GRADUATE SCHOOL OF / FACULTY OF SCIENCE KYOTO UNIVERSITY

Yoshifumi Nakata and MT:
arXiv:2303.02010

Extreme Universe
A New Paradigm for Spacetime and Matter from Quantum Information
Grant-in-Aid for Transformative Research Areas (A)

Hayden-Preskill Recovery in Sachdev-Ye-Kitaev type models and spin chains

Recent Developments in Quantum Physics of Black Holes 2023

YITP, Kyoto Univ.

15:00 – 15:15, 4 April 2023

Masaki TEZUKA (Dept. of Physics, Kyoto Univ.)

Scrambling and Hamiltonian dynamics

Information delocalization in black holes

e.g. Hayden-Preskill protocol

Holography to black holes

Maximally chaotic systems

e.g. Sachdev-Ye-Kitaev model

Haar random unitary: not realized by $\{e^{itH}\}$

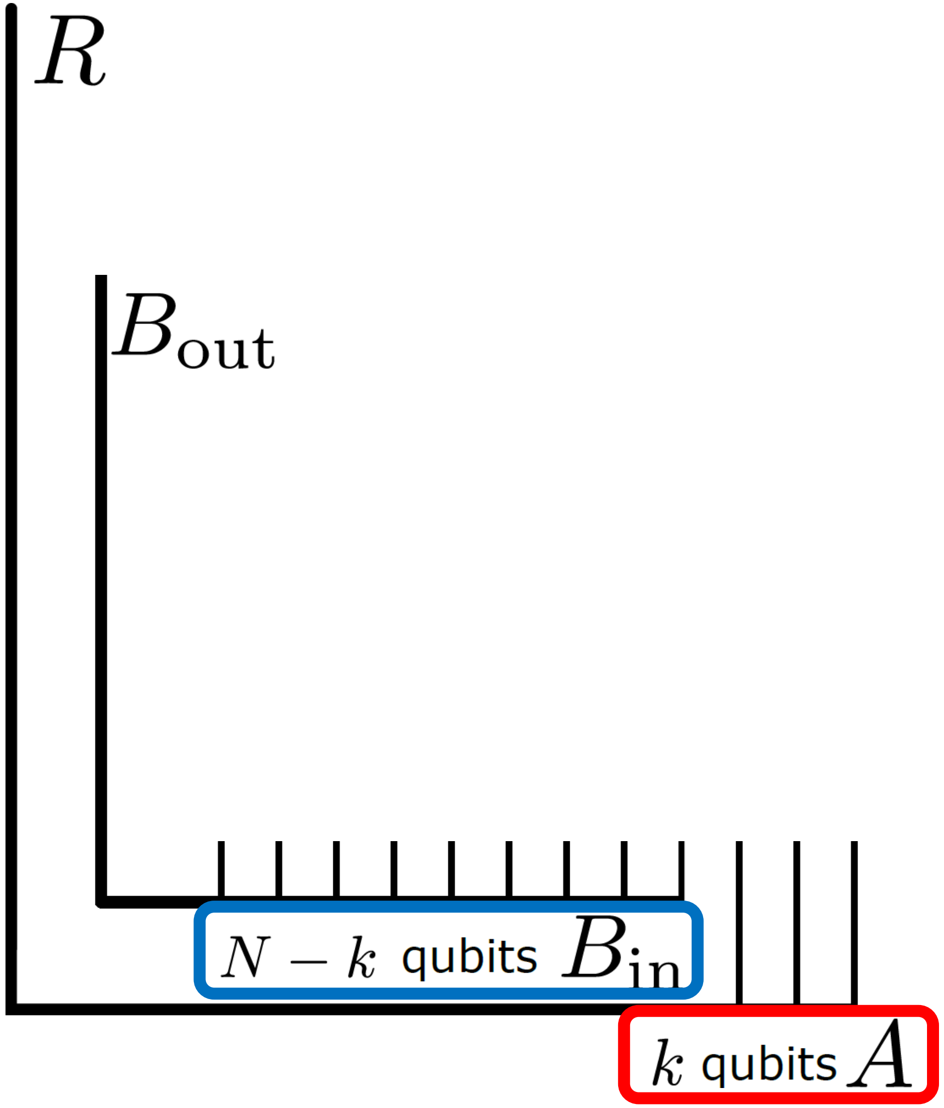
[Roberts and Yoshida, JHEP 2017]

This work:

Quantitatively study scrambling by Hamiltonian dynamics

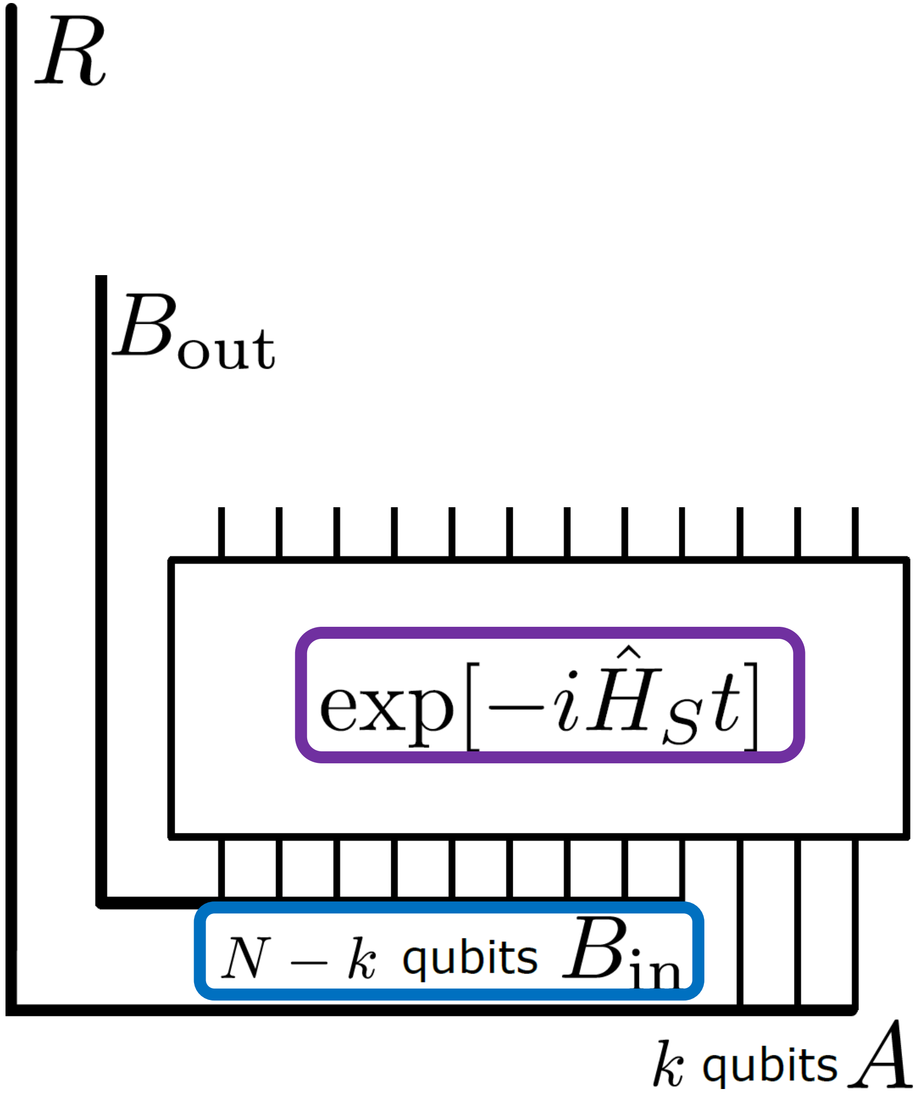
- **SYK model and its variants (sparse, add mass term)**
- **One-dimensional spin chains (Ising, Heisenberg)**

Quantum error correction: The Hayden-Preskill protocol



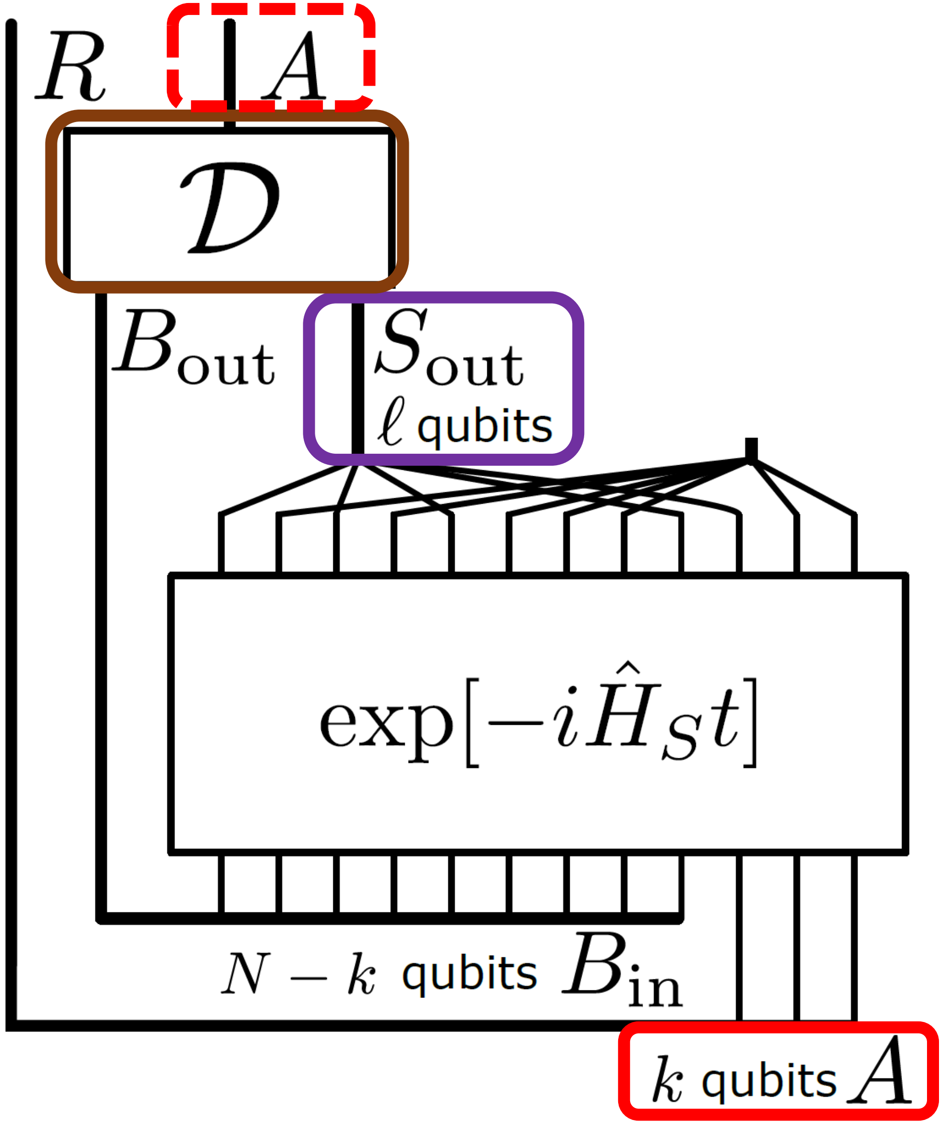
- Alice: throws k-qubit quantum information A into a box B_{in}

