SPARSE RMT AND SPARSE SYK

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Recent developments

- Random Matrix theory (RMT) is well known to have interesting connections with two-dimensional gravity. An interesting and new version of this connection has been discovered in the study of two-dimensional Jackiw-Teitelboim (Saad, Shenker, Stanford'19).
- It has been shown that the key features of JT gravity are correctly reproduced by the low-energy limit of the Sachdev-Ye-Kitaev (SYK) model (Sachdev, Ye'92, Kitaev'15), which is a quantum mechanical model of N flavors of Majorana fermions with random couplings.
- The Hamiltonian for SYK model with N flavours of Majorana fermions can then be thought of as a $L \times L$ dimensional Hermitian matrix, with $L = 2^{\frac{N}{2}}$, acting on a tensor product of Hilbert spaces of $N/2$ qubits.
- Quite remarkably, this model reproduces many aspects of JT gravity- the pattern of symmetry breaking, the Schwarzian action for the time reparametrizations, the resulting thermodynamics and the behaviour of OTOCs - are all shared by JT gravity. (Maldacena, Stanford '16, Polchinski, Rosenhaus'16).

Motivation

These observations raise several interesting questions:

- How much randomness is needed for agreement with gravity?
- What happens when we start from the GUE and begin reducing the randomness, by decreasing the number of Gaussian random variables?
- Is the resulting behaviour, at low-energies, dependent on only the number of random variables or also on which variables have been retained?
- When do we get the behaviour at low-energies to agree with JT gravity? Etc.

GUE VS SYK

- The partition function for the random matrix theory is given by

$$Z_{RMT} = \int DM \exp\left(-\frac{1}{2\sigma^2}\mathrm{Tr}M^2\right) ,$$

Here M are $L \times L$ Hermitian Matrices and where σ is given by $\sigma = \frac{1}{\sqrt{L}}$

- In the limit $L \to \infty$, the density of states ρ is given by

$$\rho(\lambda) = \frac{L}{2\pi}\sqrt{4-\lambda^2} .$$

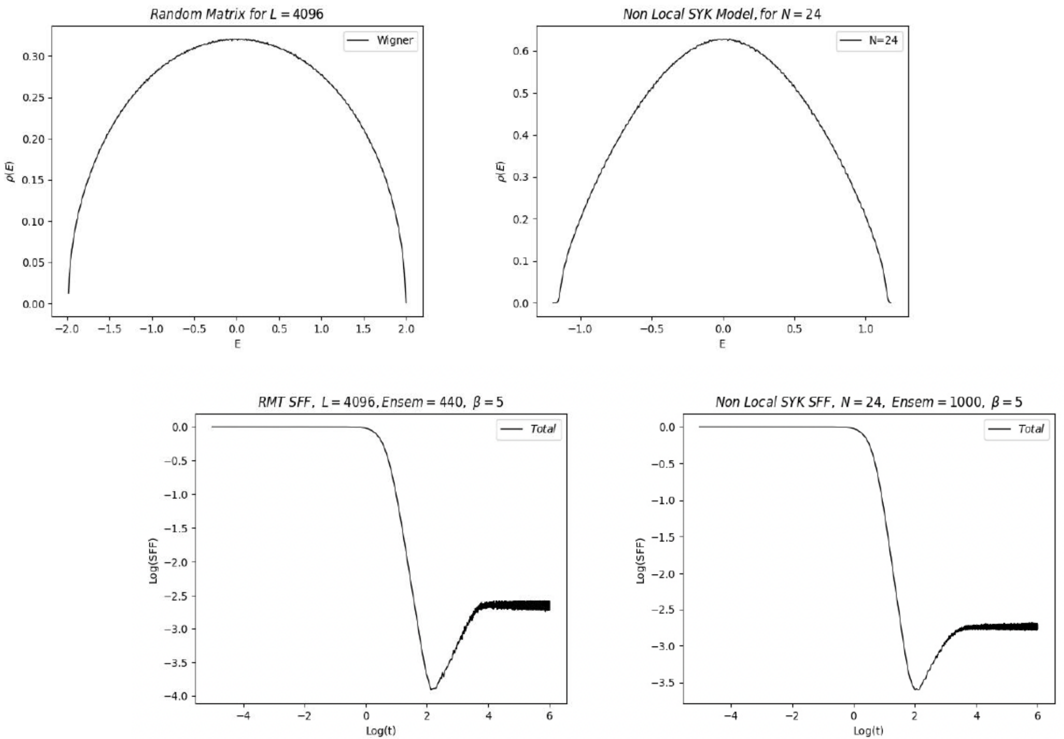
- The Hamiltonian for the SYK is given by

