Random State and Hawking Radiation

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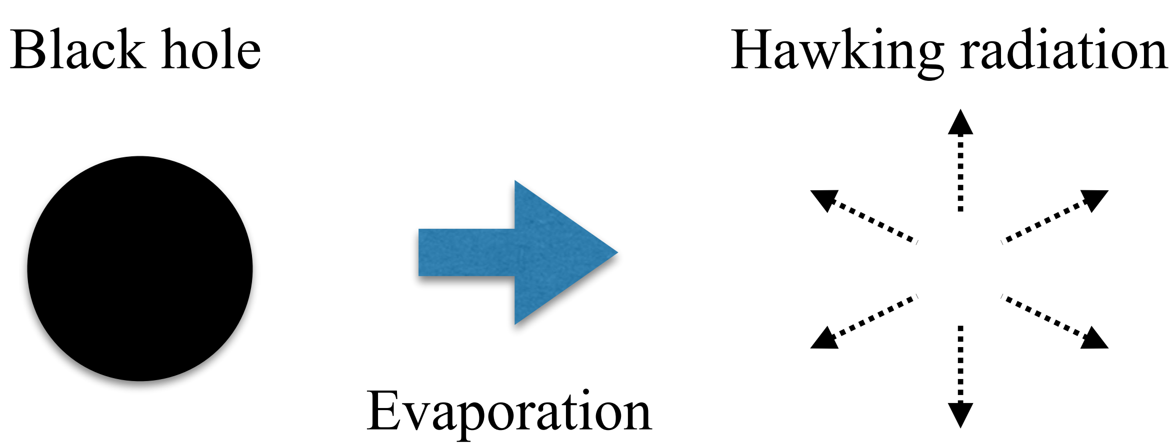
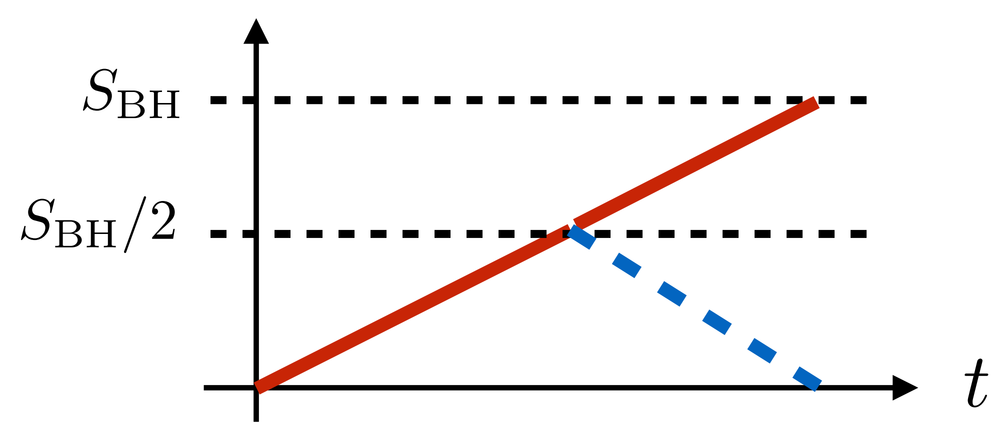
Information Paradox

Black hole information paradox: [Hawking (1975)]
The entropy of Hawking radiation predicted from semiclassical calculation can be arbitrarily large, violating unitarity

- There should be missing large contributions in semiclassical method
- The entropy should be given by the Page curve [Page (1993)]





Entropy of radiation predicted by semiclassical approach

Page Curve from Replica Wormhole

[Penington (2019)][Almheiri et.al (2019)]
[Almheiri et.al (2019)][Almheiri et.al (2019)][Penington et.al (2019)]