Quantum error correction: The Hayden-Preskill protocol



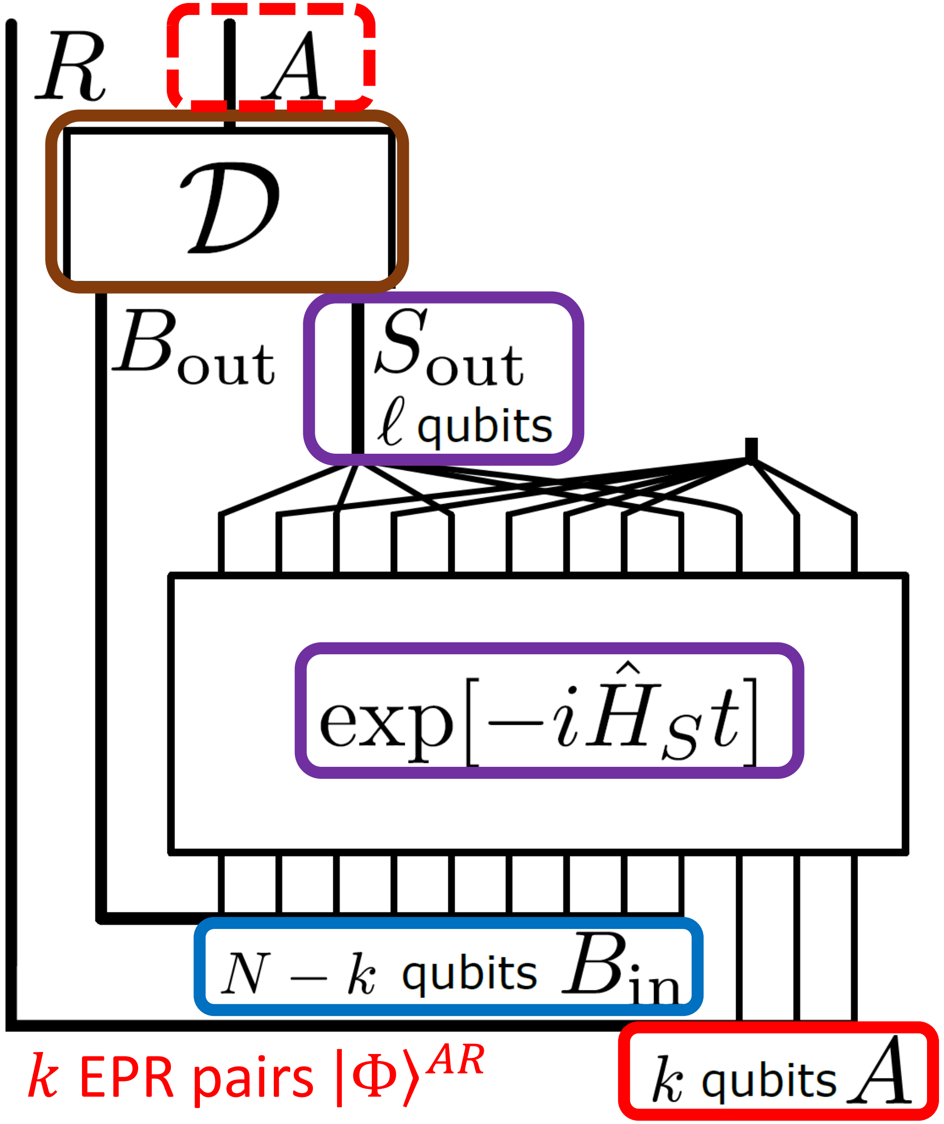
- Alice: throws k-qubit quantum information A into a box B_{in}
- Bob: knows the original state of B_{in} and the Hamiltonian $\hat{H}_S$ of $S = A + B_{\text{in}}$

Quantum error correction: The Hayden-Preskill protocol



- Alice: throws k-qubit quantum information A into a box $B_{\rm in}$
- Bob: knows the original state of $B_{\rm in}$ and the Hamiltonian $\widehat{H}_S$ of $S = A + B_{\rm in}$
- Bob obtains ℓ qubits $S_{\rm out}$ after time t. Can Bob decode ($\mathcal{D}$) Alice's secret?

Quantum error correction: The Hayden-Preskill protocol



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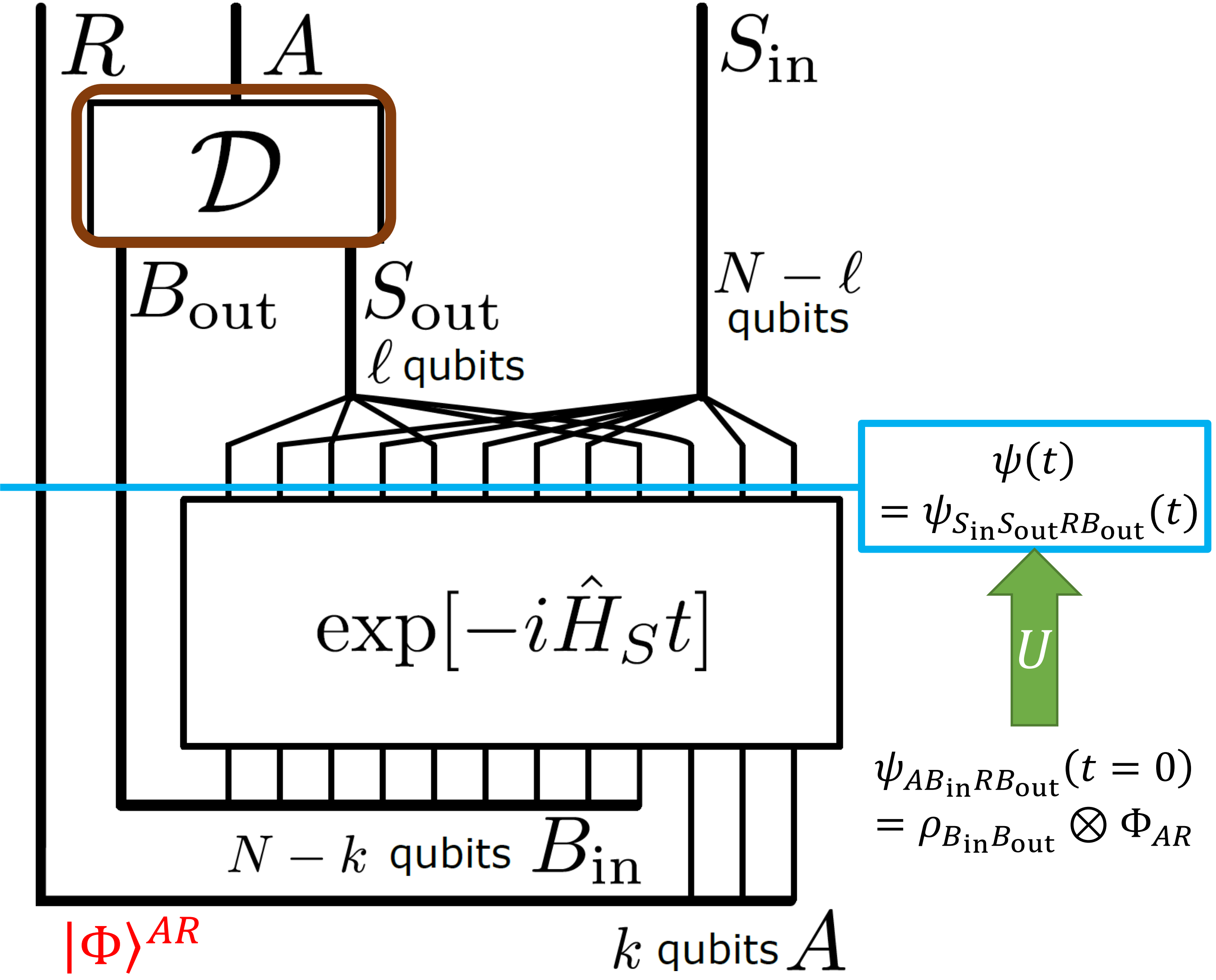
Black holes: information recovery for $\ell \sim k$

[Hayden and Preskill, JHEP 2007]

Circular unitary (Haar) ensemble was assumed

Quantum error correction: The Hayden-Preskill protocol



Recovery error $\Delta_{\hat{H}}(t,\beta)$ among any $\mathcal{D}$ is hard to compute…

Decoupling approach

For $\mathcal{D}$ to succeed, no correlation is allowed between S_{in} and R

$$\rho_{S_{\mathrm{in}}R} = \mathrm{Tr}_{B_{\mathrm{out}},S_{\mathrm{out}}} |\psi(t)\rangle\langle\psi(t)|$$

