Timelike Entanglement Entropy

Jonathan Harper

Yukawa Institute for Theoretical Physics, Kyoto University
jonathan.harper@yukawa.kyoto-u.ac.jp

April 4th, 2023

Recent Developments in Quantum Physics of Black Holes 2023

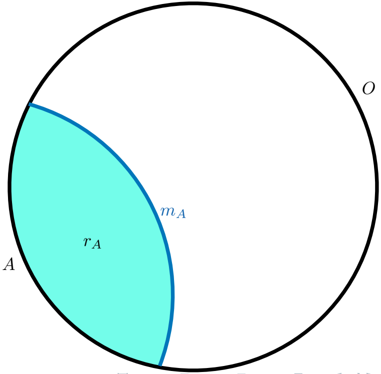
Based on arXiv:2302.11695, 2210.09457 in collaboration with: K. Doi, A. Mollabashi, T. Takayanagi, and Y. Taki

Holographic entanglement entropy

For a static time slice Σ of a holographic spacetime choose a boundary region A: The holographic entanglement entropy (EE) is given by the Ryu-Takayanagi (RT) formula ('06):

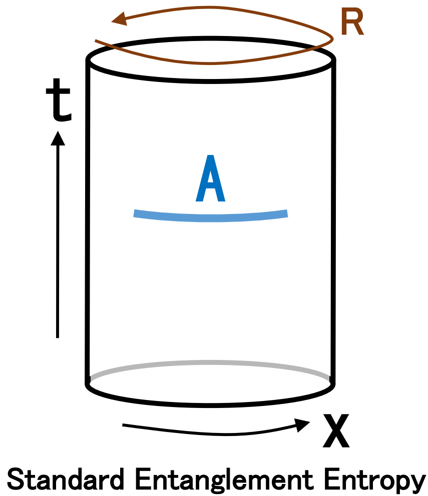
$$S_A = \min_{m \sim A} area(m) = \text{area}(m_A).$$

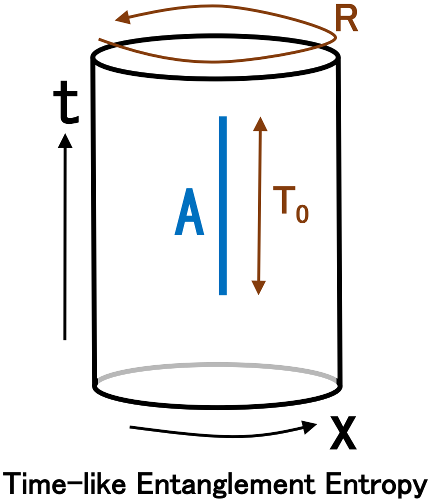
- Split the boundary into two regions A and its complement O.
- m_A is the minimal area surface *homologous* to A ("RT surface").
- Homologous: Exists *homology region* r_A which interpolates between A and m_A.



Timelike entanglement entropy

Can we generalize the entanglement entropy to timelike regions?





Timelike entanglement entropy

The entanglment entropy can be calculated in the boundary CFT using the replica trick for general points $P = (t_P, x_P)$ and $Q = (t_Q, x_Q)$:

$$S_A = S_A^{(1)} = \frac{c}{3} \log \left[\frac{\sqrt{(x_P - x_Q)^2 - (t_P - t_Q)^2}}{\epsilon} \right].$$

Thus, if our region is purely in the time direction $\Delta x = 0$

$$S_A^{(\mathrm{T})} = \frac{c}{3} \log \frac{T_0}{\epsilon} + \frac{c\pi}{6} i.$$

Similarly for a thermal state with inverse temperature β:

$$S_A^{(\mathrm{T})} = \frac{c}{3} \log \left[\frac{\beta}{\pi\epsilon} \sinh \frac{\pi T_0}{\beta} \right] + \frac{i\pi c}{6}.$$

Can this quantity be understood geometrically using holography?