$$H_{\mathrm{SYK}_q} = i^{q/2} \sum_{1 \le i_1 < i_2 < \cdots i_q \le N} j_{i_1 i_2 \cdots i_q} \psi_{i_1} \psi_{i_2} \cdots \psi_{i_q}$$

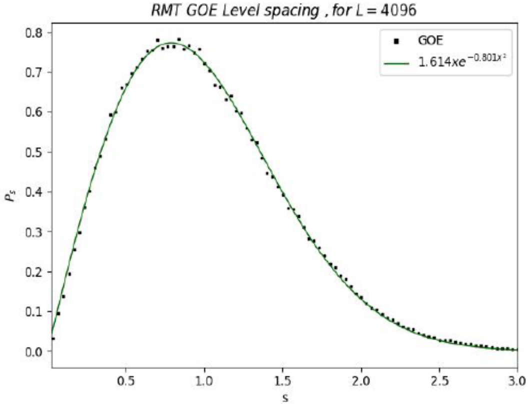
with the couplings $j_{i_1 \ldots i_q}$ drawn from a Gaussian ensemble with variance

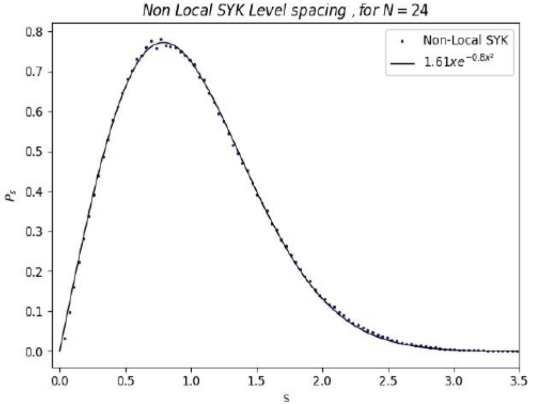
$$\langle j_{i_1 i_2 \cdots i_q}, j_{j_1 j_2 \cdots j_q} \rangle = \frac{(q-1)! J^2}{N^{q-1}} \delta_{i_1, j_1} \delta_{i_2, j_2} \cdots \delta_{i_q, j_q}$$

- The density of states and the spectral form factor of GOE and SYK are



- The level spacing for the nearest neighbor eigenvalues is same,

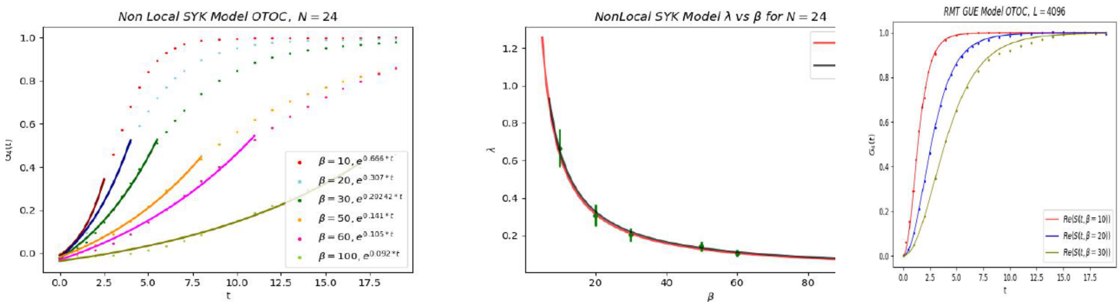




Cotler, Gur-Ari, Hanada, Polchinski, Saad, Shenker, Stanford, Streicher, Tezuka'18

- The OTOC, given by

$$G_4 = \langle \mathrm{Tr}(e^{-\beta H}\psi_i(t)\psi_j(0)\psi_i(t)\psi_j(0))\rangle$$



