Pseudo Entropy in Quantum Many-Body Systems and Holography

Kotaro Tamaoka (YITP)

Based on
2005.13801 (PRD) with **Yoshifumi Nakata**, **Tadashi Takayanagi**, **Yusuke Taki**, and **Zixia Wei**
2011.09648 (PRL) with **Ali Mollabashi**, **Noburo Shiba**, **Tadashi Takayanagi**, and **Zixia Wei**

+ work in progress

@ Recent progress in theoretical physics based on quantum information theory, YITP Kyoto, March 2021

This talk: a holography-inspired QI-quantity and its applications

Pseudo Entropy = Entanglement Entropy for "Transition Matrix"

$$\rho^{\psi} = |\psi\rangle\langle\psi| \longrightarrow \mathcal{T}^{\psi|\varphi} = \frac{|\psi\rangle\langle\varphi|}{\langle\varphi|\psi\rangle}$$

Plan

1. **Interpretation** 2. **Gravity dual** 3. Application as **order parameter**

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Background: Entropy and Area in (Quantum) Gravity

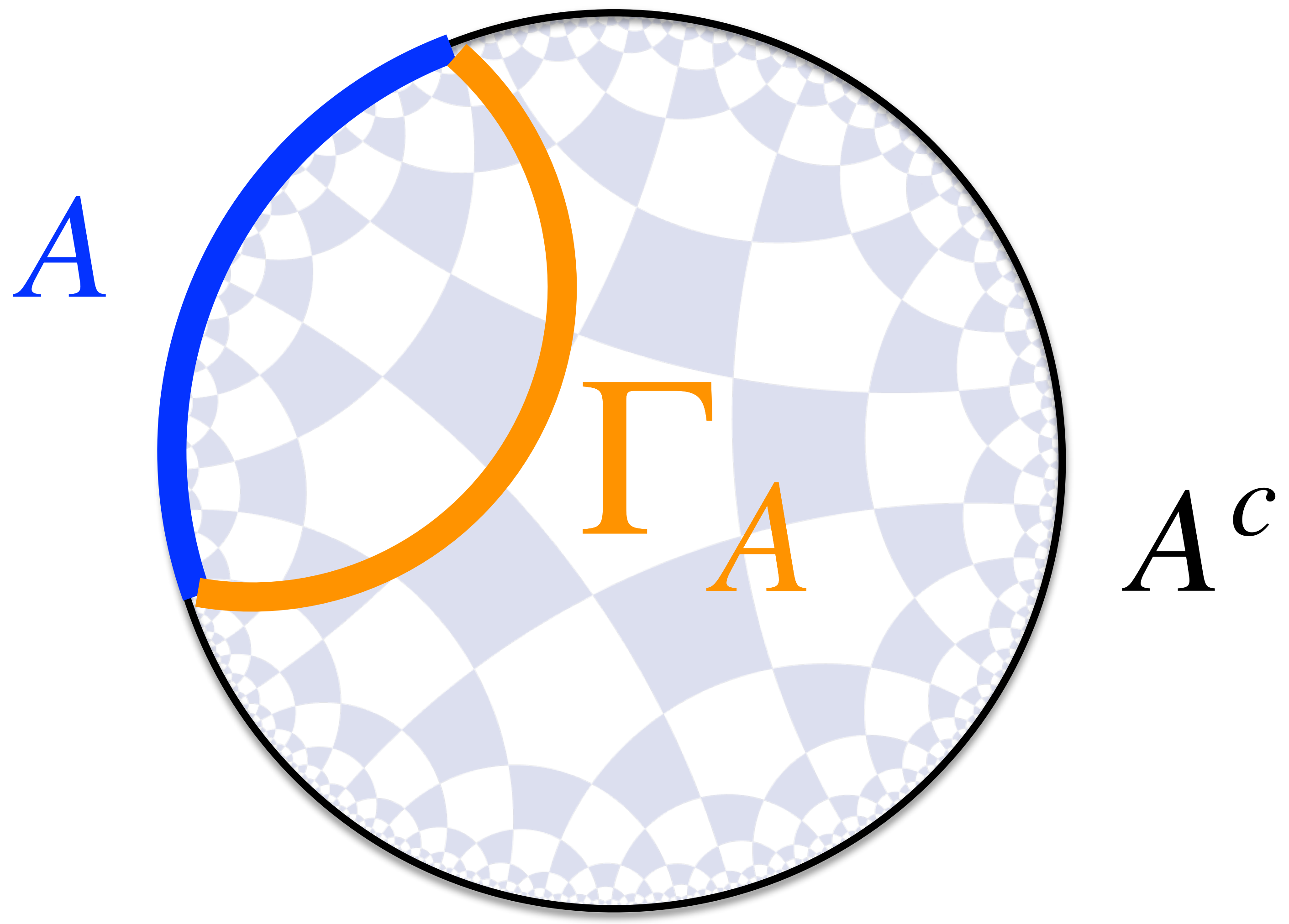
Thermodynamical Entropy and Area [Bekenstein '72] and [Hawking '74]

$$S = \frac{A}{4G_N}$$

! Volume law in lower dimension (The idea of **holographic principle**)

Entanglement entropy and Area [Ryu-Takayanagi '06],…

$$S(\rho_A) \equiv -\mathrm{Tr}(\rho_A \log \rho_A) = \frac{\mathrm{Area}(\Gamma_A)}{4G_N}$$



Microscopic entropy ↔ Area (in Lorentzian spacetime)