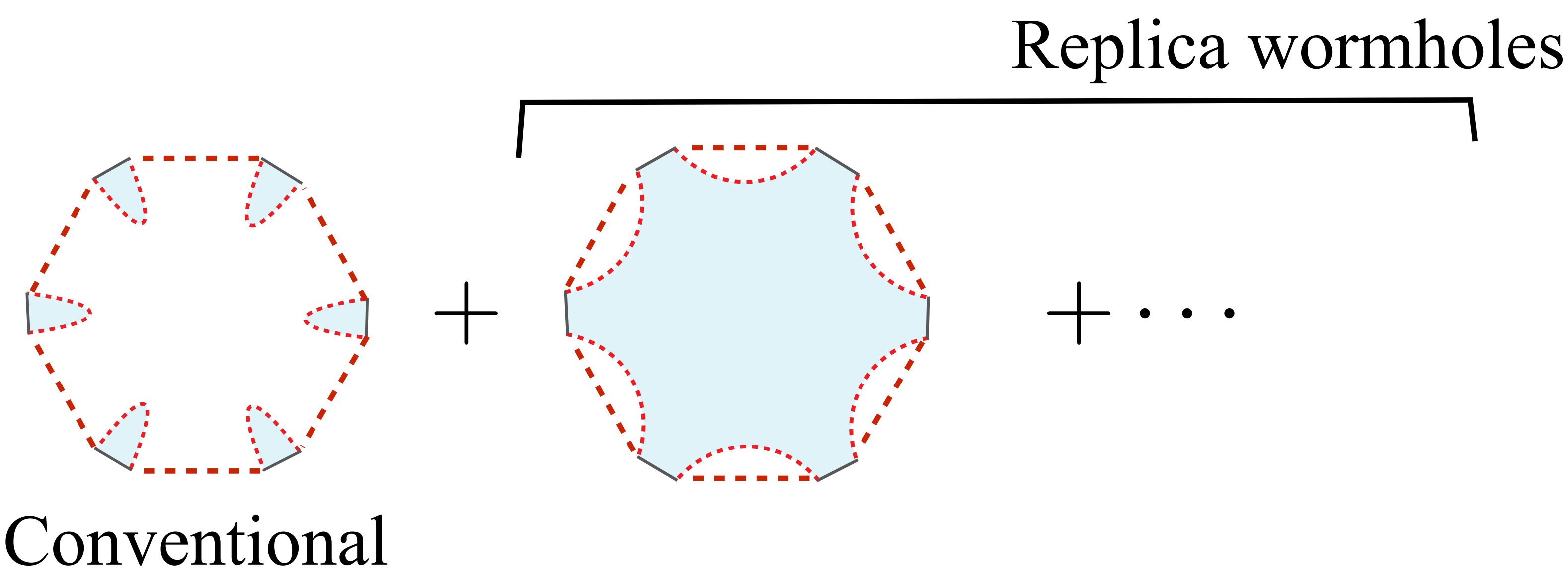
Replica Wormhole:
Once we include non-trivial topology called replica wormhole, semiclassical computation yields correct answer consistent with QM

Replica trick:

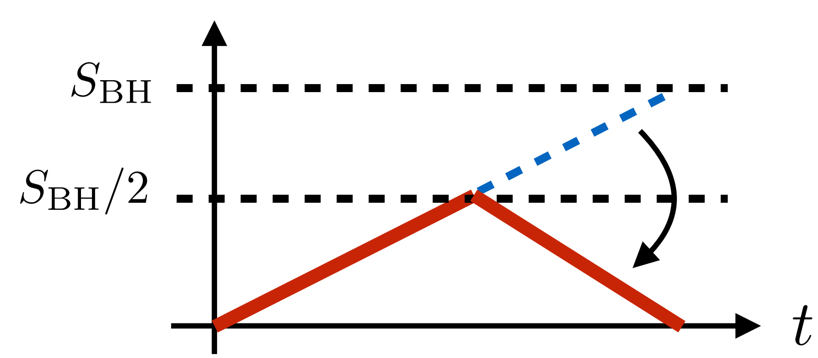
$$S = -\mathrm{Tr}\left[\rho \log \rho\right] = \lim_{n \to 1} \frac{\log\, \mathrm{Tr}[\rho^n]}{n-1}$$

Replica wormhole:

$$\mathrm{Tr}[\rho^n] =$$



Page curve:



Ensemble Average of Gravity Theories

Ensemble Average: [Saad et al. (2019)][Bousso, Wildenhain (2020)]