Decoding error estimate

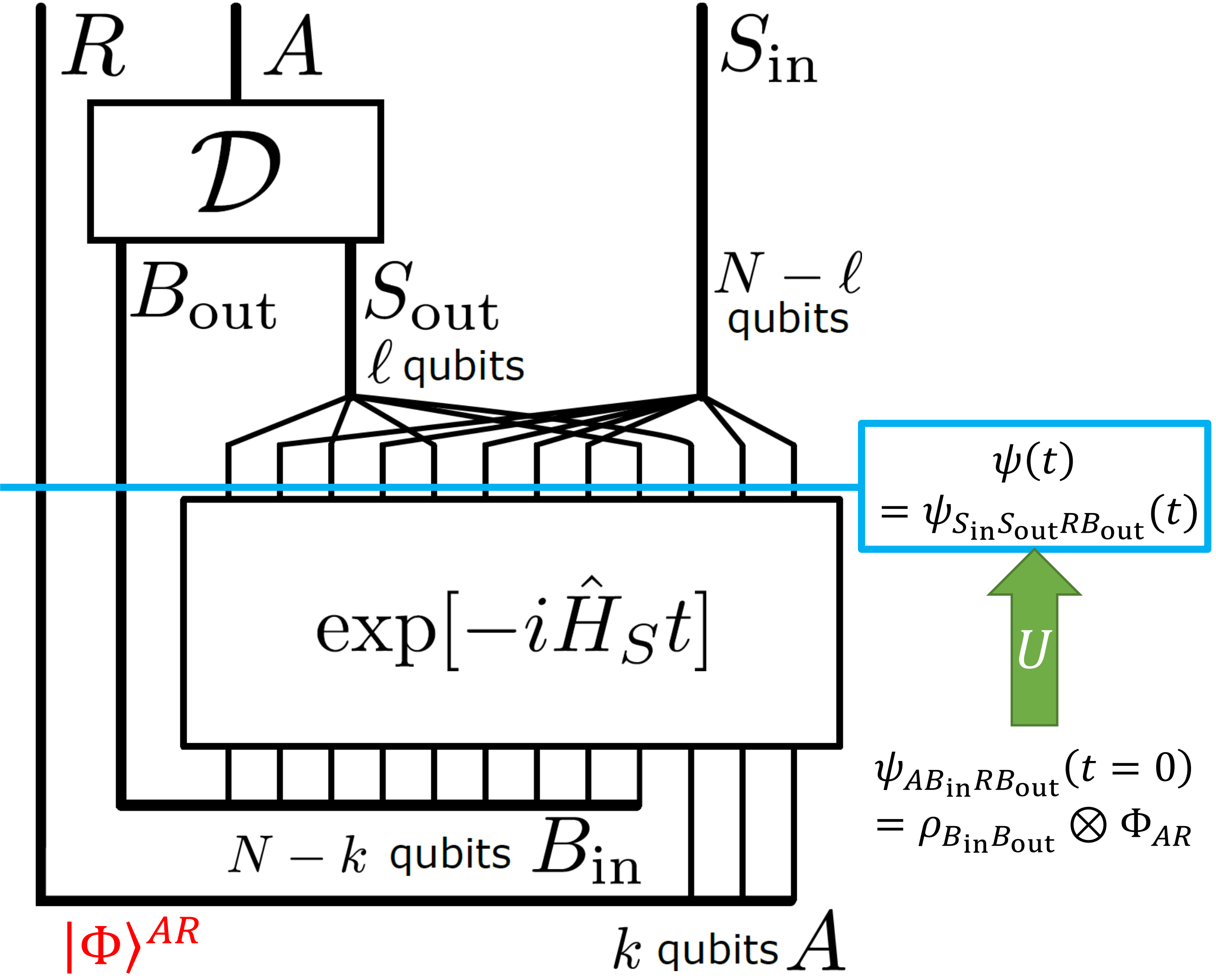
$$\overline{\Delta}_{\hat{H}}(t,\beta) \equiv \min\left\{1, \sqrt{\left\| \rho_{S_{\mathrm{in}}R} - \rho_{S_{\mathrm{in}}} \otimes \frac{I_R}{d_R} \right\|_1}\right\}$$

$$(\geq \Delta_{\hat{H}}(t,\beta))$$

$$\rho_{S_{\mathrm{in}}} = \mathrm{Tr}_R \rho_{S_{\mathrm{in}}R}$$

$$\|M\|_1 \equiv \mathrm{Tr}\sqrt{M^\dagger M}$$

Quantum error correction: The Hayden-Preskill protocol



<u>Haar random unitary case</u>:

$$\bar{\Delta}_{\rm Haar}(\beta) = \min\left\{1, 2^{\frac{1}{2}(\ell_{\rm Haar,th}(\beta)-\ell)}\right\}$$

$$\ell_{\rm Haar,th}(\beta) = \frac{N+k-H(\beta)}{2} \xrightarrow{\beta\to 0} k$$

$\bar{\Delta}_{\rm Haar}$ exponentially decreases as function of ℓ after $\ell \approx k$ [HP recovery]

P. Hayden and J. Preskill, JHEP 2007

Our numerical study:

- **SYK-type Hamiltonians**
- **One-dimensional spin chains**

➔ Characterization of chaotic Hamiltonian dynamics

The SYK model

[Kitaev 2015][Sachdev & Ye 1993]

$$\hat{H}_{\mathrm{SYK4}} = \sum_{1 \le a < b < c < d \le 2N_{\mathrm{q}}} J_{abcd} \hat{\chi}_a \hat{\chi}_b \hat{\chi}_c \hat{\chi}_d$$

J_{abcd} : Gaussian random couplings

$\hat{\chi}_{a=1,2,\ldots,2N_{\mathrm{q}}}$: Majorana fermions

- Maximally chaotic at low T [Maldacena, Shanker, and Stanford 2016] ($\lambda_{\mathrm{Lyapunov}} \to 2\pi k_{\mathrm{B}} T/\hbar$: chaos bound)
- Correspondence to 1+1d gravity, random matrix

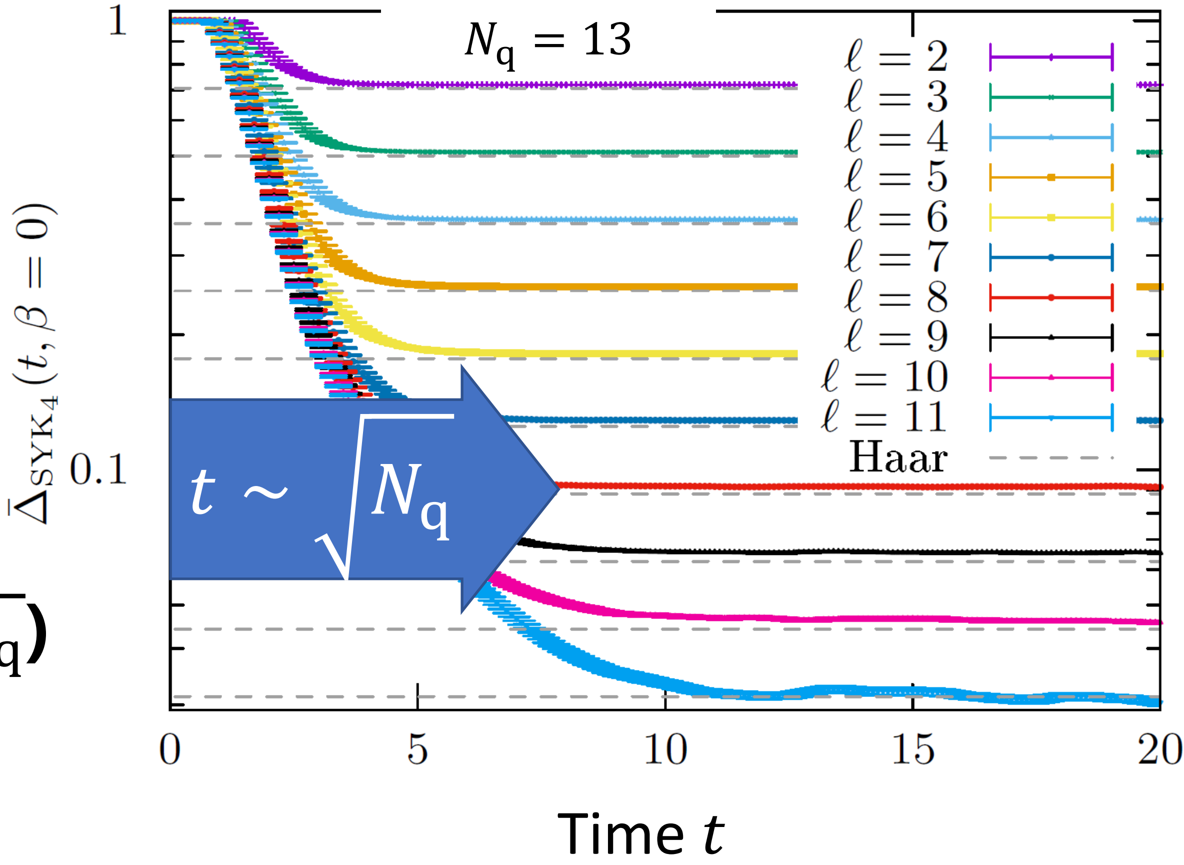
$\overline{\Delta}$ reaches the Haar value quickly ($t \sim \sqrt{N_{\mathrm{q}}}$)

Normalization here: Half-bandwidth $\sqrt{\frac{\langle \mathrm{Tr}\, \hat{H}^2 \rangle}{2^{N_{\mathrm{q}}}}} = 1.$

If we choose to scale $\overline{J_{abcd}^2} \propto N_q^{-3}$, $t \sim$ const.