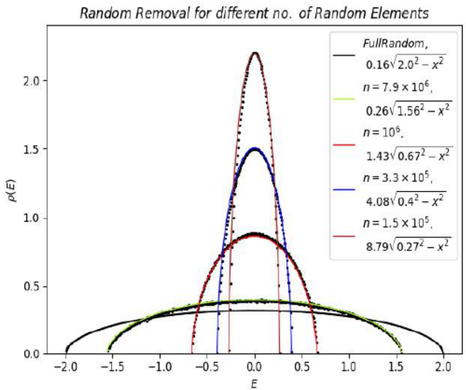
$$G_4^{SYK} \simeq 1 - \frac{1}{N}e^{\lambda_L t},$$

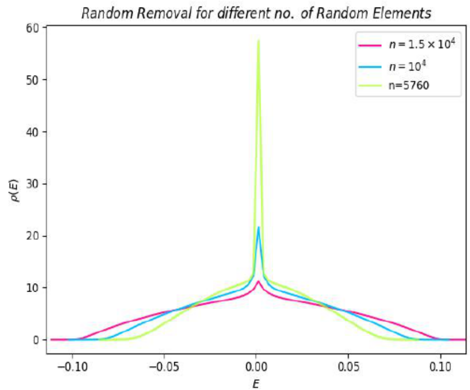
$$G_4^{RMT} \simeq \frac{F^{RMT}(t)}{F^{RMT}(0)}, \quad F^{RMT}(t) = \left| \frac{J_1\left(2(t - i\frac{\beta}{4})\right)}{t - i\frac{\beta}{4}} \right|^4$$

Cotler, Hunter-Jones, Liu, Yoshida ' 17

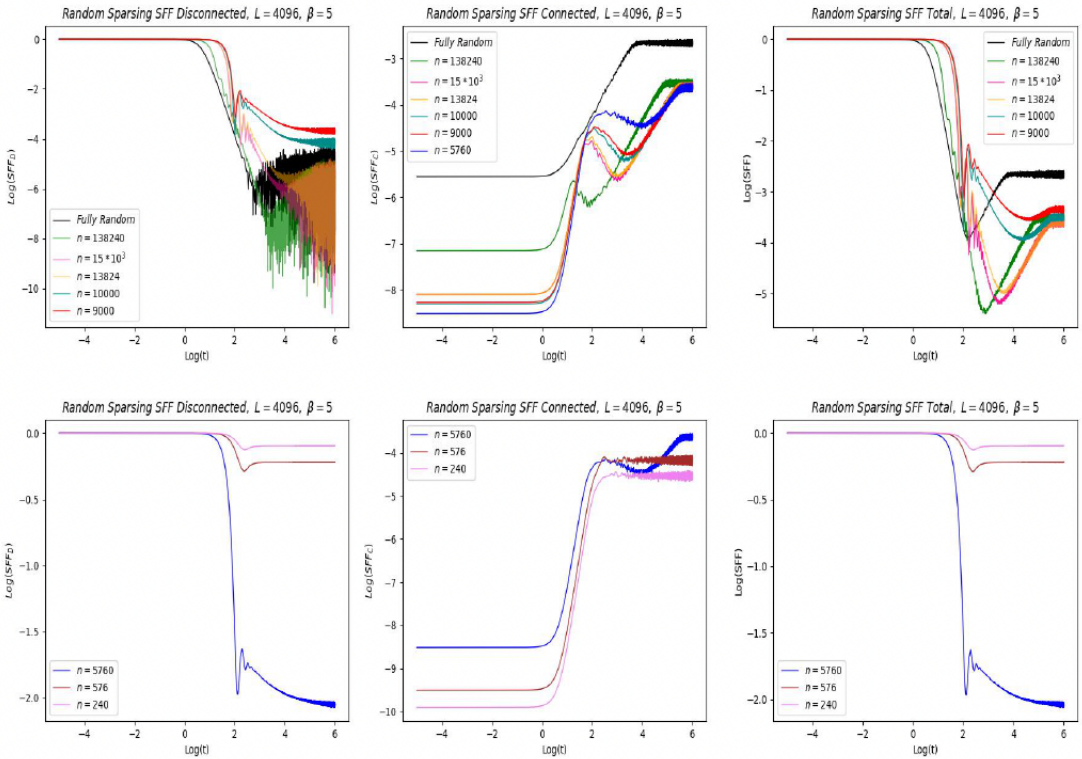
SPARSE RMT

- Explore the consequences of considering random matrices which are not fully random.
- Pick randomly and uniformly, n off-diagonal matrix elements, M_{ij}, with $i < j$.
- The value of each of these matrix elements is chosen independently with its real and imaginary parts being drawn from a Gaussian distribution with variance $\frac{1}{2L}$. All other off-diagonal matrix elements of M are also set to zero.
- Finally the matrix is made Hermitian by taking it to be $\frac{1}{2}(M + M^\dagger)$.
- The density of states is given by