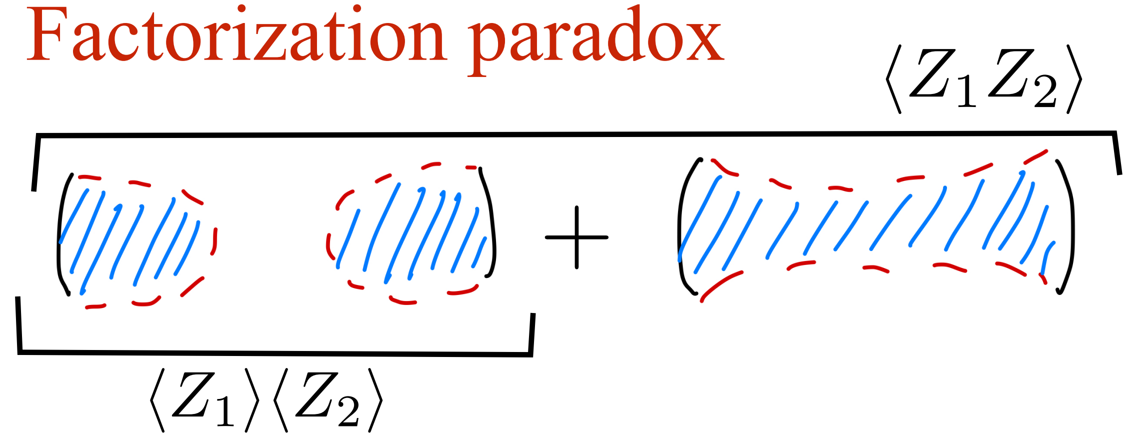
Gravity used in the derivation is an ensemble of gravity theories

- JT gravity is 2d gravity, dual to an ensemble of boundary Hamiltonians

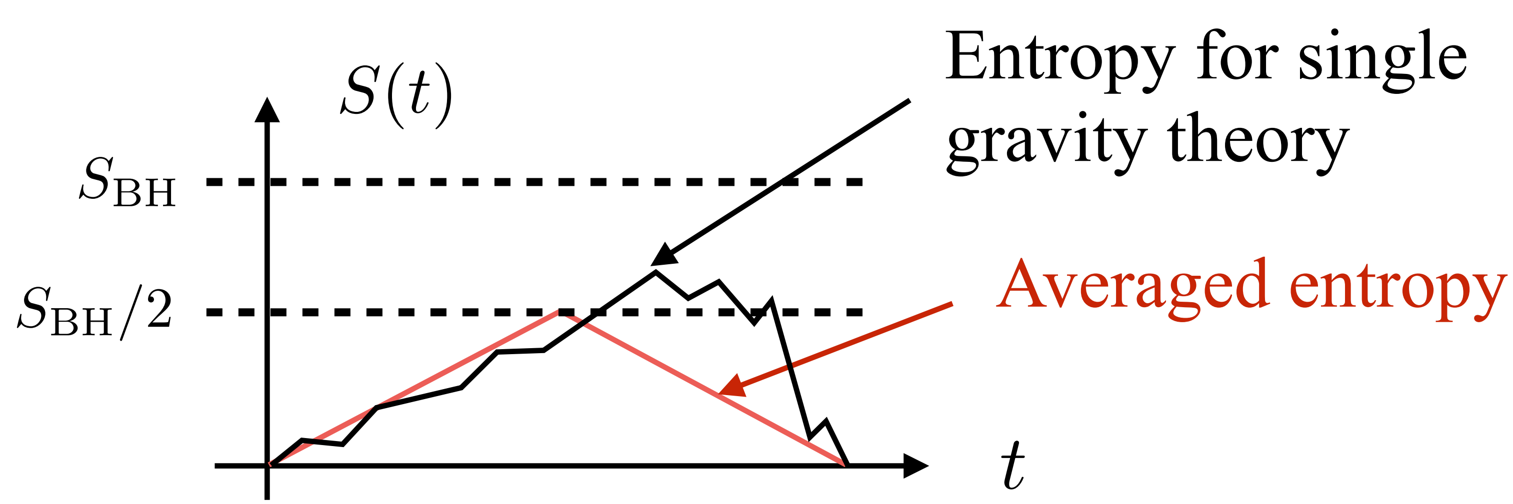
$$Z_{\rm JT} = \int dH e^{-V(H)}$$

- Wormholes indicates ensemble: Factorization paradox

$$\langle Z_1 Z_2 \rangle \neq \langle Z_1 \rangle \langle Z_2 \rangle$$



- The Page curve from the replica wormhole is the entropy averaged over ensemble of theories, which can differ significantly from that of single theory

