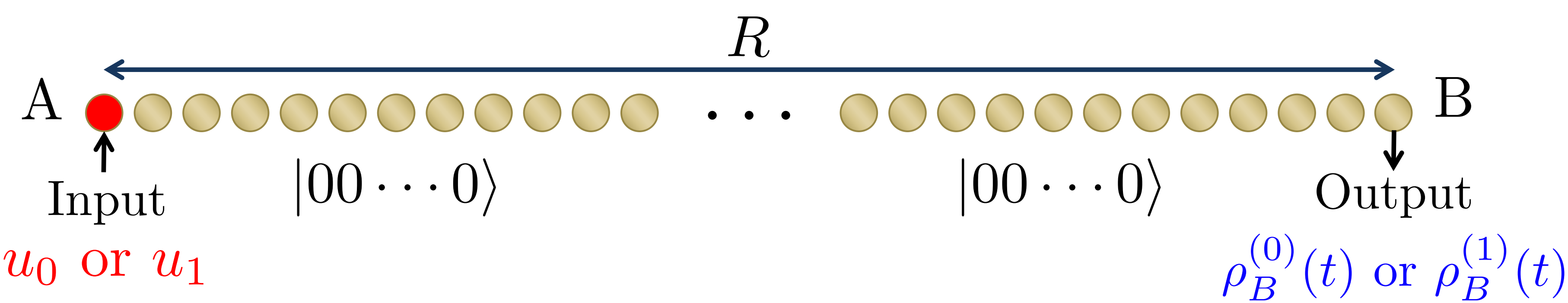
$$\rho_B^{(s)}(t) := \mathrm{tr}_{B^c}\left[U_t^\dagger (u_s^\dagger \rho_0 u_s) U_t\right]$$

$U_t = \mathcal{T} e^{-i\int_0^t H(\tau)d\tau}$

$H(\tau)$: interaction decay of $r^{-\alpha}$

$s = 0$ or 1

➡ Can we distinguish $\rho_B^{(0)}(t)$ and $\rho_B^{(1)}(t)$?



Optimality of the bound

- Lieb-Robinson bound gives the upper bound

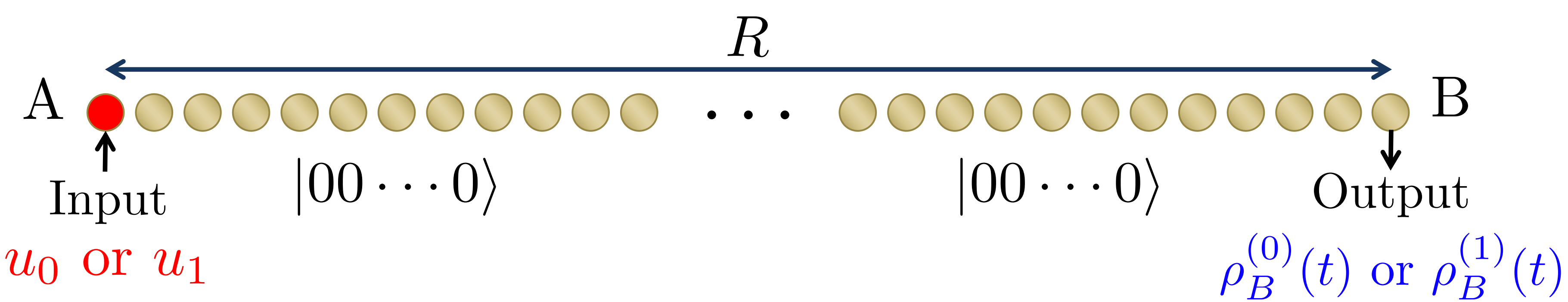
$$\|\rho_B^{(1)}(t) - \rho_B^{(0)}(t)\|_1 \le \sup_{O_B : \|O_B\|=1} \|[u_1, O_B(t)]\| \lesssim R^{-\alpha}|t|^{2D+1}$$

(using our theorem)

- Explicit quantum dynamics that achieves

$$\|\rho_B^{(1)}(t) - \rho_B^{(0)}(t)\|_1 \gtrsim R^{-\alpha}|t|^{2D+1}$$

➡ The dynamics consists of CNOT-type short-range interactions and long-range Ising interactions