(cf. fast scrambling conjecture: $t = \mathcal{O}(\ln N_{\mathrm{q}})$)

[arXiv:2303.02010]



Sparse (or pruned) SYK

$$\hat{H} = \sum_{a<b<c<d} x_{abcd} J_{abcd} \hat{\chi}_a \hat{\chi}_b \hat{\chi}_c \hat{\chi}_d \,, x_{abcd} = \begin{cases} 1 & \text{(probability } p) \\ 0 & \text{(probability } 1-p) \end{cases}, P(J_{abcd}) = \frac{\exp\left(-\frac{J_{abcd}^2}{2J^2}\right)}{\sqrt{2\pi J^2}}$$

$K_{\mathrm{cpl}} = \binom{2N_{\mathrm{q}}}{4} p$: Number of non-zero x_{abcd}

$K_{\mathrm{cpl}} \sim \mathcal{O}(1) N_{\mathrm{q}}$ enough for
- Random matrix-like behavior
- Large entropy per fermion at low T !

$$p \sim \frac{4!}{(2N_{\mathrm{q}})^3} = \mathcal{O}(N_{\mathrm{q}}^{-3})$$

- Talk by Brian Swingle (September 2019)
- A. M. García-García, Y. Jia, D. Rosa, J. J. M. Verbaarschot, Phys. Rev. D **103**, 106002 (2021)
- S. Xu, L. Susskind, Y. Su, and B. Swingle, arXiv:2008.02303

M. Tezuka, O. Oktay, E. Rinaldi, M. Hanada, and F. Nori, PRB **107**, L081103 (2023) (arXiv:2208.12098)

Sparse (or pruned) SYK with interaction $= \pm 1$

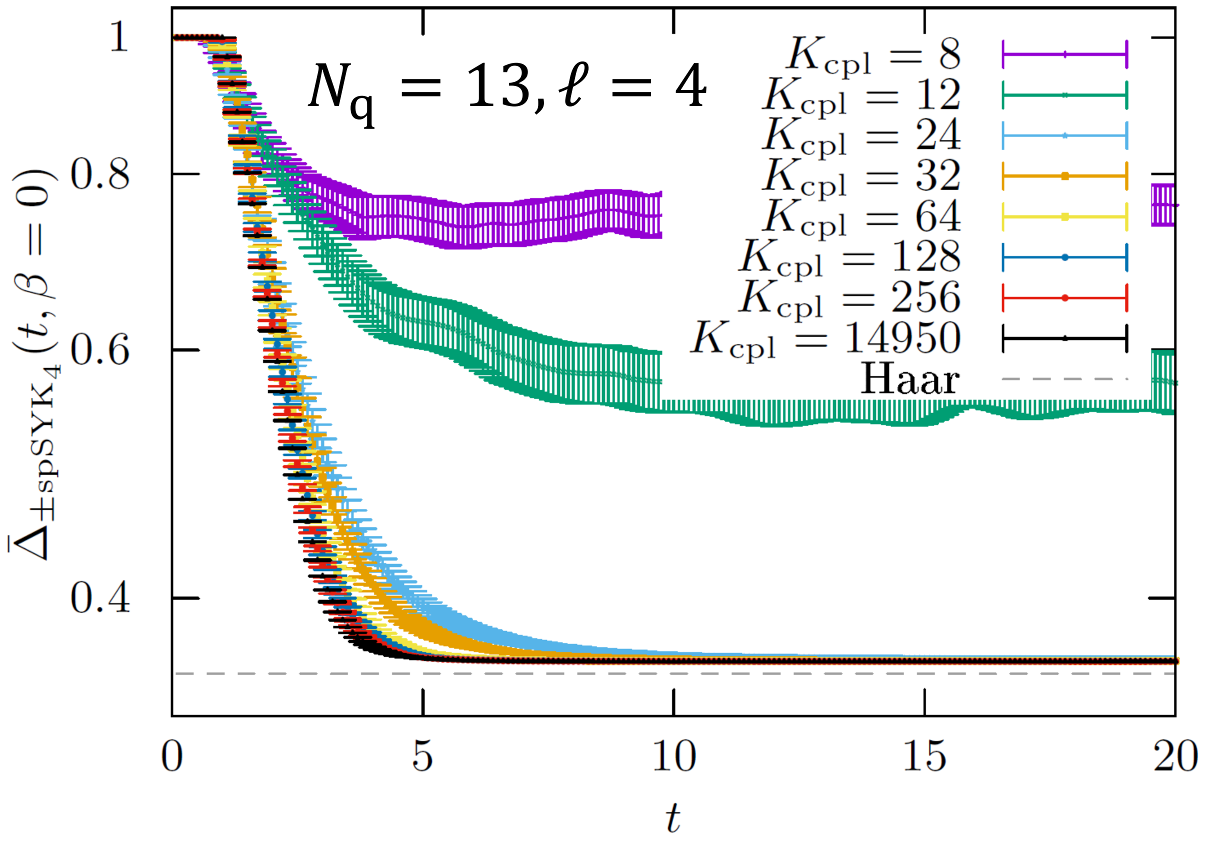
$$\hat{H}_{\pm\mathrm{spSYK4}} \propto \sum_{1\leq a<b<c<d\leq 2N_q} x_{abcd}\hat{\chi}_a\hat{\chi}_b\hat{\chi}_c\hat{\chi}_d\,, x_{abcd} = \begin{cases} 1 & \text{(probability } p/2) \\ -1 & \text{(probability } p/2) \\ 0 & \text{(probability } 1-p) \end{cases}$$

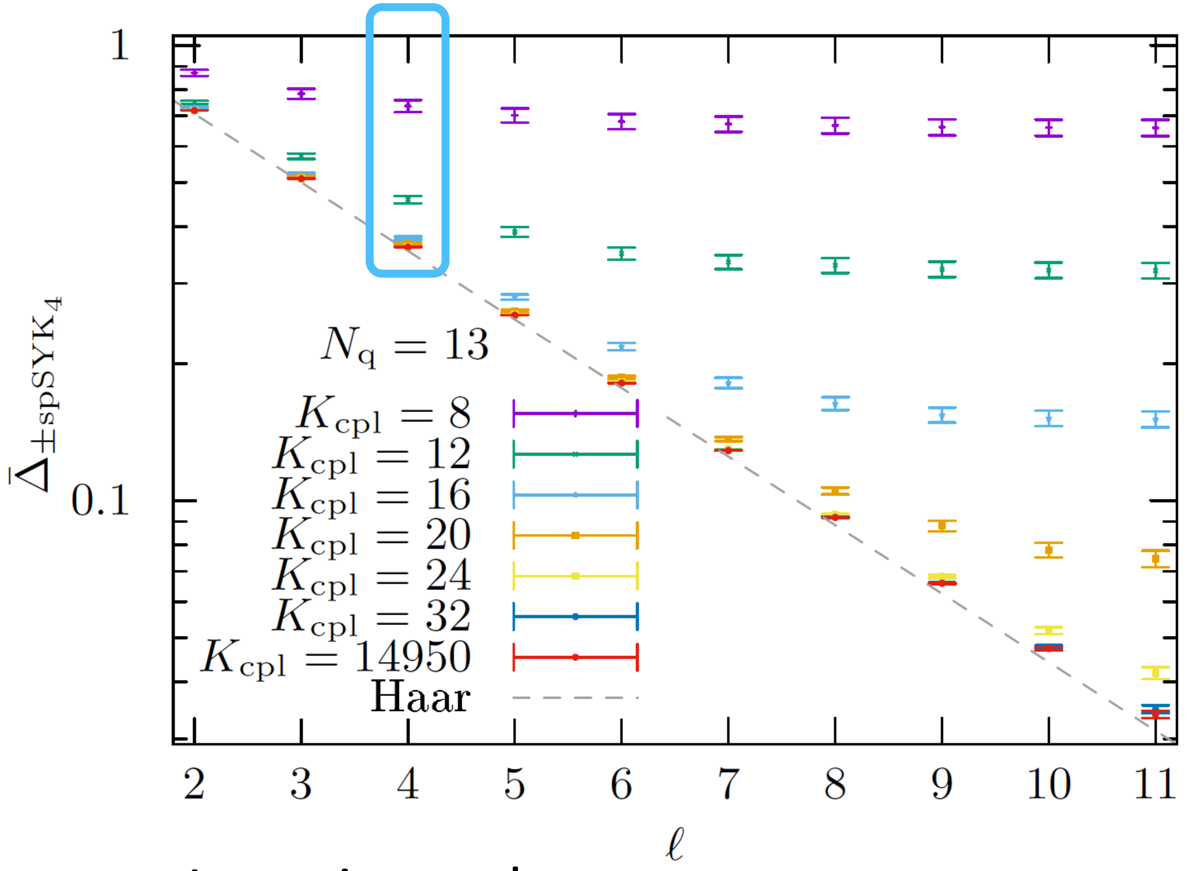
Random-matrix statistics for $K_{\mathrm{cpl}} = \binom{2N_\mathrm{q}}{4} p \gtrsim 2N_\mathrm{q}$.

x_{abcd} can be taken to be +1 at finite $p \ll 1$ (unary sparse SYK) [see appendix of our papar], where for $p = 1$, the model is not chaotic [Lau, Ma, Murugan, and Tezuka, J. Phys. A. 2021]

cf. for non-sparse (all-to-all coupling) SYK,
Non-Gaussian disorder average has been studied in [Krajewski, Laudonio, Pascalie, and Tanasa, PRD 2019];
Possibility of ± 1 couplings mentioned already in Kitaev's talk (2015)

$\overline{\Delta}_{\hat{H}}(t,\beta)$ for binary-coupling sparse SYK



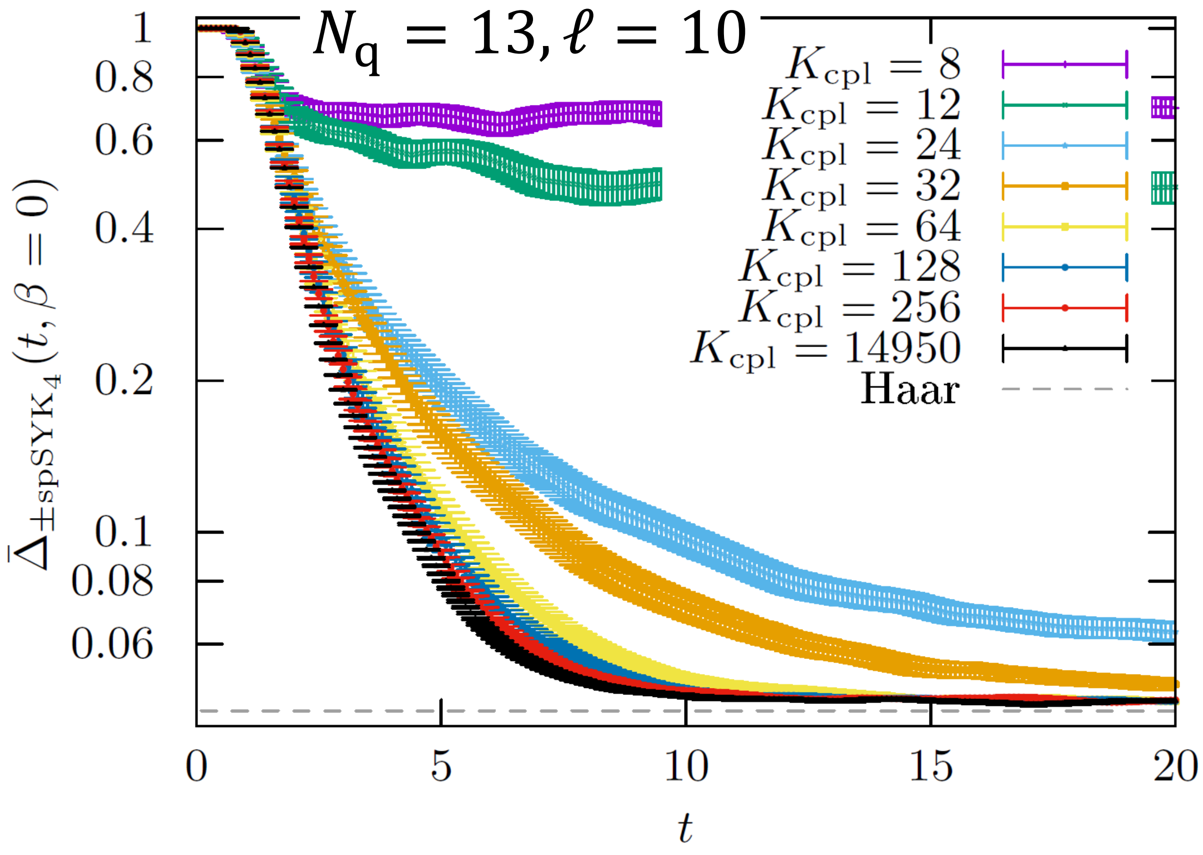


Time dependence:
approach (binary-coupling & Gaussian) dense model as K_{cpl} is increased