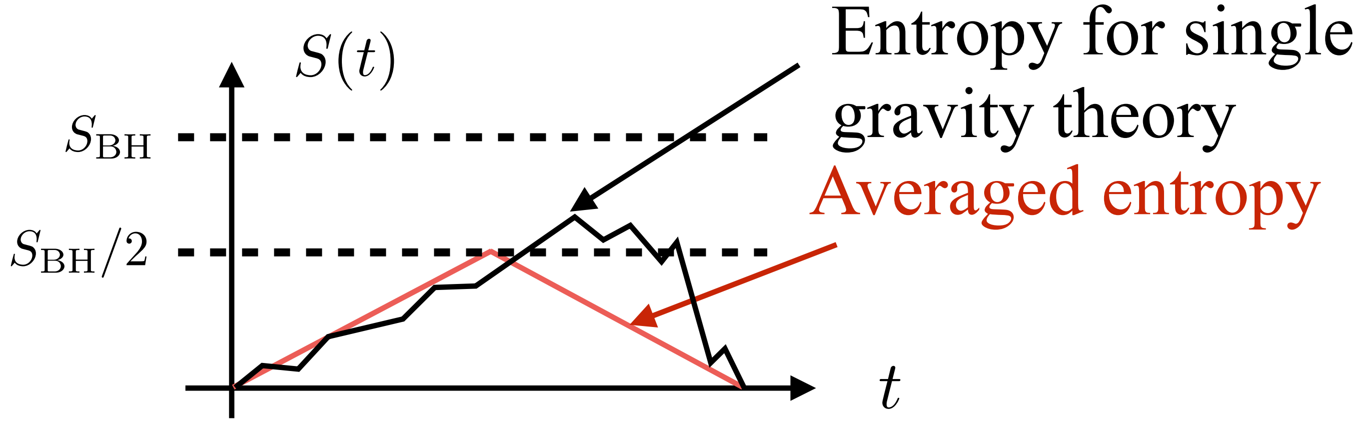


Questions we address

Average vs Typicality:
Is the averaged entropy typical?
→ Otherwise, replica wormhole is not very useful!

Entropy Fluctuation
This captures typical magnitude for entropy fluctuation

$$\delta S := \sqrt{\langle S^2 \rangle - \langle S \rangle^2}$$



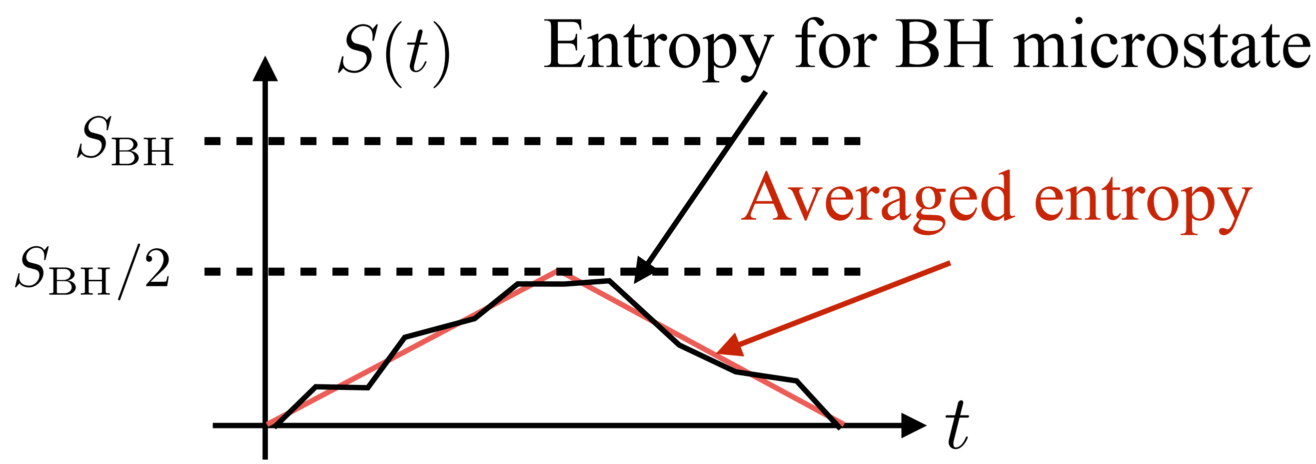
- We ask how small this is

Random state:
Random state, which is a typical state on the Hilbert space, has been thought to model the Hawking radiation since Page's argument. Is this assumption correct, in particular for the entropy fluctuation?