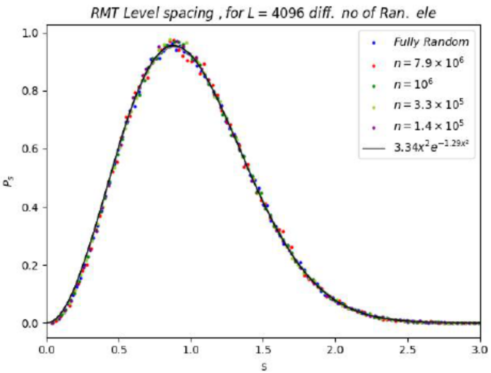


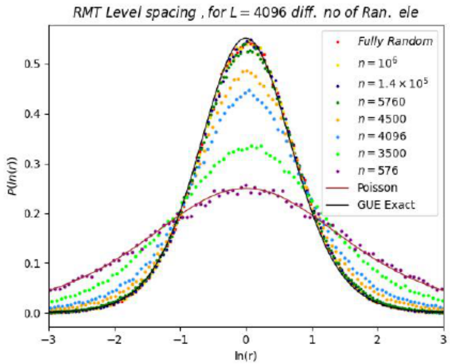


- SFF shows interesting characteristics.



- Level spacing shows that the eigenvalue differences become uncorrelated for sufficiently sparse random matrix theory.



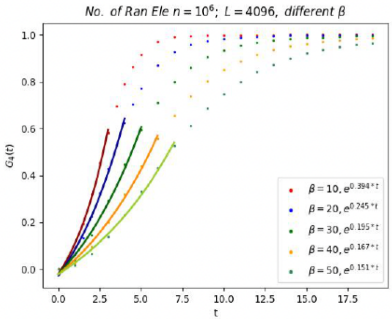


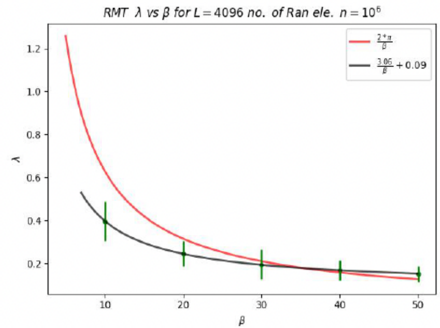
$$\frac{32}{\pi^2} s^2 e^{-\frac{4s^2}{\pi}}$$

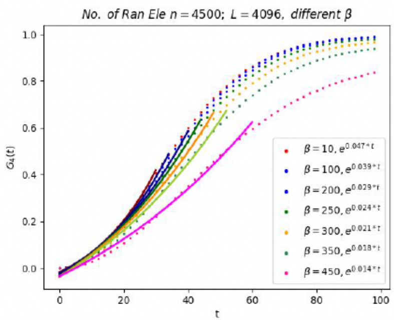
$$P(\ln r) = \frac{1}{K} \frac{r(r+r^2)^\beta}{(1+r+r^2)^{1+\frac{3\beta}{2}}},$$ (You, Ludwig, Xu'16)

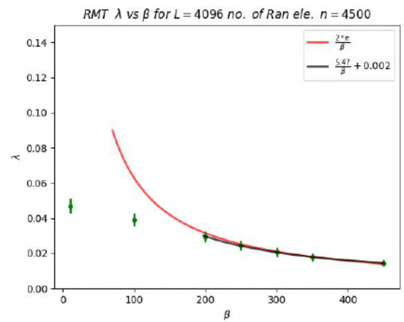
where $r_i = \frac{\Delta\lambda_i}{\Delta\lambda_{i+1}}$

- OTOC show a slower growth, perhaps indicating a possible exponential growth at early times at sufficient sparseness.