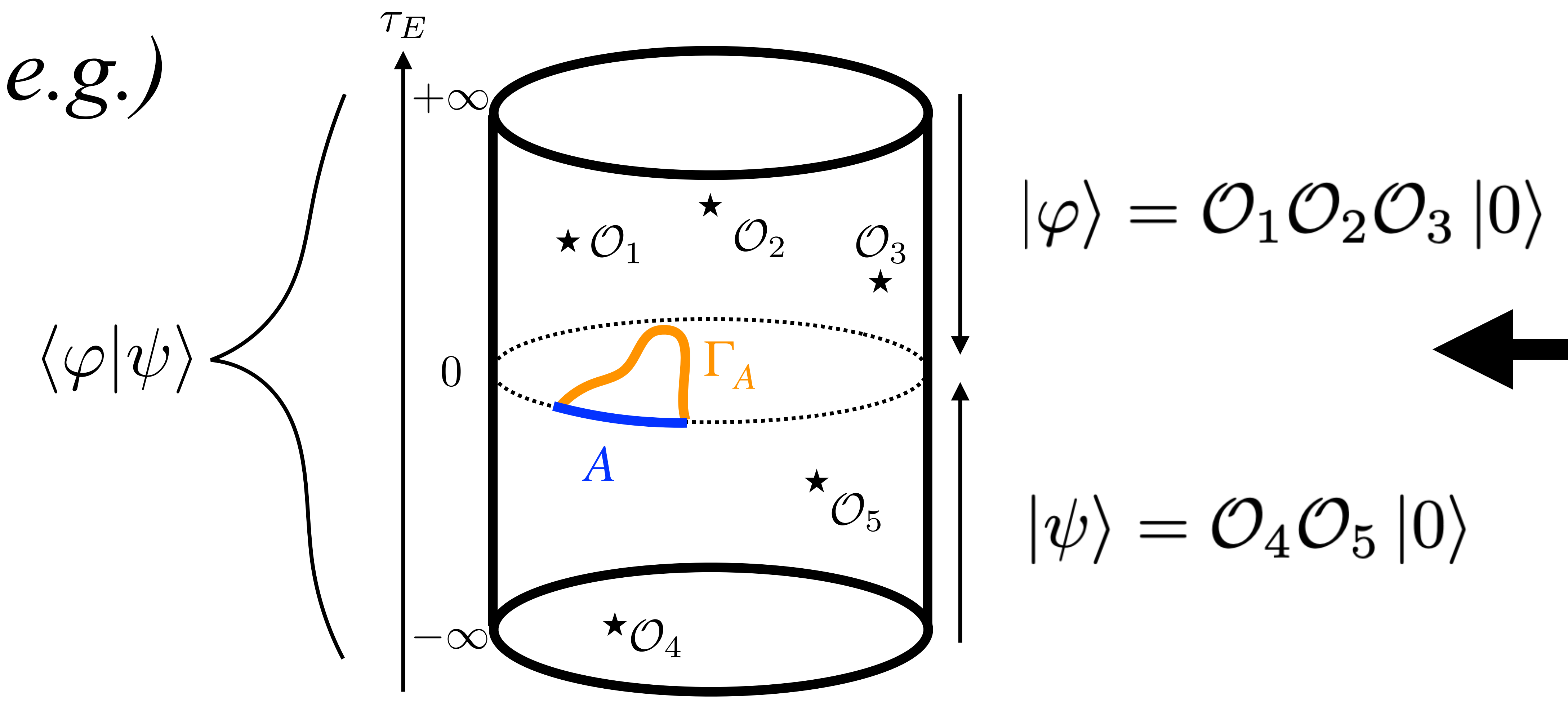
(We will focus on the classical part, but quantum/non-perturbative corrections are also important)

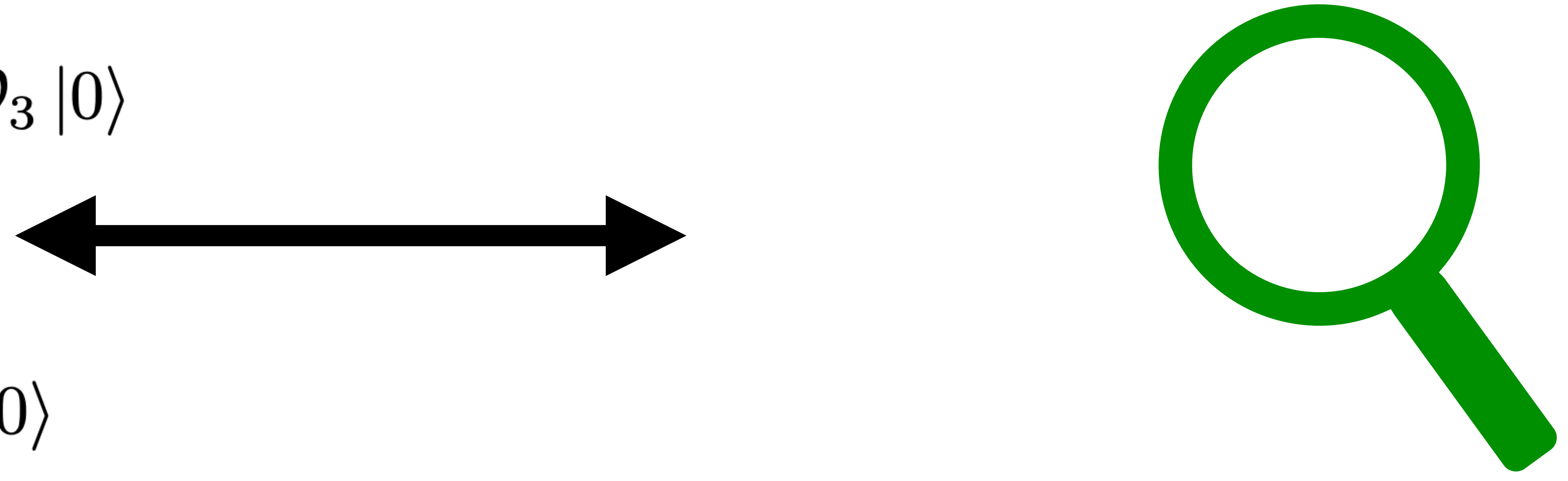
Motivation from Gravity

Minimal surfaces in Euclidean AdS
("time-dependent" due to φ≠ψ)

Quantum information theoretical quantity?
(Perhaps new one?)

e.g.)





- <φ|ψ> (overlap in CFT) has a sharp gravity dual based on AdS/CFT
- What is the **boundary dual** of **minimal area** in this geometry?

(If φ=ψ, the entanglement entropy)

→ Our answer: Pseudo Entropy!