Summary of Results

Averaged entropy is typical, the error is exponentially suppressed:
The deviation from the averaged entropy is exponentially small

$$\delta S_{\mathbf{R}} = \begin{cases} \dfrac{1}{\sqrt{2}e^{S(E)}} & (k \ll e^{S(E)}) \\ \sqrt{\dfrac{1-\frac{e^{S(E)}}{k}}{\pi e^{2S(E)}}} & (k \gg e^{S(E)}) \end{cases}$$



Random state differs from BH after the Page time:
Due to the fluctuation of the gravity Hilbert space dimension, the entropy fluctuation behaves differently

Stronger Measure Concentration of Subsystem Entropy:
Subsystem entropy fluctuation of the random state at large dimensions is much smaller than the upper bound from Levy's lemma

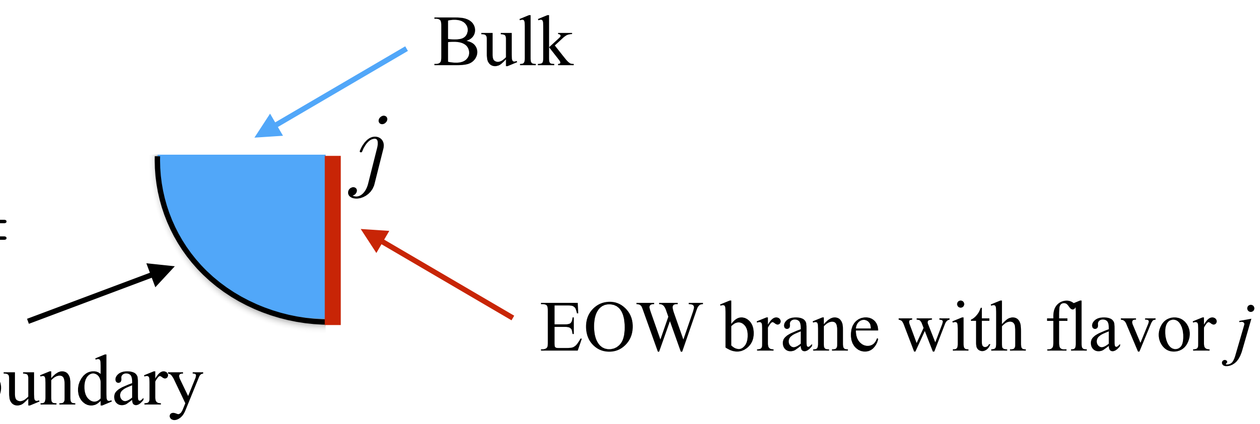
Model of BH evaporation

Model of BH evaporation: [Penington et.al (2019)]

We use 2d gravity model called PSSY model in microcanonical window

- JT gravity + k species of the End-of-the-world branes

$$|j\rangle_{\mathbf{EOW}} =$$



- Radiation and the gravity is naively maximally entangled

$$|\Psi\rangle = \sum_{i=1}^{k} |j\rangle_{\mathbf{R}} |j\rangle_{\mathbf{EOW}}$$

- k: Hilbert space dimension of the Hawking radiation **R**
  e^S: Dimension of the gravity

Page curve:

$$S_{\mathbf{R}} = \begin{cases} \log\, k - \dfrac{k}{2e^{S(E)}} & (k < e^{S(E)}) \\ S(E) - \dfrac{e^{S(E)}}{2k} & (k > e^{S(E)}), \end{cases}$$

Applies to both random state and BH

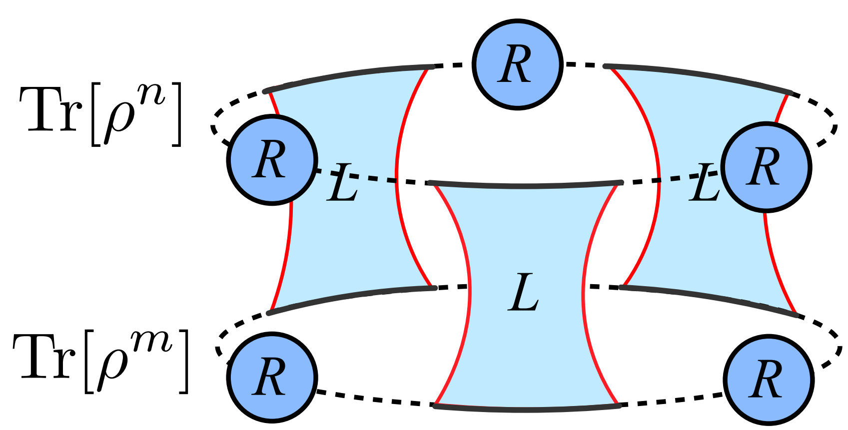
Outline of our method

Method:
By including wormhole contributions, we can compute the average of the entropy fluctuation squared