Holographic timelike entanglement entropy

Bulk computation proceeds as a generalization of the RT formula:

- We consider paths $\{\Gamma_A\}$: Collections of spacelike and timelike extremal surfaces which are homologous to A.
- Spacelike surfaces have real area while timelike have imaginary.
- The total complex area of the paths is varied w.r.t. to the joining points.
- The area of the resulting optimal path Γ_A^* gives the TEE $S_A^{(\mathrm{T})}$ of A.

$$S_A^{(\mathrm{T})} = \frac{c}{6}\,\mathrm{area}(\Gamma_A^*)$$

BTZ black hole

$2+1$d spacetime geometry at inverse temperature β with two asymptotic boundary regions

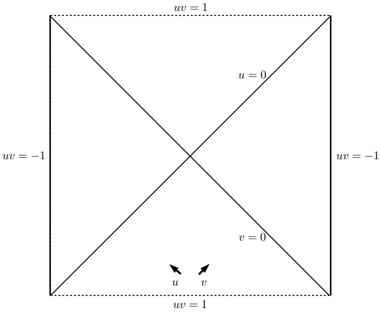
- We work in Kruskal Coordinates:

$$ds^2 = -4\frac{du dv}{(1+uv)^2} + \frac{(1-uv)^2}{(1+uv)^2} r_+^2 d\phi^2$$

- For a constant ϕ slice geodesics are simple and given by

$$u(v) = \frac{c_1 + c_2 v}{1 + c_1 v}$$

and are full determined by specifying two points.

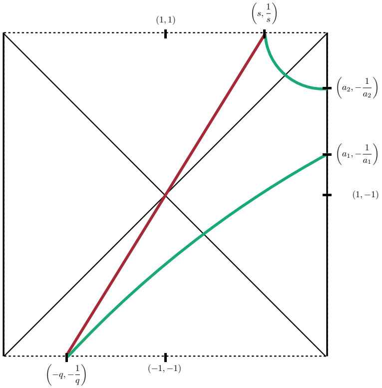


BTZ black hole

- We consider two points on the right boundary $u = a_1, a_2$ where

$$a_i = e^{r_+ T_i}.$$

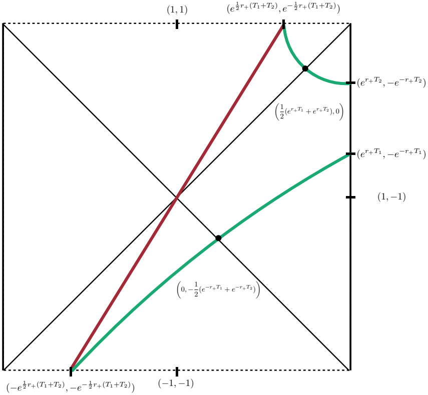
- The two points are *only* connected by spacelike geodesics which end on opposite singularities.
- Construct a family of paths which consist of two spacelike and one timelike geodesic.
- Paths are determined by the location of the joining points s, q on the future past singularity.



BTZ black hole

- Varying with respect to s, q determines the optimal path Γ_A^* with $s^2 = q^2 = a_1 a_2$ and area

$$\log\left(\frac{1}{\epsilon^2}\frac{(a_2 - a_1)^2}{a_1 a_2}\right) + i\pi.$$



The timelike entanglement entropy is

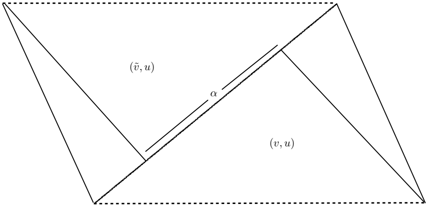
$$S_A^{(\mathrm{T})} = \frac{c}{3}\log\left(\frac{\beta}{\pi\epsilon}\sinh\left(\frac{\pi}{\beta}(T_2 - T_1)\right)\right) + \frac{c}{6}i\pi$$

Consider a slightly more complicated geometry: A low energy shockwave along $u = 0$ causes a shift $\tilde{v} = \alpha + v$

$$ds^2 = -4\frac{dudv}{(1+u(v+\alpha\theta(u)))^2} + \frac{(1-u(v+\alpha\theta(u)))^2}{(1+u(v+\alpha\theta(u)))^2} r_+^2 d\phi^2$$