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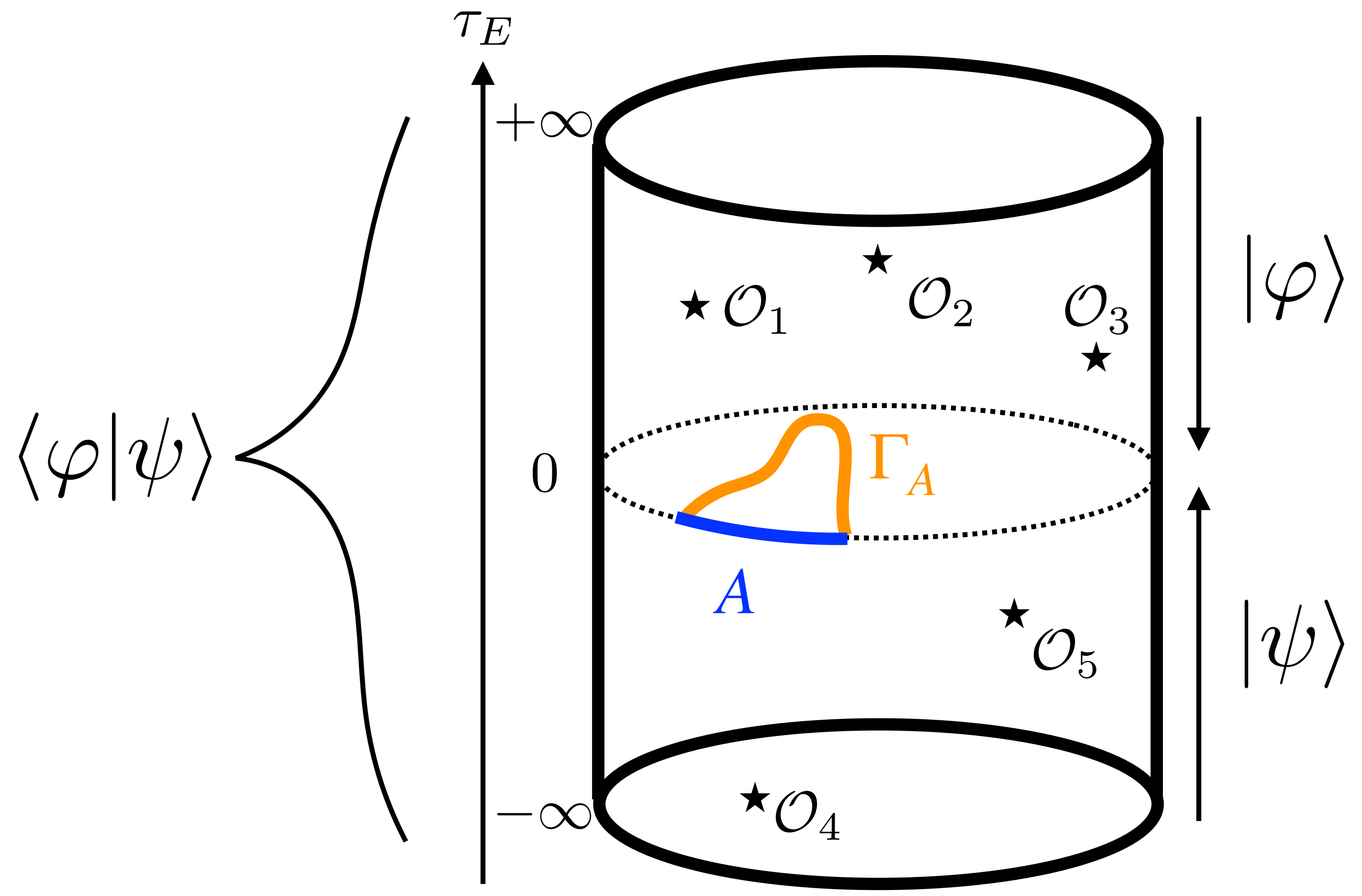
1. Interpretation

# of distillable EPR pairs
under LOCC+ **post-selection**

$|\psi\rangle \longrightarrow |\varphi\rangle$

Initial **Final**

2. Gravity dual



3. Order parameter

$|\varphi\rangle$ & $|\psi\rangle$

In the same quantum phase?

Density Matrix (pure state)

$$\rho^{\psi} = \frac{|\psi\rangle\langle\psi|}{\langle\psi|\psi\rangle}$$

<u>Expectation value</u>

$$\langle A\rangle_{\psi} = \mathrm{Tr}[A \cdot \rho^{\psi}] = \frac{\langle\psi|A|\psi\rangle}{\langle\psi|\psi\rangle}$$

Transition Matrix

$$\mathcal{T}^{\psi|\varphi} = \frac{|\psi\rangle\langle\varphi|}{\langle\varphi|\psi\rangle}$$

<u>Weak value: complex value in general</u>

$$\frac{\langle\varphi|A|\psi\rangle}{\langle\varphi|\psi\rangle} = \mathrm{Tr}[A \cdot \mathcal{T}^{\psi|\varphi}]$$

[Aharanov-Albert-Vaidman '88]

Pseudo (Entanglement) Entropy

[Nakata-Takayanagi-Taki-KT-Wei '20]

$$S(\mathcal{T}_A^{\psi|\varphi}) = -\mathrm{Tr}\left[\mathcal{T}_A^{\psi|\varphi} \log \mathcal{T}_A^{\psi|\varphi}\right]$$

where $\mathcal{T}_A^{\psi|\varphi} = \mathrm{Tr}_{A^c}\mathcal{T}^{\psi|\varphi}$

Precise definition: defined via eigenvalues (Jordan normal form),

"Renyi entropy" : $S^{(n)}(\mathcal{T}_A^{\psi|\varphi}) = \dfrac{1}{1-n}\log \mathrm{Tr}[(\mathcal{T}_A^{\psi|\varphi})^n]$

Pseudo Entropy can be defined as n→1 limit

- **Complex-valued in general**

- **In some nice class of states, (Pseudo Entropy) ≧ 0**

  - Ground states of spin systems (*e.g.* transverse Ising model)

$$\mathcal{T}^{1|2} = |0_{J_1,h_1}\rangle\langle 0_{J_2,h_2}| \qquad H_{1,2} = -J_{1,2}\sum_{i=0}^{N-1}\sigma_i^z\sigma_{i+1}^z - h_{1,2}\sum_{i=0}^{N-1}\sigma_i^x$$

  - Holographic states (CFT states dual to semi-classical geometry in AdS/CFT)

$$\vdots$$

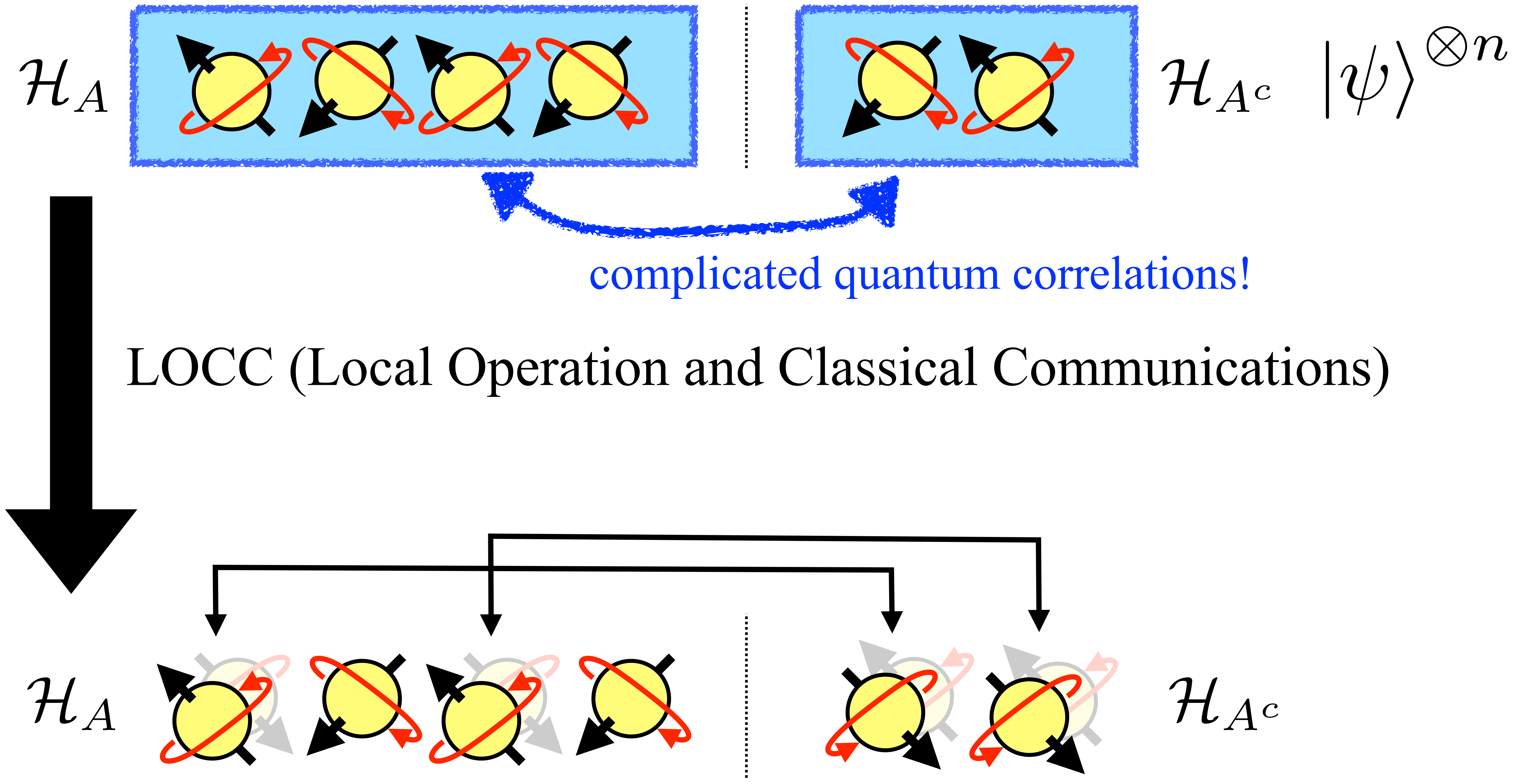
(non-Hermitian "modular Hamiltonian" !?)