Late-time value:

very close to the Haar value $2^{\frac{1-\ell}{2}}$, indistinguishable for $K_{cpl} \gtrsim 3N_q$

$\bar{\Delta}_{\widehat{H}}(t,\beta)$ for binary-coupling sparse SYK



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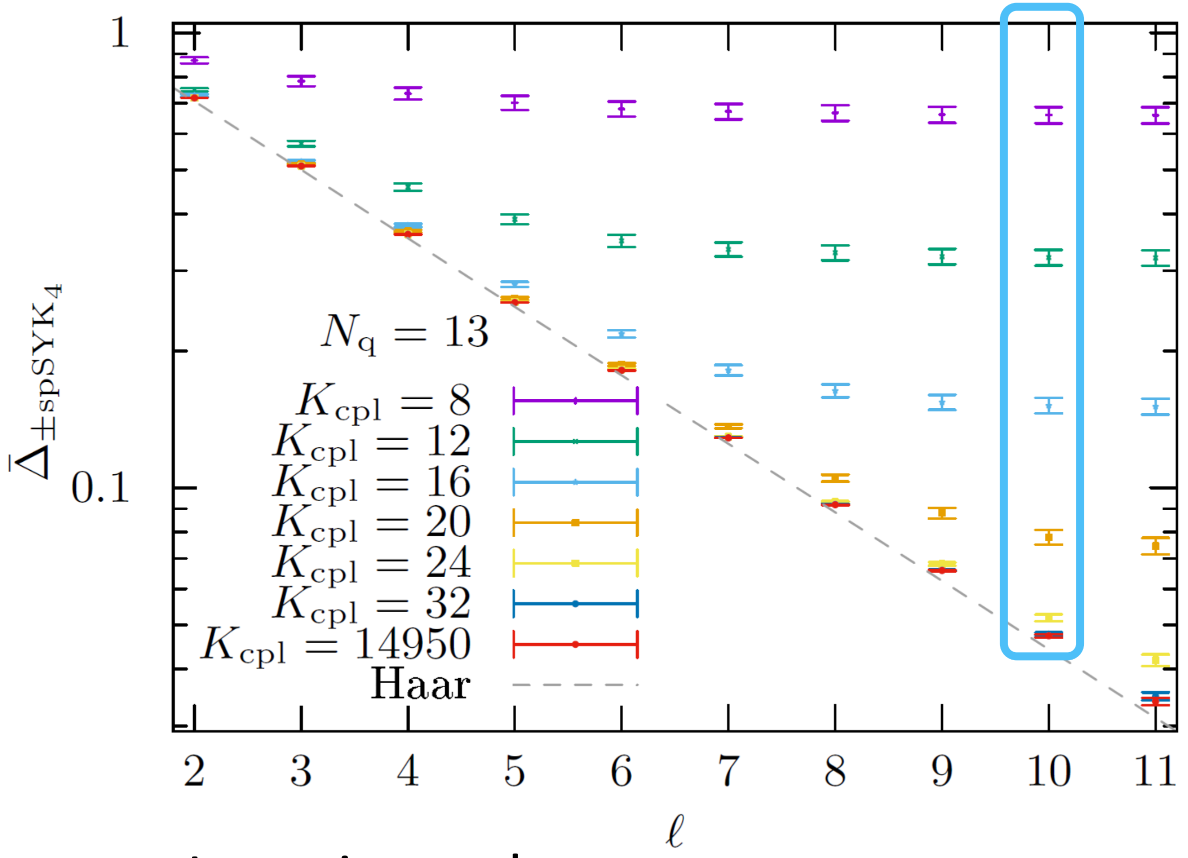

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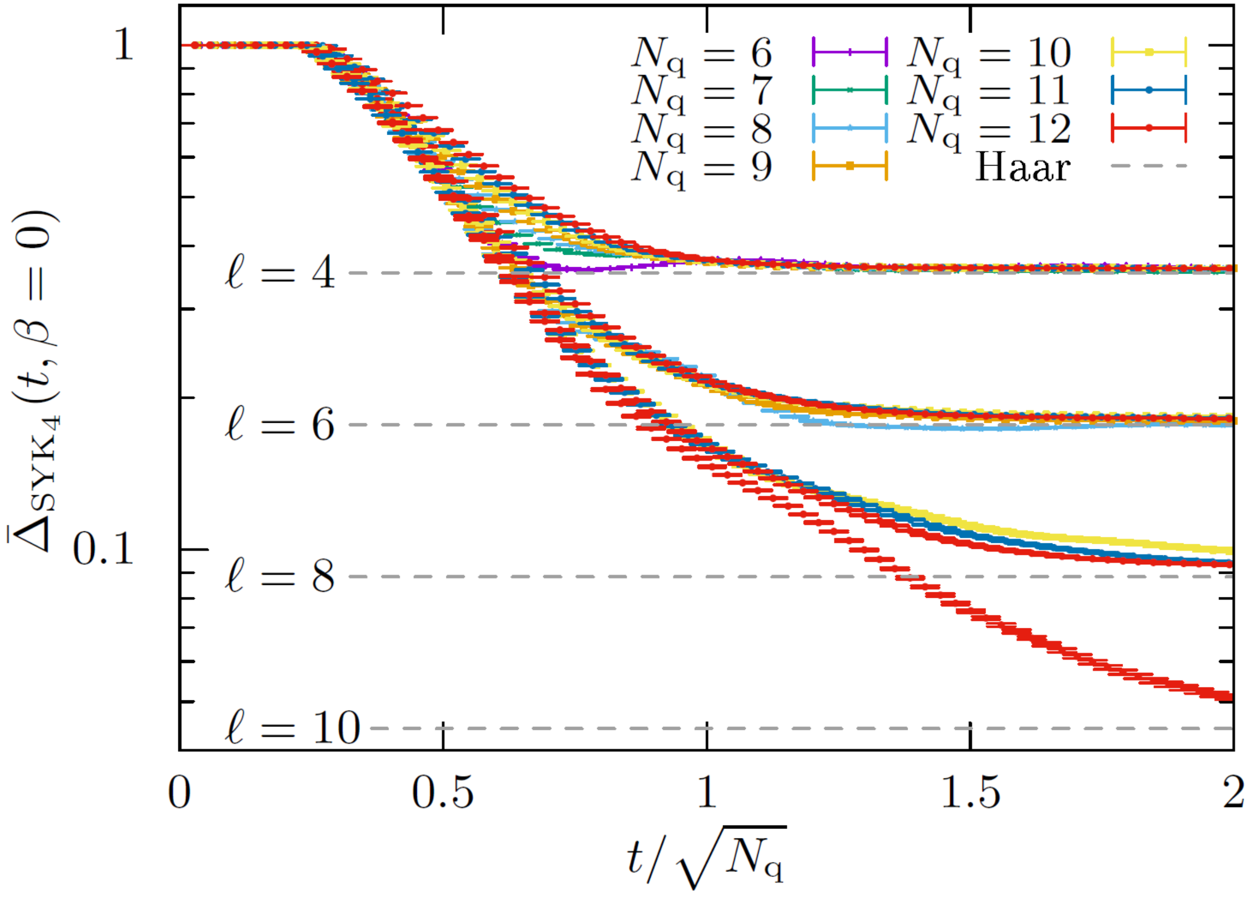
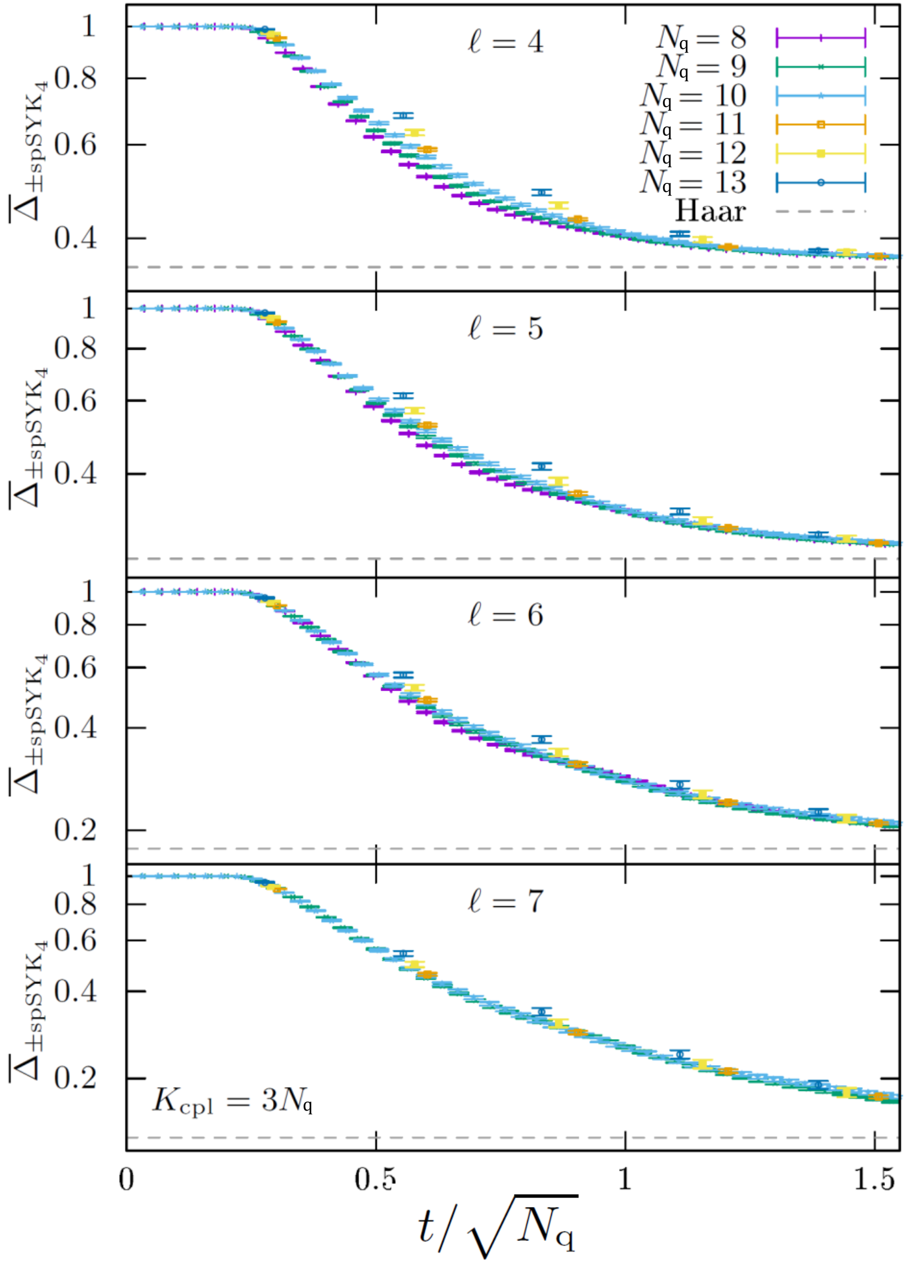
Scaling