- The spacelike entanglement of a region A is unaffected by the shockwave

$$S_A = \frac{c}{3}\log\left(\frac{\beta}{\pi\epsilon}\sinh\left(\frac{\pi}{\beta}(\phi_2 - \phi_1)\right)\right)$$

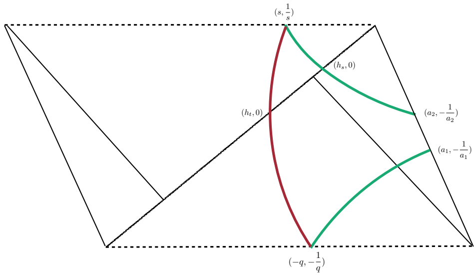


What about the timelike entanglment entropy?

Shockwave

We consider the same procedure for determining the TEE:

- The geodesics must be joined across the horizon at the location of the shockwave.
- Construct a family of paths which consist of four spacelike and two timelike geodesic.
- Paths are determined by the location of the joining points s, q on the future past singularity and the joining points on the horizon h_s, h_t

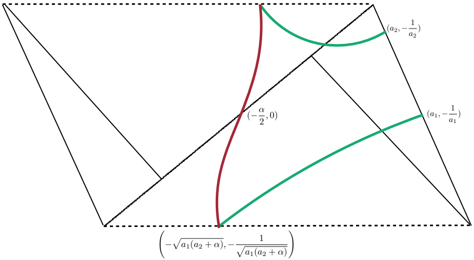


Shockwave

Varying with respect to s, q, h_s, h_t determines the optimal path and TEE:

$$S_A^{(T)} = \frac{c}{3}\log\left(\frac{\beta}{\pi\delta}\left(\sinh(\frac{\pi}{\beta}(T_2 - T_1)) + \frac{\alpha}{2}e^{-\frac{\pi}{\beta}(T_2+T_1)}\right)\right) + \frac{c}{6}i\pi.$$

- Result explicitly depends on shift α. TEE is sensitive to the shockwave unlike spacelike entanglement entropy.
- Can be confirmed by careful treamtment of CFT under local quench.

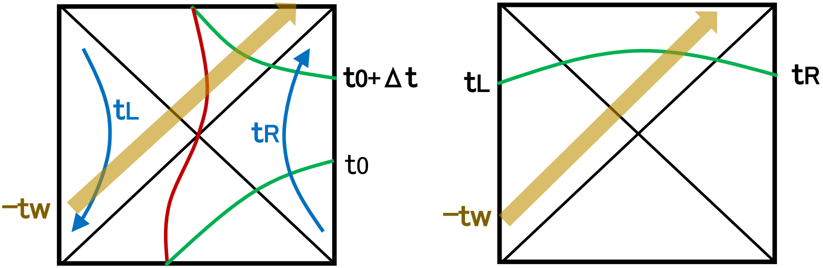


Shockwave

- Times on the left and right boundary are related by $t_R = t_L + i\frac{\beta}{2}$
- In the shockwave geometry geodesics between the left and right boundary have length (Shenker, Stanford '13)

$$\frac{c}{3}\log\left[\frac{\beta}{\pi\delta}\left(\cosh\frac{\pi}{\beta}(t_R - t_L) + \frac{\alpha}{2}e^{-\frac{\pi}{\beta}(t_R+t_L)}\right)\right].$$

- Under this substitution our result is exactly reproduced.



Conclusions

- We defined a new quantity the *timelike entanglement entropy* $S_A^{(\mathrm{T})}$ for a boundary time interval A.
- Holographically determined by the complex area of a stationary combination of space *and* timelike surfaces.
- These surfaces must form a closed path homologous to the boundary region.
- Future directions:
  - Multiple regions $\longrightarrow$ multiple saddles, phase transitions.
  - General boundary regions $\longleftrightarrow$ More exotic paths?
  - Examples of geometries with other values for imaginary component?