$$\langle(\delta S)^2\rangle = \frac{\partial}{\partial n}\frac{\partial}{\partial m}\Big|_{n=m=1}\Big(\langle\mathrm{Tr}[\rho^n]\mathrm{Tr}[\rho^m]\rangle - \langle\mathrm{Tr}[\rho^n]\rangle\langle\mathrm{Tr}[\rho^m]\rangle\Big)$$

- We can show it is indeed given by the sum of wormhole contributions

$\langle\mathrm{Tr}[\rho^n]\mathrm{Tr}[\rho^m]\rangle =$



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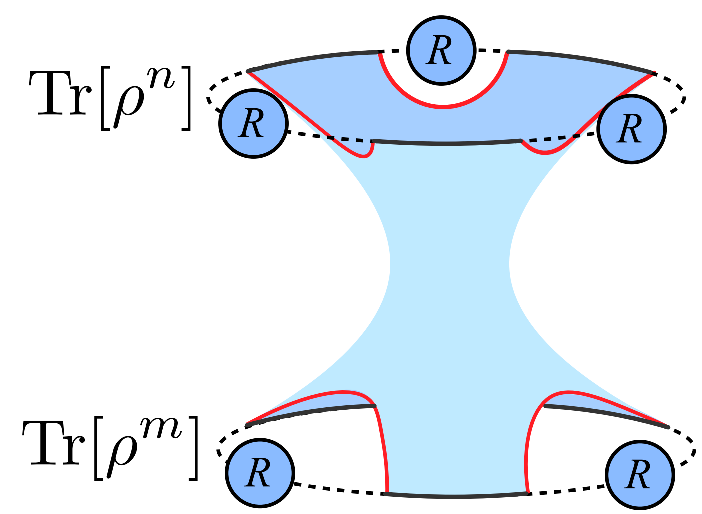
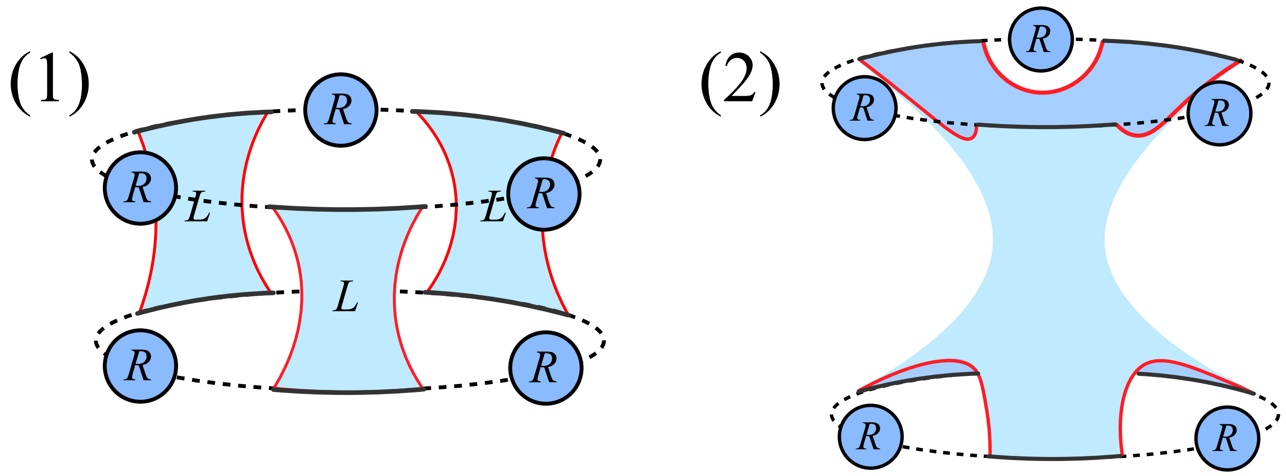