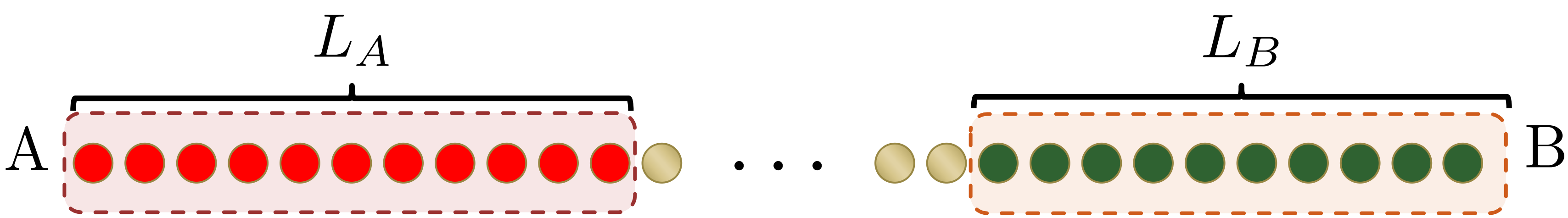
Optimality of the bound

- Step 1 (t/3) : copy the input state, prepare GHZ states

(Implementable by CNOT type short range interaction)

$|L_A|, |L_B| = \mathcal{O}(t)$

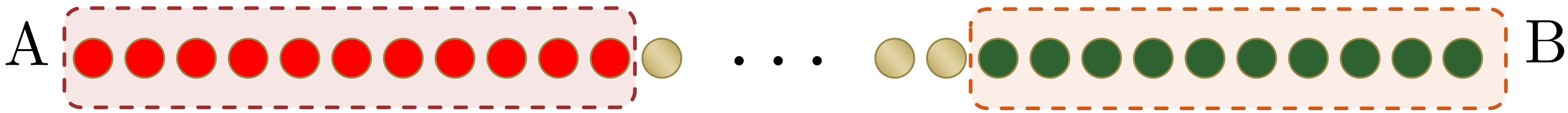
Z. Eldredge, et al., PRL 119, 170503 (2017)



$|00\cdots0\rangle$ or $|11\cdots1\rangle$

GHZ state: $\frac{|00\cdots0\rangle+|11\cdots1\rangle}{\sqrt{2}}$

- Step 2 (2t/3) : long-range Ising interaction $e^{-iH_{\mathrm{Ising}}t/3}$



$|00\cdots0\rangle$ or $|11\cdots1\rangle$

Rotated GHZ state: $\frac{|00\cdots0\rangle+e^{\pm i\theta}|11\cdots1\rangle}{\sqrt{2}}$

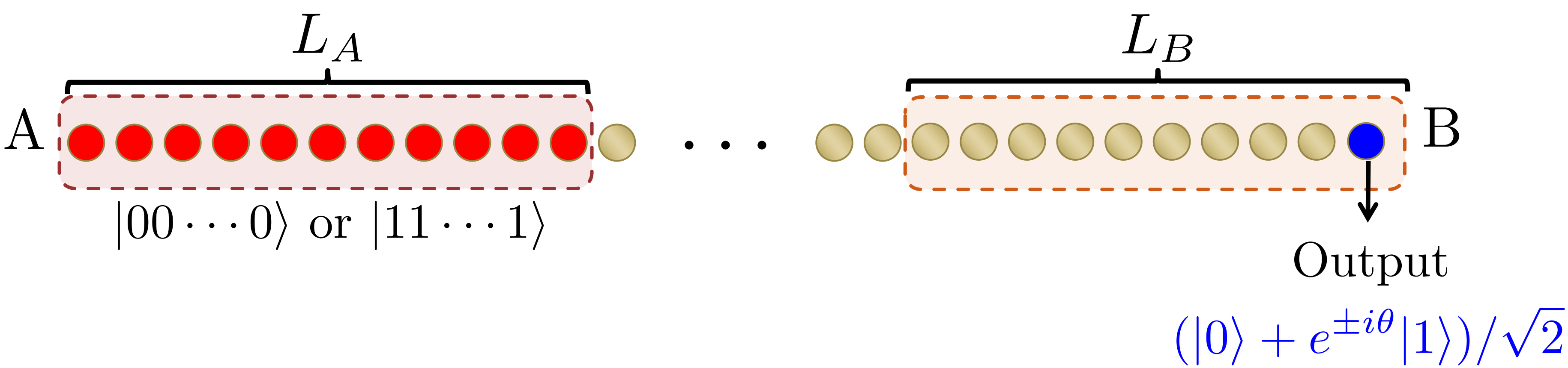
$$\boldsymbol{\theta \propto t \cdot |L_A| \cdot |L_B| / R^{\alpha}}$$