Normalization: SYK half-bandwidth $\sqrt{\frac{\langle \mathrm{Tr}\,\hat{H}^2 \rangle}{2^{N_q}}} = 1$, $\hbar = 1$

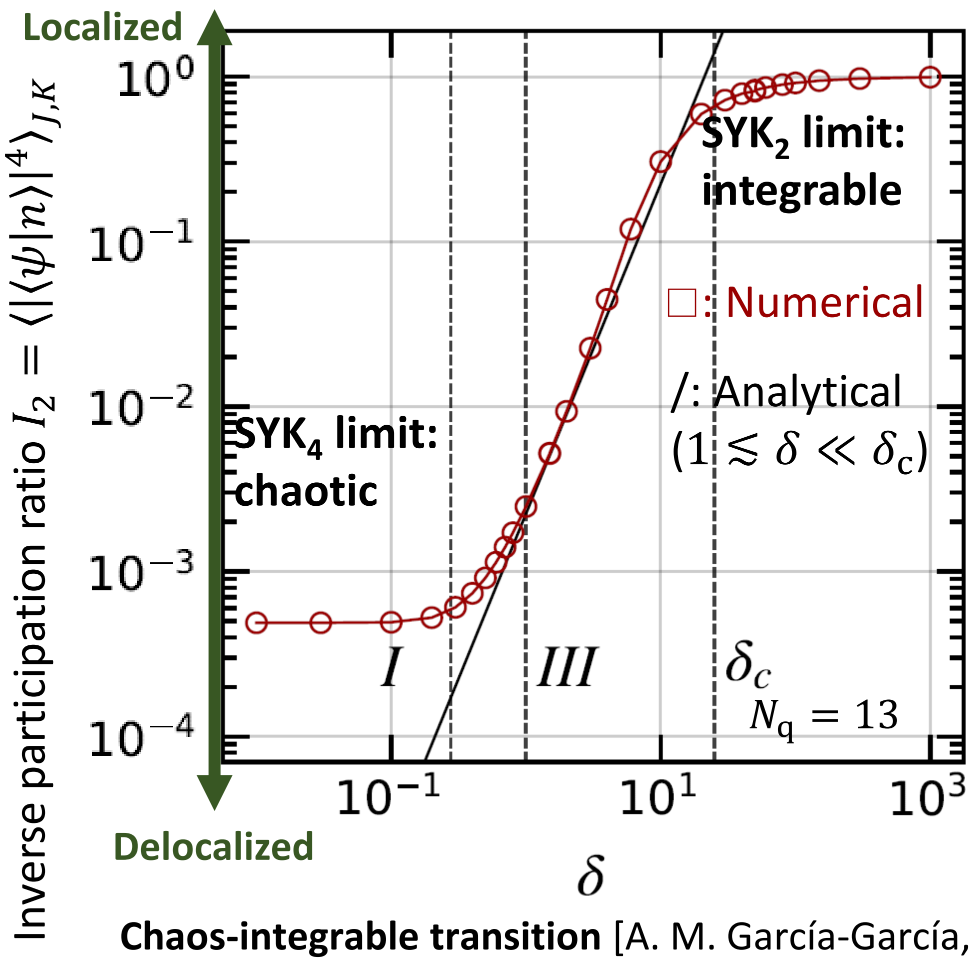
- The Haar value $\bar{\Delta} = 2^{\frac{k-\ell}{2}}$ is reached after $t \sim \mathcal{O}\left(\sqrt{N_q}\right)$

SYK_{4+2}

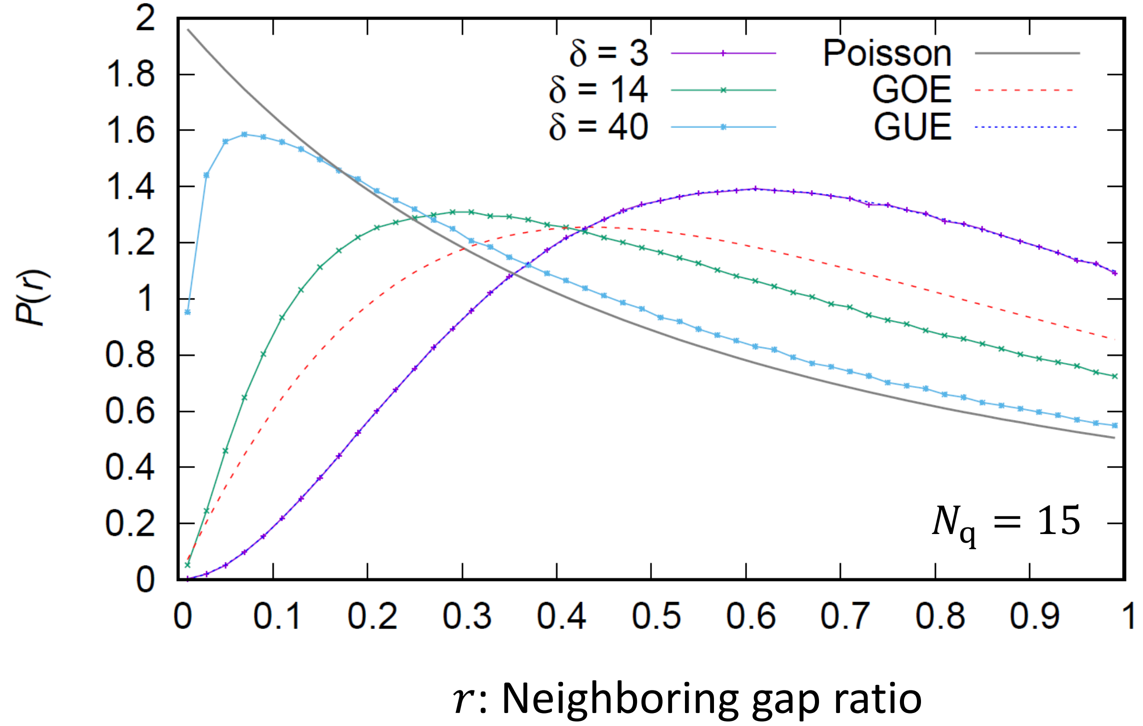
$$\hat{H} = \sum_{1\le a<b<c<d}^{N_{\mathrm{Maj}}=2N} J_{abcd}{}'\hat{\chi}'_a\hat{\chi}'_b\hat{\chi}'_c\hat{\chi}'_d + i\sum_{1\le a<b}^{N_{\mathrm{Maj}}} K_{ab}\hat{\chi}'_a\hat{\chi}'_b = -\sum_{1\le a<b<c<d}^{2N} J_{abcd}\hat{\chi}_a\hat{\chi}_b\hat{\chi}_c\hat{\chi}_d + \sum_{1\le j\le N}^{N} v_j(2\hat{n}_j - 1)$$

Normalization of J_{abcd}, v_j (mass of complex fermion $(\hat{\chi}_{2j-1} + i\hat{\chi}_{2j})/\sqrt{2}$):
SYK_4 bandwidth = 1, width of v_j distribution = δ

Eigenstate localization in the Fock space



Spectral correlation



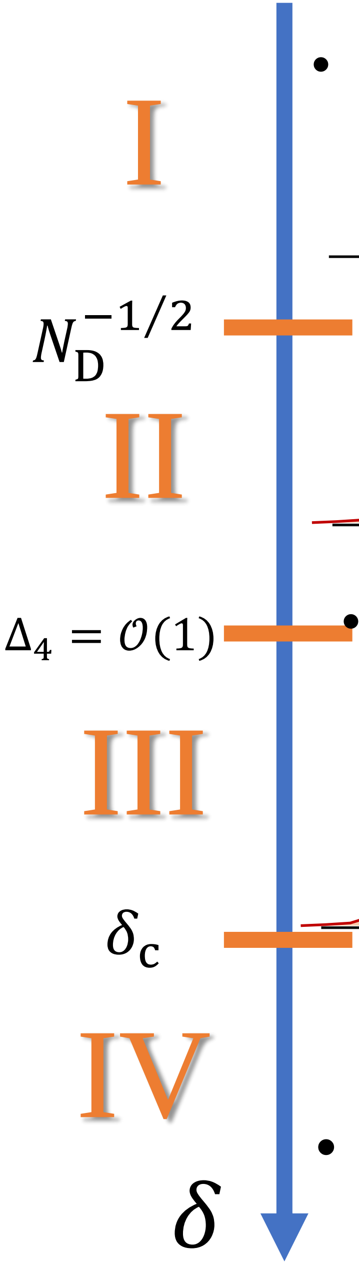
Random-matrix like even when eigenstates are nearly localized in the Fock space for $\delta > 1$