Local SYK

- Consider a variation of the SYK model where the number of random couplings is vastly reduced and is only $\mathcal{O}(N)$.
- The Hamiltonian is given by

$$H_{\mathrm{local-SYK}} = i^{q/2} \sum_{i_1, i_2, \cdots i_q} j_{i_1 i_2 \cdots i_q} \psi_{i_1} \psi_{i_2} \cdots \psi_{i_q} \,,$$

$$j_{i_1 i_2 \cdots i_q} = 0 \quad (\text{except } i_1 = i_2 + 1 = i_3 + 2 = \cdots = i_q + q - 1, \\ \text{and} \;\; i_1 = i_2 - 1 = i_3 - 2 = \cdots = i_q - (q-1)),$$

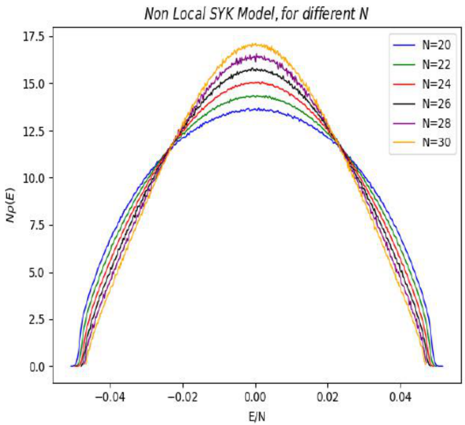
with the fermions satisfying periodic boundary conditions:

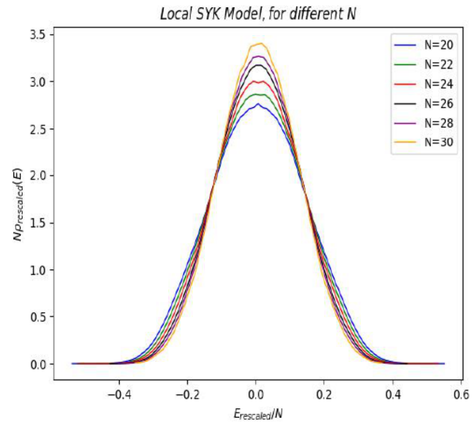
$$\psi_{i=N+1} = \psi_{i=1} \,.$$

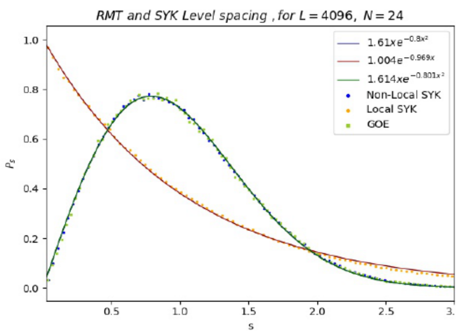
The couplings $j_{i_1 i_2 \ldots i_q}$ present are taken to be random with vanishing mean and variance

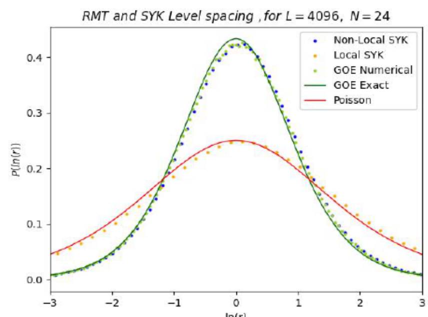
$$\langle j_{i_1 i_2 \cdots i_q} j_{j_1 j_2 \cdots j_q} \rangle = (q-1)! \hat{J}^2 \delta_{i_1, j_1} \delta_{i_2, j_2} \cdots \delta_{i_q, j_q}$$

- The density of states is more narrower and level spacing is Poisson-like.