- **Real part of PE:**

∃ A nice interpretation based on "distillable EPR pairs"

(Next slide)

Entanglement Entropy = # of Distillable EPR pairs under LOCC



$$\rho_A = \mathrm{Tr}_{A^c} |\psi\rangle\langle\psi|$$

$$|\mathrm{EPR}\rangle^{\otimes N}$$

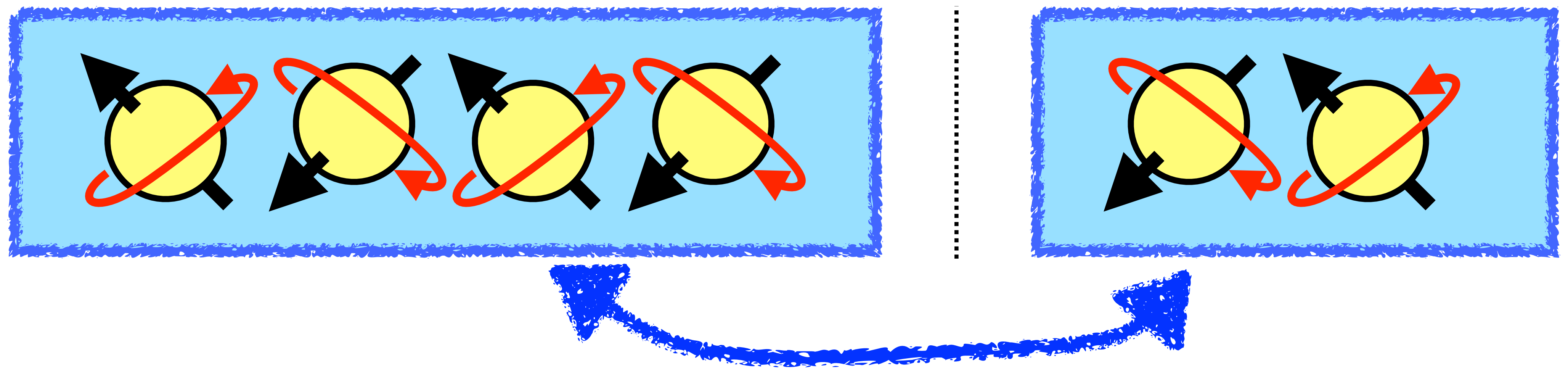
Bunch of maximally entangled states (EPR pairs)

$$S(\rho_A) = \lim_{n\to\infty} \frac{N}{n}$$

Pseudo Entropy = # of Distillable EPR pairs under LOCC + post-selection

(⚠ Proven only when the reduced transition matrix is Hermitian and real-positive)

$\mathcal{H}_A$



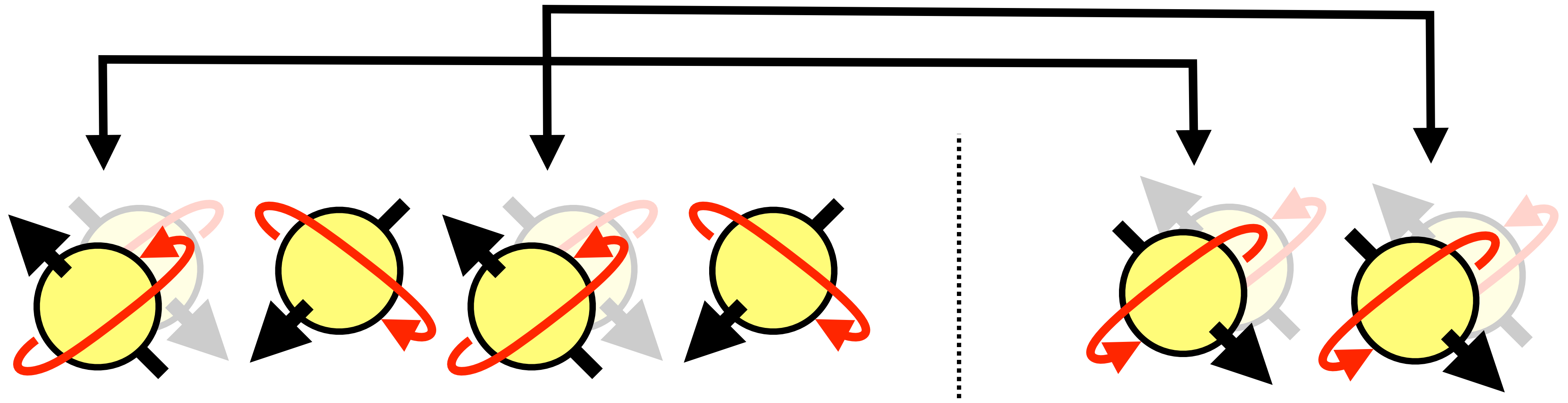
$\mathcal{H}_{A^c}$ $|\psi\rangle^{\otimes n}$

complicated quantum correlations!