Chaos-integrable transition [A. M. García-García, A. Romero-Bermúdez, B. Loureiro, and MT, PRL **120**, 241603 (2018)]
Localization in many-body Fock-space [F. Monteiro, T. Micklitz, MT, and A. Altland, PRResearch **3**, 013023 (2021)]
Quantum ergodicity before localization [F. Monteiro, MT, A. Altland, D. A. Huse, and T. Micklitz, PRL **127**, 030601 (2021)]

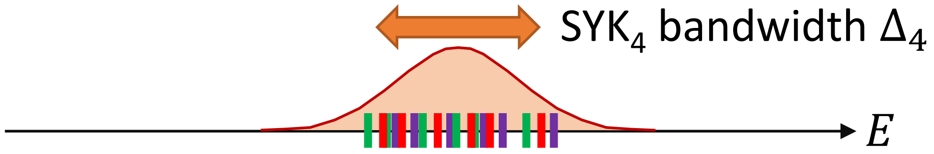
Four regimes of SYK_{4+2} [PRR **3**, 013023 (2021)]

$$\hat{H}_{\mathrm{SYK}_{4+2}} = \hat{H}_{\mathrm{SYK}_4} + \delta \hat{H}_{\mathrm{SYK}_2}$$

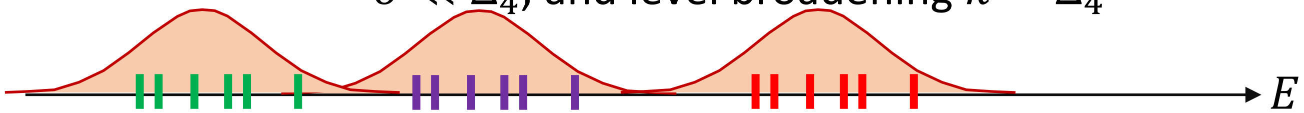
Bars with the same color:
many-body states in SYK_2–diagonal basis
directly connected by SYK_4 term



- I: Uniform distribution of wave functions



Nearest neighbors remain energetically close,
$\delta \ll \Delta_4$, and level broadening $\kappa = \Delta_4$



- II, III: Global DoS $D\nu \approx \frac{D}{\sqrt{2\pi N_{\mathrm{q}}}\delta}$

Only $\mathcal{O}\left(\left(\frac{\Delta_4}{\delta}\right)^2\right)$ of nearest neighbors remain in resonance,
level broadening reduced to $\kappa \sim \Delta_4^2/\delta$

$$\delta_{\mathrm{c}} = \frac{N_{\mathrm{q}}^2}{4\sqrt{3}} \log N_{\mathrm{q}} \text{ for large } N_{\mathrm{q}}$$

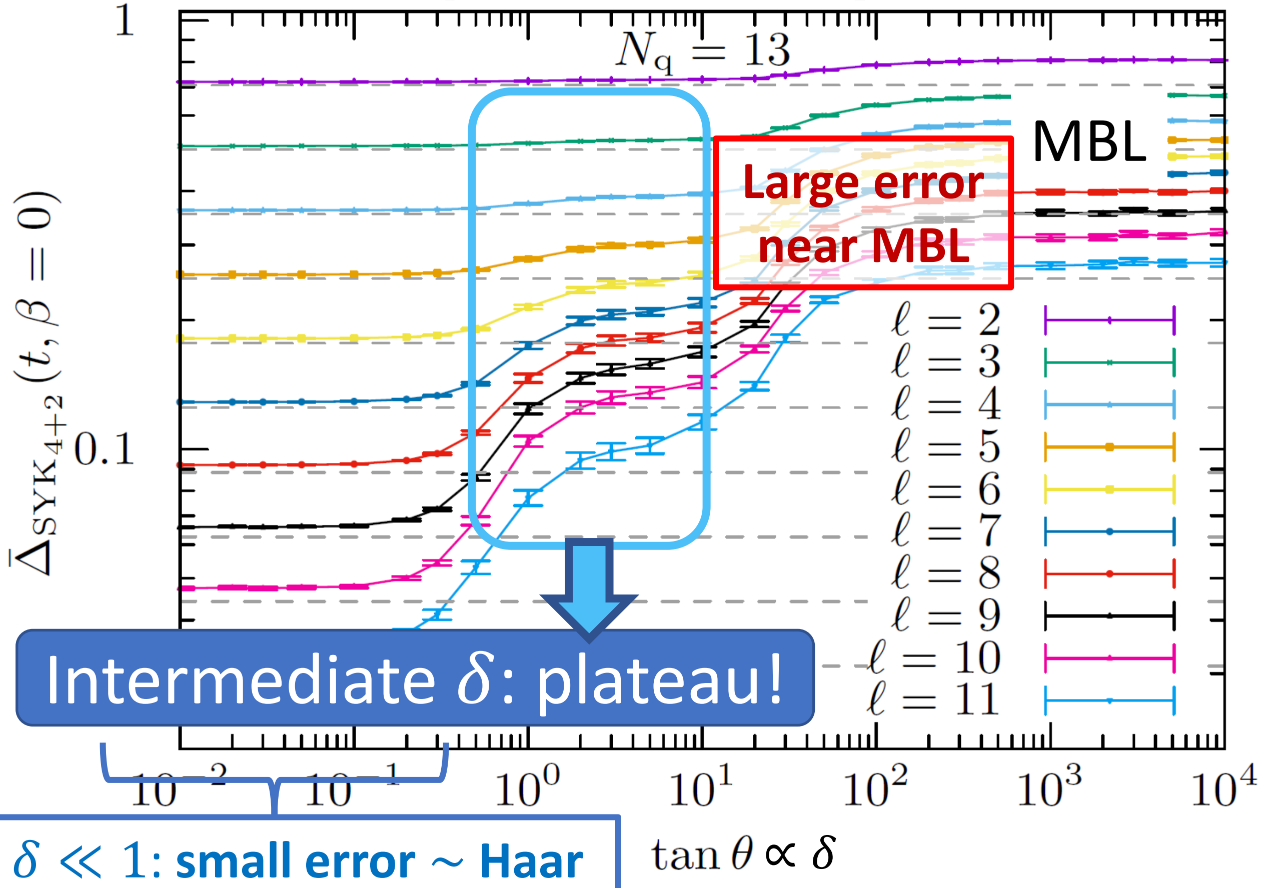
- IV: All eigenstates localized to $\mathcal{O}(1)$ sites:
many-body localization (MBL) in Fock space

Late-time error estimate for SYK4+2

$$\hat{H}_{\mathrm{SYK}_{4+2}} = \cos\theta\,\hat{H}_{\mathrm{SYK}_4} + \sin\theta\,\hat{H}_{\mathrm{SYK}_2}$$

Recovery error estimate

[arXiv:2303.02010]



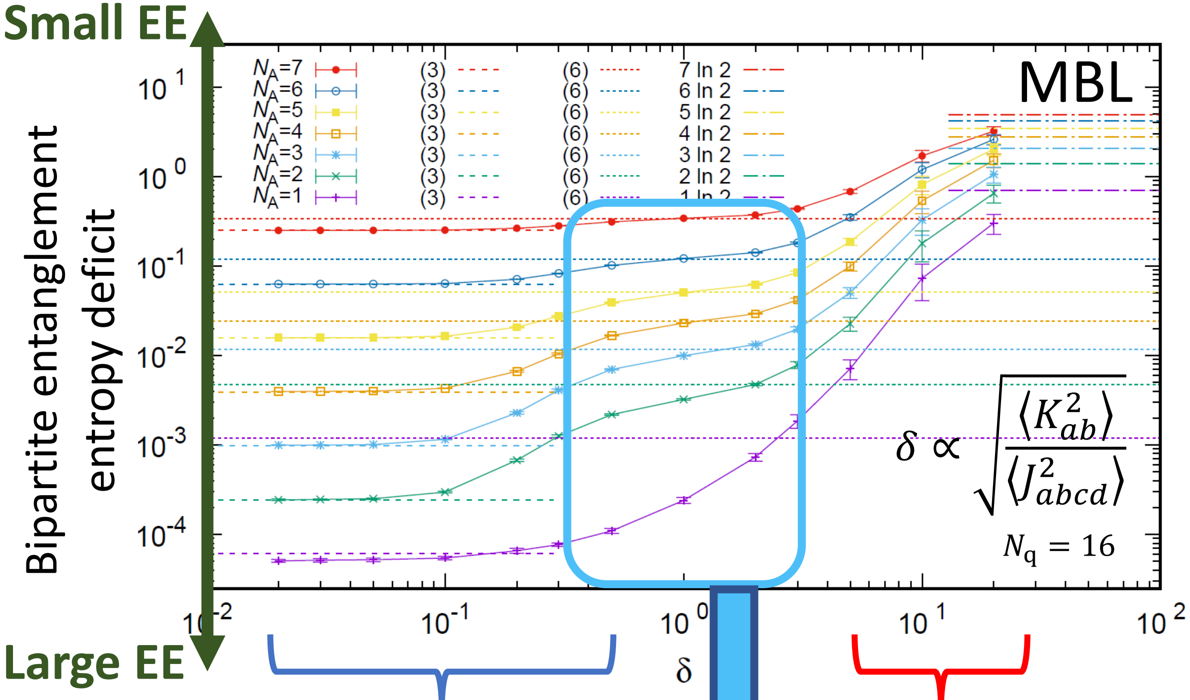
Large error near MBL

Intermediate δ: plateau!