Optimality of the bound

- Step 3 (3t/3) : disentangle the rotated GHZ state

$|L_A|, |L_B| = \mathcal{O}(t)$

$\theta \propto t \cdot |L_A| \cdot |L_B| / R^\alpha$



$$\|\rho_B^{(1)}(t) - \rho_B^{(0)}(t)\|_1 = \sin(2\theta) \gtrsim t|L_A| \cdot |L_B|/R^\alpha \propto t^3/R^\alpha$$

➡ D-dimensional systems, $|L_A|, |L_B| = \mathcal{O}(t^D)$

➡ More explicit lower bound has been recently obtained.

M. C. Tran, et al., PRX 10, 031009 (2020).

Further discussion
Lieb-Robinson bound for α<2D+1

- Current results : Polynomial light cone ($\alpha > 2\mathrm{D}$) $R \propto t^{\frac{\alpha - D}{\alpha - 2D}}$

M. C. Tran, et al., PRX **9**, 031006 (2019).
Kuwahara and Saito, PRL **126**, 030604 (2021).

- What happens for $\alpha \leqq 2\mathrm{D}$?

➡ (General cases) there is a protocol to achieve sub-exponential light cone

$$R \propto \exp\left[\mathcal{O}(t^{\kappa_\alpha})\right] \ \ (\kappa_\alpha < 1)$$

M. C. Tran, et al., arXiv: 2010.02930v1

➡ (Special cases, OTOC) Polynomial light cone exists for ($\alpha > \mathrm{D}$)

$$R \propto t^{\frac{2\alpha - D + 1}{2\alpha - 2D}}$$

Kuwahara and Saito, PRL **126**, 030604 (2021)

$$\text{Out-of-time-order correlators (OTOCs)}: \frac{1}{\mathrm{tr}(\hat{1})}\mathrm{tr}([W_i(t), V_{i'}]^\dagger [W_i(t), V_{i'}])$$

Summary

- Long-range interaction: $1/r^{\alpha}$
- Linear light cone: what is the critical α_c ?
- Tight Lieb-Robinson bound

$$\|[O_i(t), O_j]\| \lesssim t^{2D+1}(d_{i,j} - \bar{v}t)^{-\alpha} \Rightarrow \boldsymbol{\alpha_c = 2D + 1}$$

$$O_i(t) \simeq O_i(t, i[vt])$$

$$(\alpha > \alpha_c)$$

- Open questions
  - Dose the linear light cone break down for $\alpha = \alpha_c$?
  - Further elaboration for the case of $\alpha < 2D$

Kuwahara and Saito, PRL **126**, 030604 (2021)
M. C. Tran, et al., arXiv: 2010.02930v1

Thank you for listening

Proof idea : connecting different length scale

- H_ℓ : interaction length up to ℓ

$$\ell_1 \to \ell_2 \to \cdots \to \ell_{q^*} \to \infty$$

$$\ell_1 = \mathcal{O}(1), \quad \ell_q = e^{e^{\mathcal{O}(q)}}$$

$$\ell_{q^*} = |t|^{\tilde{\eta}}, \quad \tilde{\eta} := 1 - \frac{\alpha - 2D - 1}{2(\alpha - D)}$$

$$H_{\ell_1} \to H_{\ell_2} \to \cdots \to H_{\ell_{q^*}} \to H_\infty$$

Short range | Middle range | Long range

$$e^{-iH_{\ell_q} t} = e^{-iH_{\ell_{q-1}} t} \mathcal{T} \exp\left[-i \int_0^t e^{iH_{\ell_{q-1}}\tau} (H_{\ell_q} - H_{\ell_{q-1}}) e^{-iH_{\ell_{q-1}}\tau} d\tau\right]$$

U_1 connect U_2

$$\|[U_1 U_2 O_i U_2^\dagger U_1^\dagger, O_j]\|$$

- The connection should be performed very carefully!!