Outline of Our Results

Results:

- We found there are two types of wormholes, corresponding to (1) the random state and (2) the fluctuation of gravity Hilbert space dimension



- The entropy fluctuation squared of the Hawking radiation is given by

$$\delta S_{\mathbf{R}} = \begin{cases} \dfrac{1}{\sqrt{2}e^{S(E)}} & (k \ll e^{S(E)}) \\ \sqrt{\dfrac{1-\frac{e^{S(E)}}{k}}{\pi e^{2S(E)}}} & (k \gg e^{S(E)}) \end{cases}$$

→ <u>Thus the averaged entropy is indeed typical</u>

→ Note that entropy fluctuation of *N* independent systems grows as *N^{1/2}*

Outline of Our Results

Random state:
For random state on $H_A \otimes H_E$ with large dimensions, the entropy fluctuation is

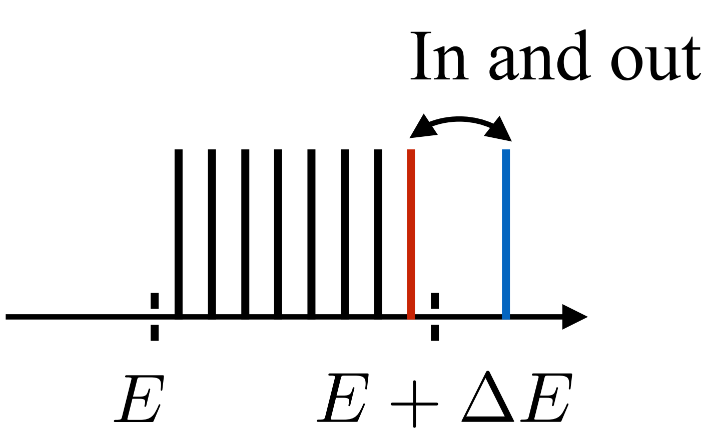
$$\delta S_A = \begin{cases} \frac{1}{\sqrt{2} d_E} & (d_A \ll d_E) \\ \frac{1}{\sqrt{2} d_A} & (d_A \gg d_E). \end{cases}$$

- Suppressed by larger Hilbert space dimension
- After the Page time, the fluctuations of the two are qualitatively different: the fluctuation does not decay at large *k* in gravity

The fluctuation of the gravity Hilbert space:
Number of states in a given gravity theory can fluctuate in an ensemble of theories, by *O(1)*

- It explains entropy fluctuation at late time
- To improve random state, one needs to consider ensemble of tensor products

$$H_A \otimes \left(\oplus_i H_{E_i} \right)$$



Outline of Our Results

Random state:
Entropy fluctuation of the random state with large dimensions is significantly smaller compared to the upper bound from measure concentration

$$\delta S_A = \frac{1}{\sqrt{2} d_E} \leq \frac{(6\pi)^{\frac{3}{2}} \log d_A}{\sqrt{d_A d_E}}$$

$$d_A \ll d_E$$

Previously known bound

$$\Pr\left(|S_A - \bar{S}_A| \geq \eta\right) \leq 2\exp\left(-\frac{d_A d_E \eta^2}{36\pi^3 (\log d_A)^2}\right)$$ [Hayden, Leung, Winter (2006)]

- If the higher point function of entropy can be neglected, thus the distribution is roughly Gaussian, then by using Chernov's bound we obtain a stronger upper bound

$$\Pr\left(|S_A - \bar{S}_A| \geq \eta\right) \overset{?}{\leq} \exp\left(-d_E^2 \eta^2\right)$$

$$d_A \ll d_E$$

Gaussian approximation

Open Questions

Immediate extensions & questions:

- Evaluate fluctuation at intermediate regime (near the Page time)
- Canonical ensemble

Canonical Typicality & Eigenstate Thermalization Hypothesis:

- Our random state result indicates that for typical state on $H_A \otimes H_E$

$d_A \ll d_E$

$$S_A = \bar{S}_A + \frac{1}{d_E} R_i$$

where *R_i* is a real random variable with zero mean and *O(1)* variance

- A manifestation of the canonical typicality [Goldstein et al. (2006)] [Popescu et al. (2006)]
- It would be interesting if we could formulate subsystem ETH [Garrison, Grover (2018)][Lashkari et al. (2018)] for subsystem entropy using our results, including fluctuating term