LOCC (Local Operation and Classical Communications)

$$\mathcal{T}_A^{\psi|\varphi} = \mathrm{Tr}_{A^c} \frac{|\psi\rangle\langle\varphi|}{\langle\varphi|\psi\rangle}$$

$\mathcal{H}_A$



$\mathcal{H}_{A^c}$

post-selection

$|\varphi\rangle^{\otimes n}$

$$|\mathrm{EPR}\rangle^{\otimes N}$$

Bunch of maximally entangled states (EPR pairs)

$$S(\mathcal{T}_A^{\psi|\varphi}) = \lim_{n\to\infty} \frac{N}{n}$$

Summary

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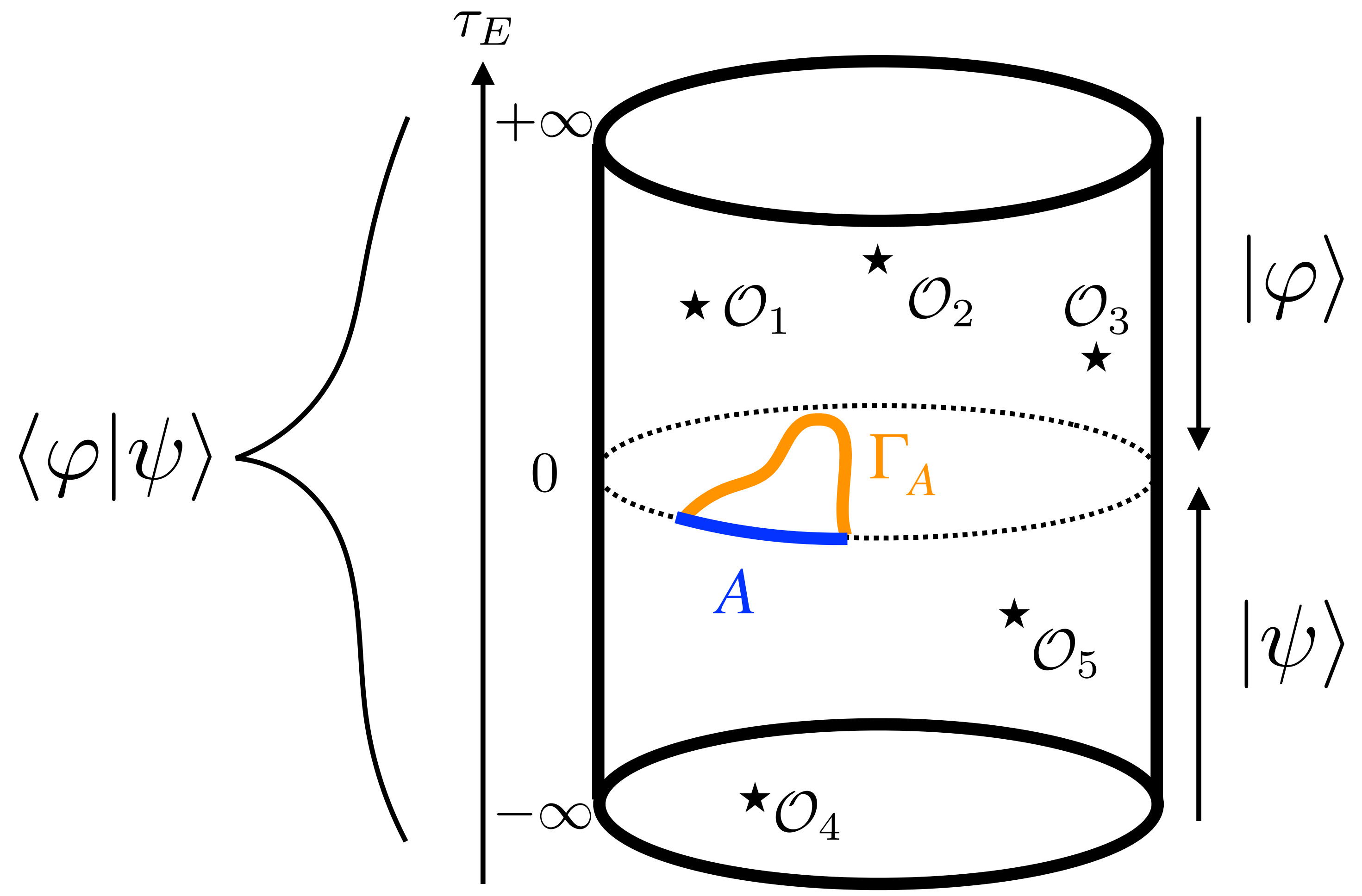