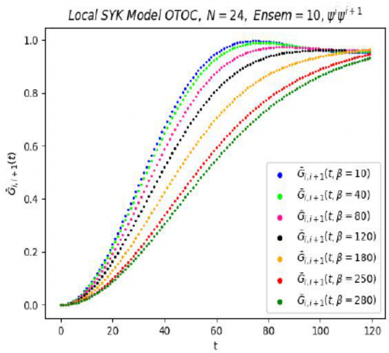




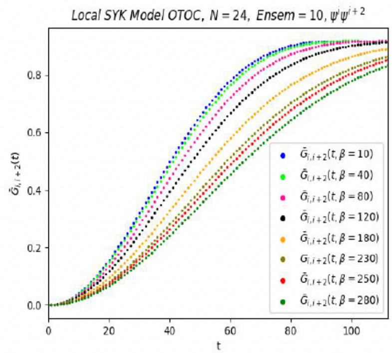




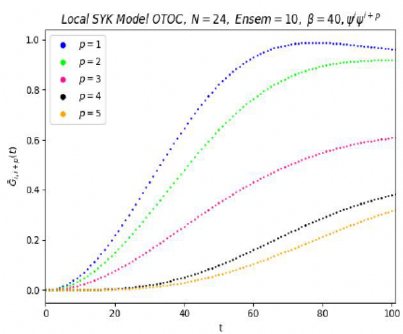
- No good large N limit exits, so Lyapunov exponent is not well defined.

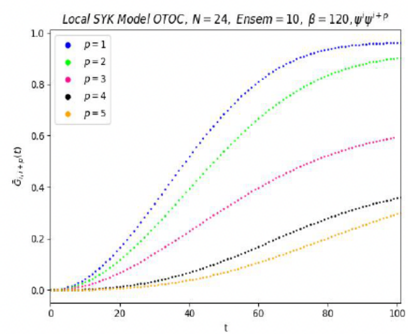


(a) Local SYK, $\bar{G}_{i,i+1}$ vs t



(b) Local SYK, $\bar{G}_{i,i+2}$ vs t





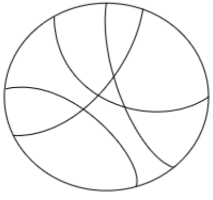
Double-Scaling limit

- Even though the local SYK model is simple enough, the model is not amenable to a saddle point analysis.
- Is there any limit in the space of (q, N) that is analytically tractable? Yes! the double scaling limit.
- For conventional SYK it corresponds to the limit (Erdős, Schröder'14, Cotler, Gur-Ari, Hanada, Polchinski, Saad, Shenker, Stanford, Streicher, Tezuka'18, Berkooz, Prithvi, Simòn '18)

$$q \to \infty, \, N \to \infty, \quad \frac{\lambda}{2} \equiv \frac{q^2}{N} = \text{fixed}$$

- Chord-diagram technique used for computing partition function and matter correlators by evaluating the moments of the form

$$m_k = \mathrm{Tr}(H^k)$$



Triple Scaling limit

- Further taking a triple scaling limit leads to the Schwarzian theory,

$$\lambda \to 0, E \to 0, \quad \frac{E}{\lambda} = \text{fixed}.$$

- Analogous double scaling limit exists for the Local SYK model.

$$q \to \infty, N \to \infty, \quad \frac{q}{N} = \text{fixed}$$

- Further taking a triple scaling limit leads to the Schwarzian theory,

$$\lambda \equiv \frac{q}{N} \to 0.$$

Back to Gaussian ensemble

- An Hermitian matrix H can be thought of as being a vector in an L^2 dimensional Hilbert space $\mathcal{H}$ of $L \times L$ Hermitian matrices. H can be expanded in any basis of $\mathcal{H}$,

$$H = \sum_{a=1}^{L^2} c_a T^a,$$

- Under a change of basis the T^a's, and H, transform by conjugation, $T^a \to UT^aU^\dagger, H \to UHU^\dagger$ where U is the $L \times L$ unitary matrix specifying the change of basis.
- Different basis can be used. (1) A standard basis corresponding to root vectors of $U(L)$. (2) A basis made out of tensor products of Pauli matrices. (3) A basis made out of products of N flavours of Majorana fermions.
- Interesting to consider the breaking of the $U(L)$ symmetry, say to $U(M) \times U(L-M)$ by

$$\frac{\mathrm{Tr}H^2}{2\sigma^2} \to \sum_{I,J=1}^{M} \frac{|H_{IJ}|^2}{2\sigma_1^2} + \sum_{\alpha,\beta=M+1}^{L} \frac{|H_{\alpha\beta}|^2}{2\sigma_2^2} + \sum_{I=1}^{M} \sum_{\alpha=M+1}^{L} \frac{|H_{I\alpha}|^2}{2\sigma_3^2}.$$

- Alternatively, can consider the class of $q = 0 \pmod 4$ terms of the SYK class.