3. Order parameter

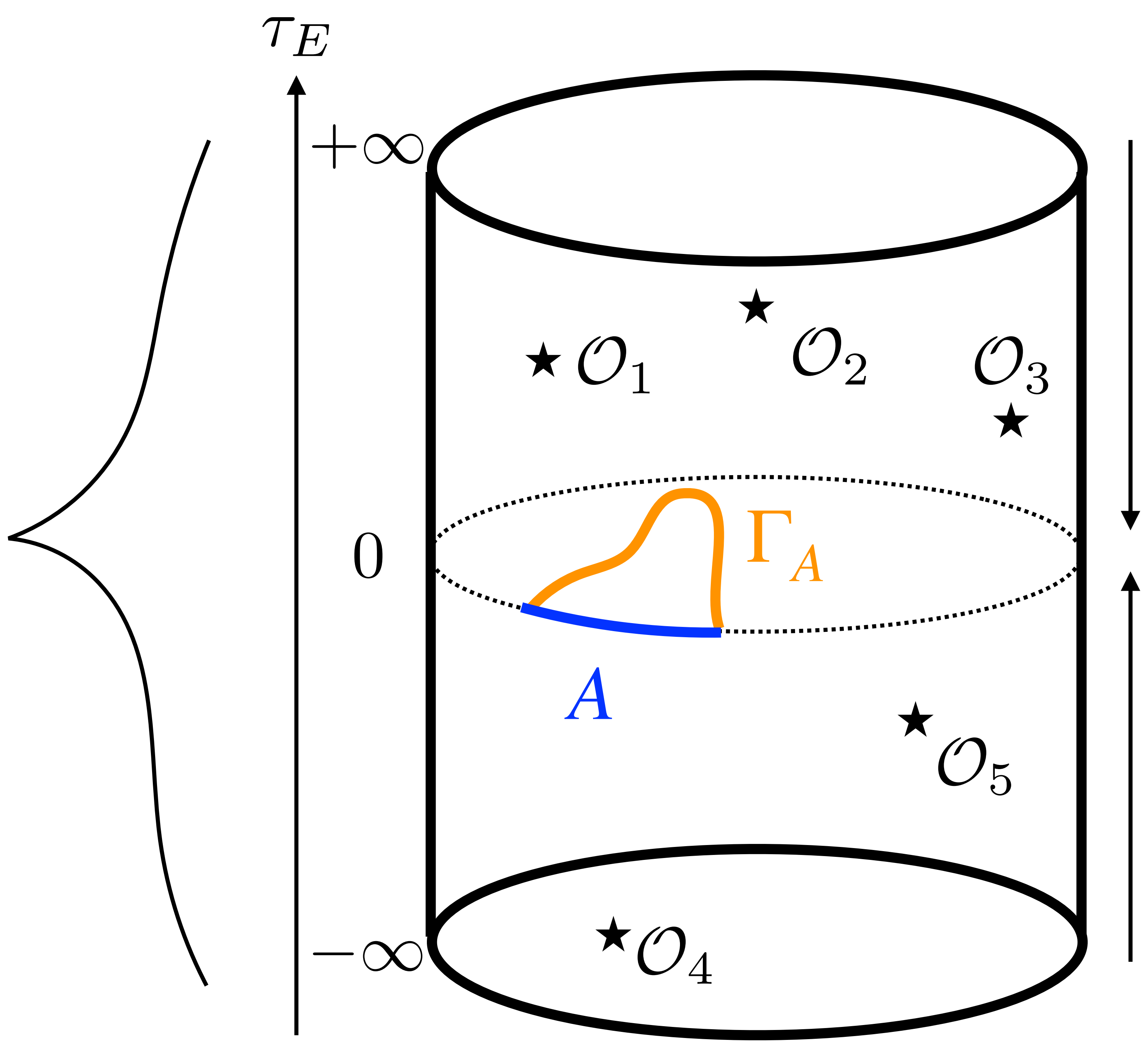
$|\varphi\rangle$ & $|\psi\rangle$

In the same quantum phase?

Holographic Pseudo Entropy (HPE)

$$S(\mathcal{T}_A^{\psi|\varphi}) = \underset{\substack{\partial\Gamma_A = \partial A \\ \Gamma_A \sim A}}{\mathrm{Min}} \left[\frac{\mathrm{Area}(\Gamma_A)}{4G_N} \right]$$

$$Z = \mathrm{Tr}(\mathcal{T}^{\psi|\varphi}) = \langle\varphi|\psi\rangle$$



$\langle\mathrm{final}| = \langle\varphi|$

$|\mathrm{initial}\rangle = |\psi\rangle$

! Can prove by reusing [Lewkowycz-Maldacena'13] argument

(Just use GKP-Witten relation to the replica manifold)

HPE as Weak Value of Area Operator

$$S(\mathcal{T}_A^{\psi|\varphi}) = \frac{\langle\varphi|\frac{\hat{A}}{4G_N}|\psi\rangle}{\langle\varphi|\psi\rangle}$$

EE for holographic states ~ expectation value of linear operator (area operator)

[Almheiri-Dong-Swingle'16]

Can confirm **linearity of PE** in holographic CFT2 :

$$|\psi\rangle = \sum_i c_i \underline{|\mathcal{O}_{H_i}\rangle}$$

$$|\varphi\rangle = \sum_j b_j \underline{|\mathcal{O}_{H_j}\rangle}$$

Heavy states

$$\Rightarrow \quad \frac{\sum_i b_i^* c_i \frac{\mathrm{Area}(\Gamma_A^{H_i})}{4G_N}}{\sum_i b_i^* c_i}$$

→ **complex-valued in general**

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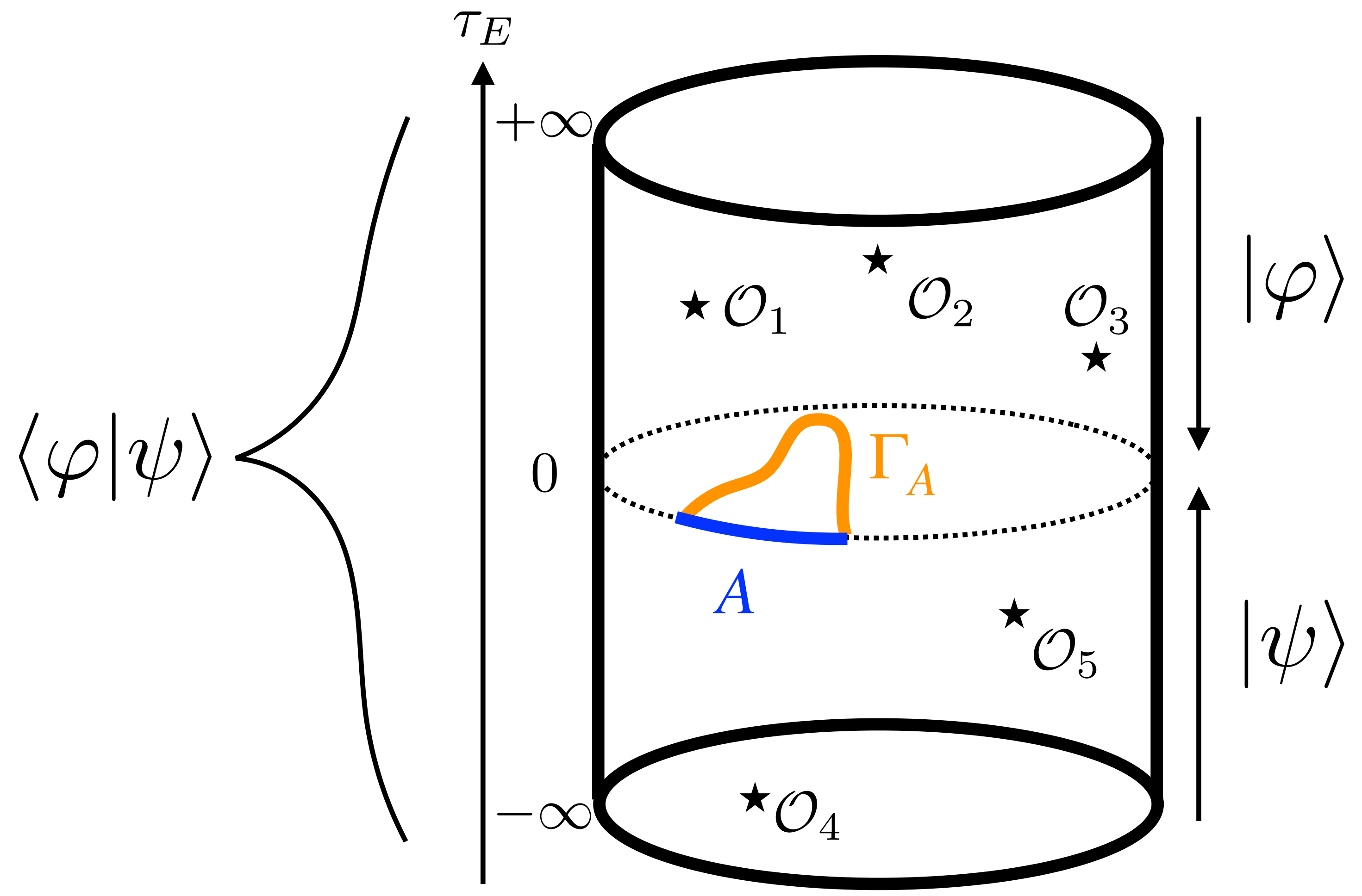

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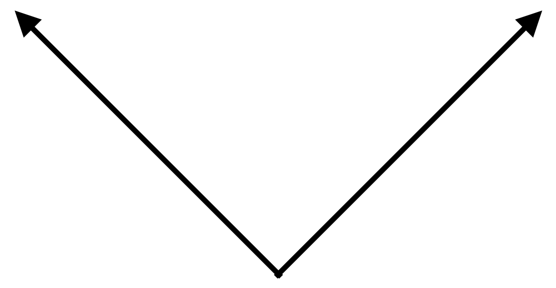


In the same quantum phase?

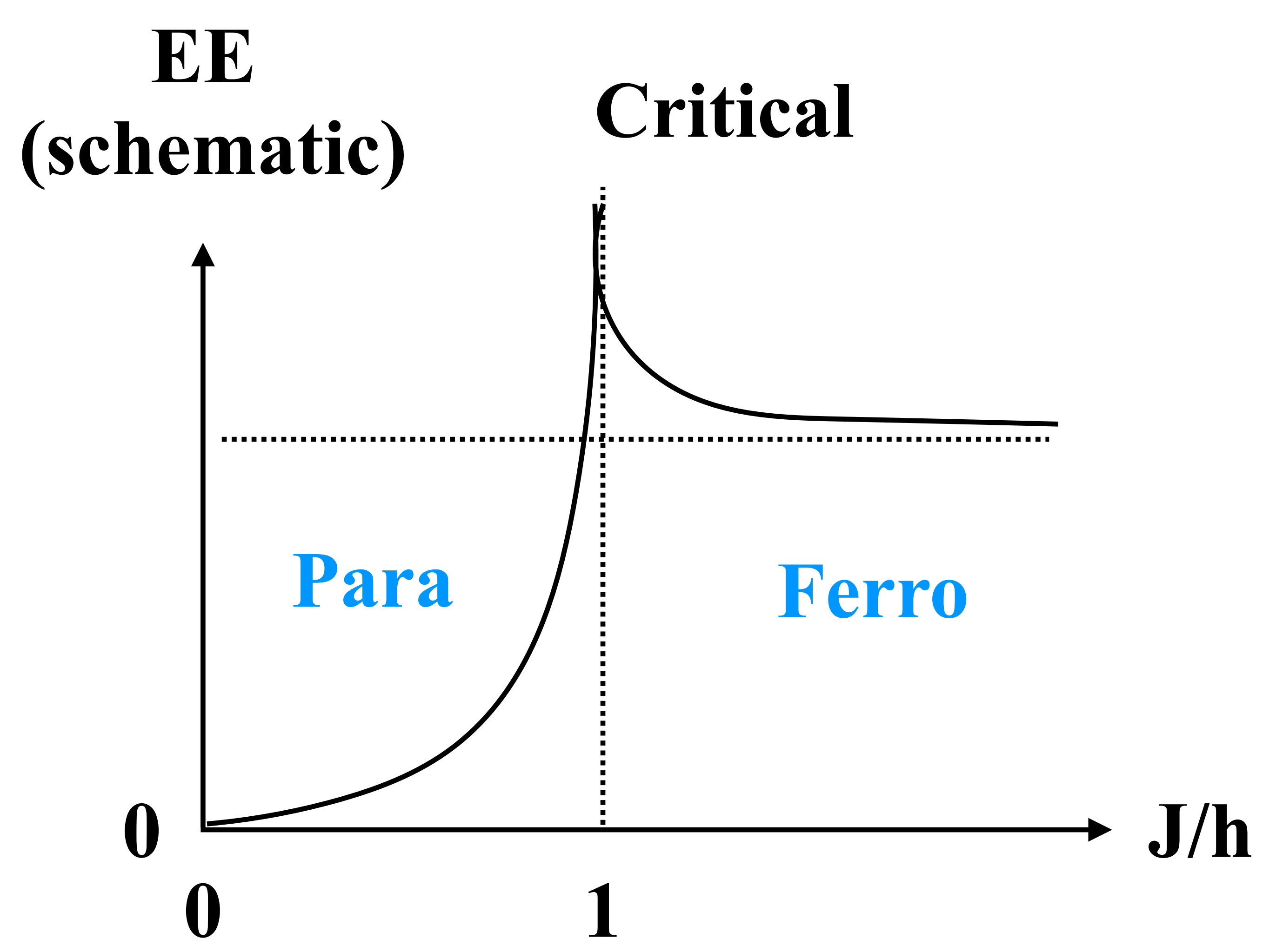
Pseudo Entropy as an order parameter

[Mollabashi-Shiba-Takayanagi-KT-Wei ’20]

Ground States in Transverse Ising model

$$H_{1,2} = -J_{1,2} \sum_{i=0}^{N-1} \sigma_i^z \sigma_{i+1}^z - h_{1,2} \sum_{i=0}^{N-1} \sigma_i^x$$

$$\Delta S_{12} \equiv S(\mathcal{T}_A^{1|2}) - S(\rho_A^{(1)})/2 - S(\rho_A^{(2)})/2$$



$\Delta S_{12} > 0$	**If two states belong to different phases**
$\Delta S_{12} < 0$	**If two states belong to the same phase**

- **generalization (*e.g.* XY model)**
- **holographic interpretation**

Mollabashi-Shiba-Takayanagi-KT-Wei in progress

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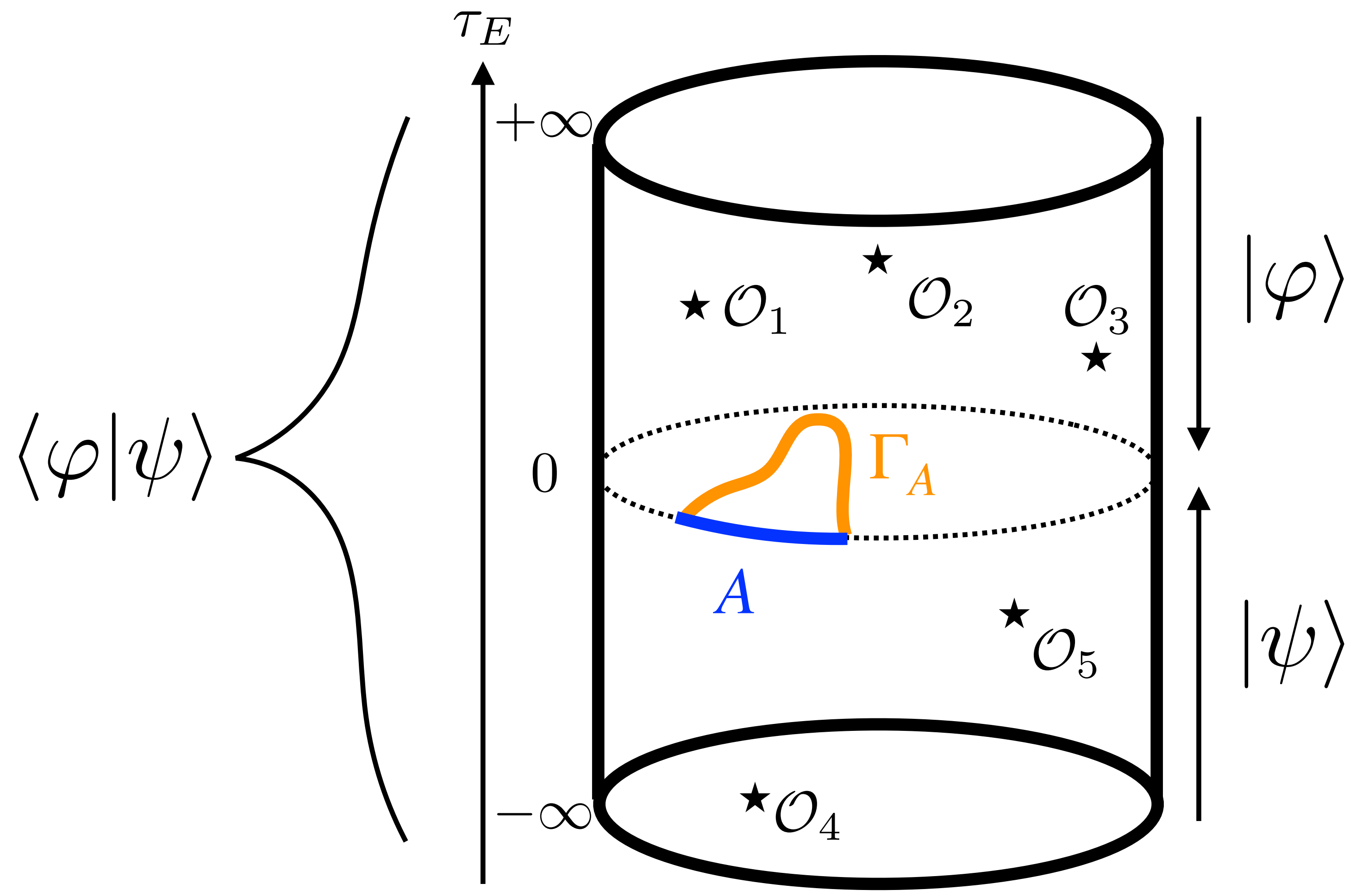

3. Order parameter

$|\varphi\rangle \,\&\, |\psi\rangle$

In the same quantum phase?

Discussion

- Interpretation of imaginary part?
- Dynamical setup (relevant to imaginary part)

  Mollabashi-Shiba-Takayanagi-KT-Wei and Goto-Nozaki-KT in progress
- Mixed state generalizations
- Further application?

⋮

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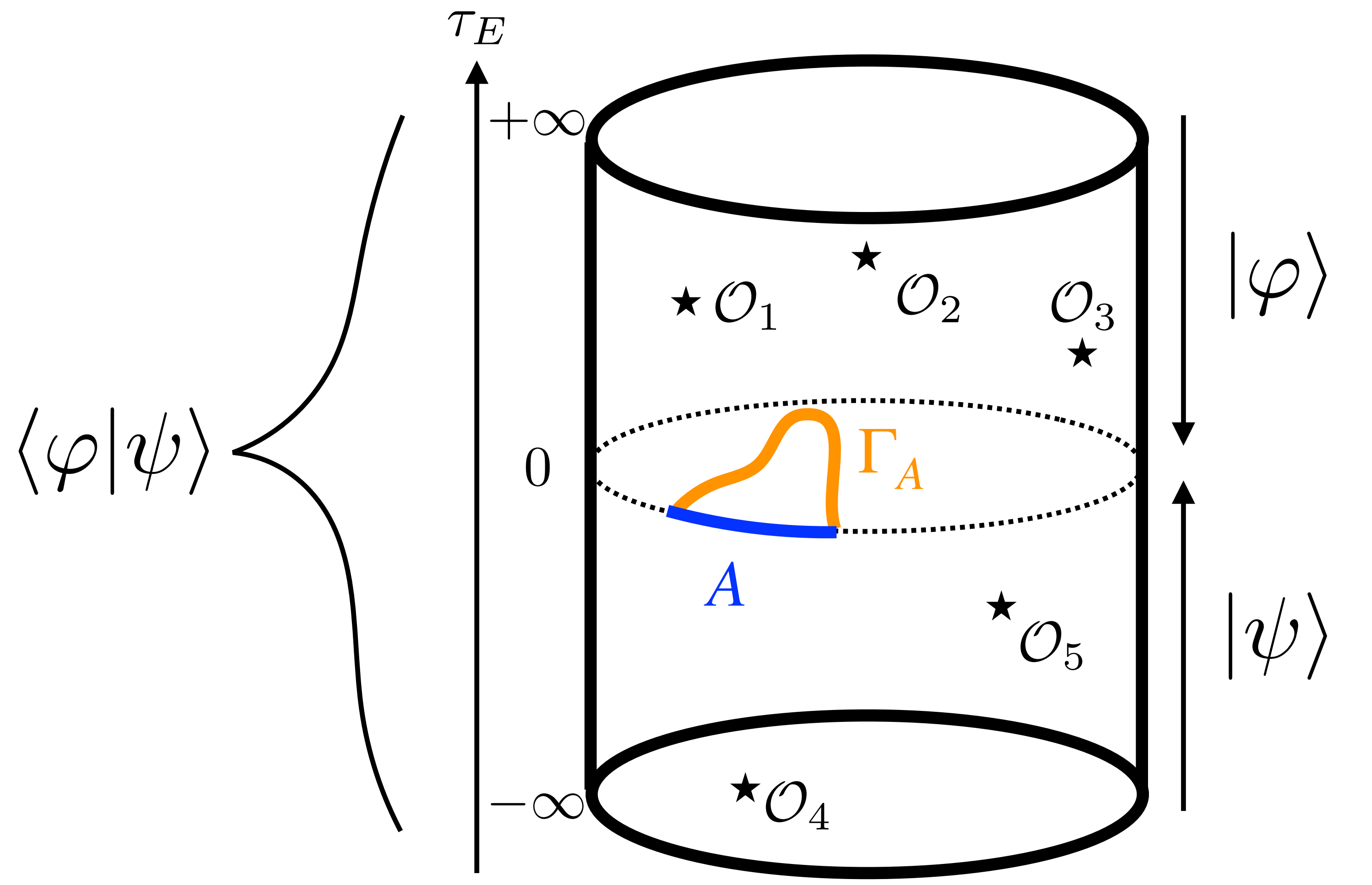

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