$\delta \ll 1$: **small error ~ Haar**

Eigenstate entanglement entropy (EE)

[Phys. Rev. Lett. **127**, 030601 (2021)]



Small δ: almost equal distribution in the entire Fock space

Large δ: each eigenstate in tiny part of the Fock space

Eigenstates in restricted part of Fock space, where they are thermally distributed

Partial decoupling, incomplete decrease of error [Nakata et al., 1903.05796, 2007.00895]

One-dimensional spin chains ($S = 1/2$)

- Ising model + uniform magnetic field

$\hat{H}_{\text{Ising}} = -J \sum_{\langle j,k \rangle} S_j^z S_k^z - g \sum_j S_j^x - h \sum_j S_j^z$

- $(g, h) = (g, 0), (0, h)$: integrable
- $(g, h) = (1.05, 0.5)$: often studied as being **far from integrability**

- Heisenberg model + random field

$\hat{H}_{\text{XXZ}} = J \sum_{\langle j,k \rangle} \boldsymbol{S}_j \cdot \boldsymbol{S}_k + \sum_j h_j S_j^z$,

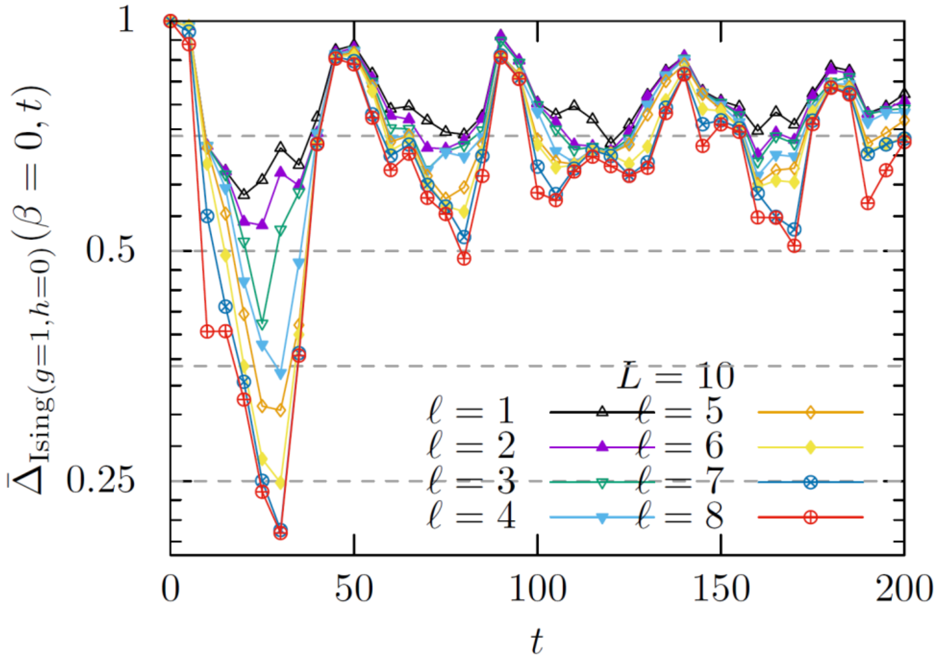
$h_j \in [-W, W]$

- $W = 0$: integrable
- $W \sim J$: **"ergodic"**
- $W \gtrsim 4J$: "MBL"

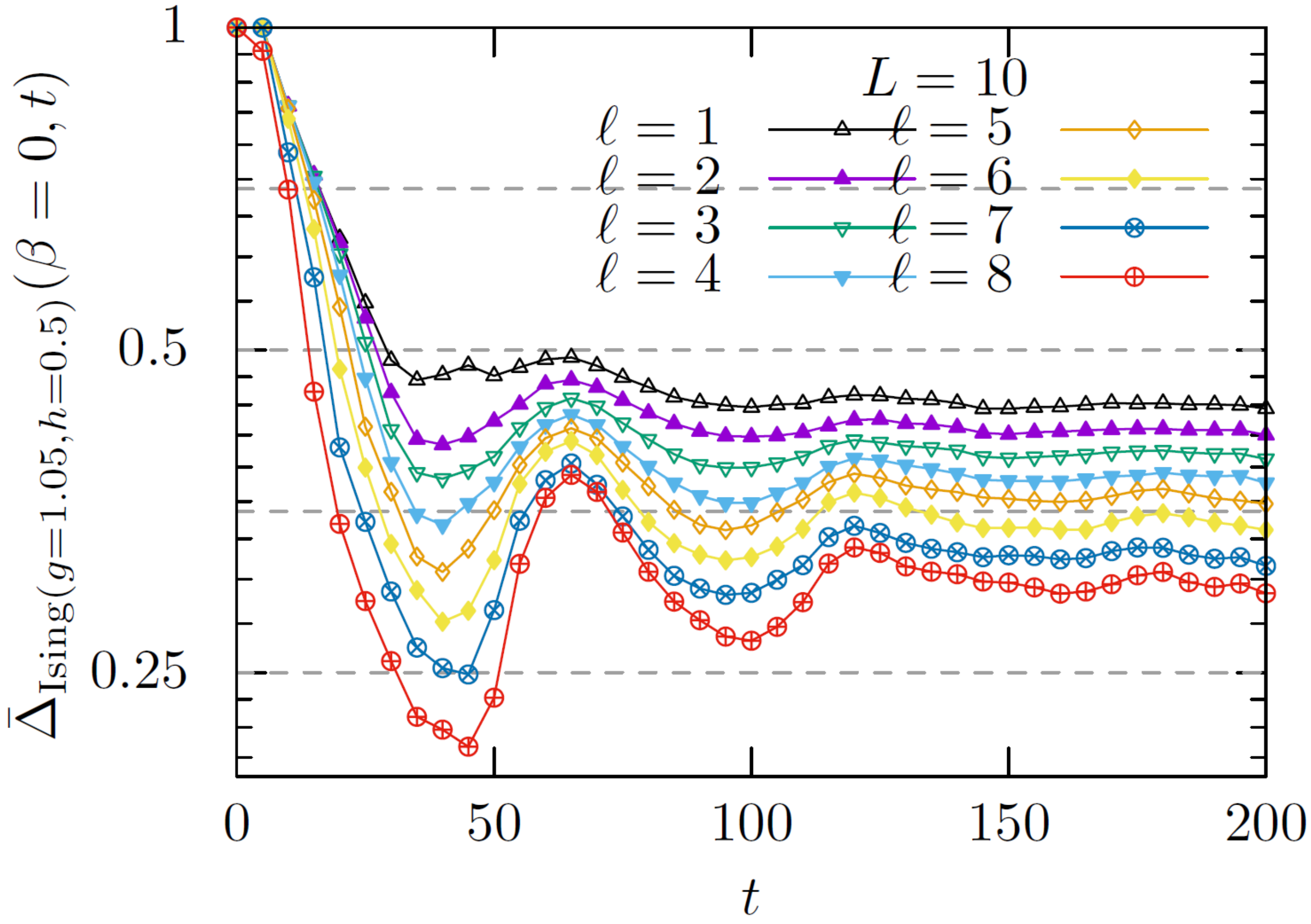
(though recently debated; see *e.g.* Morningstar *et al.*, PRB **105**, 174205 (2022))

Ising model + uniform magnetic field

- $H \parallel x$: integrable



- $(g, h) = (1.05, 0.5)$

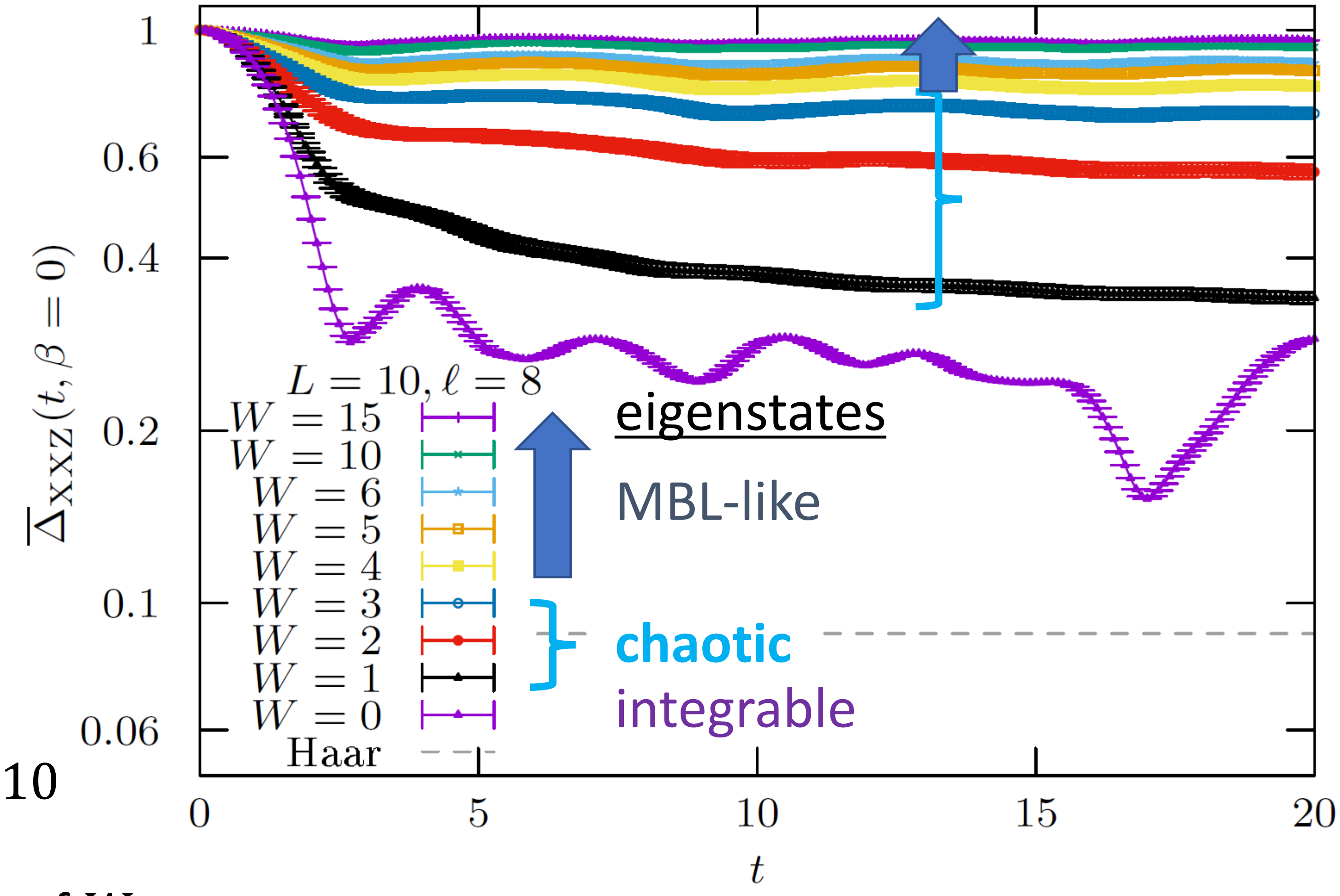


- **Very large error remains at long times in both cases**

Heisenberg model + random field

$$\hat{H}_{\mathrm{XXZ}} = J \sum_{\langle j,k \rangle} S_j \cdot S_k + \sum_j h_j S_j^z \,,$$

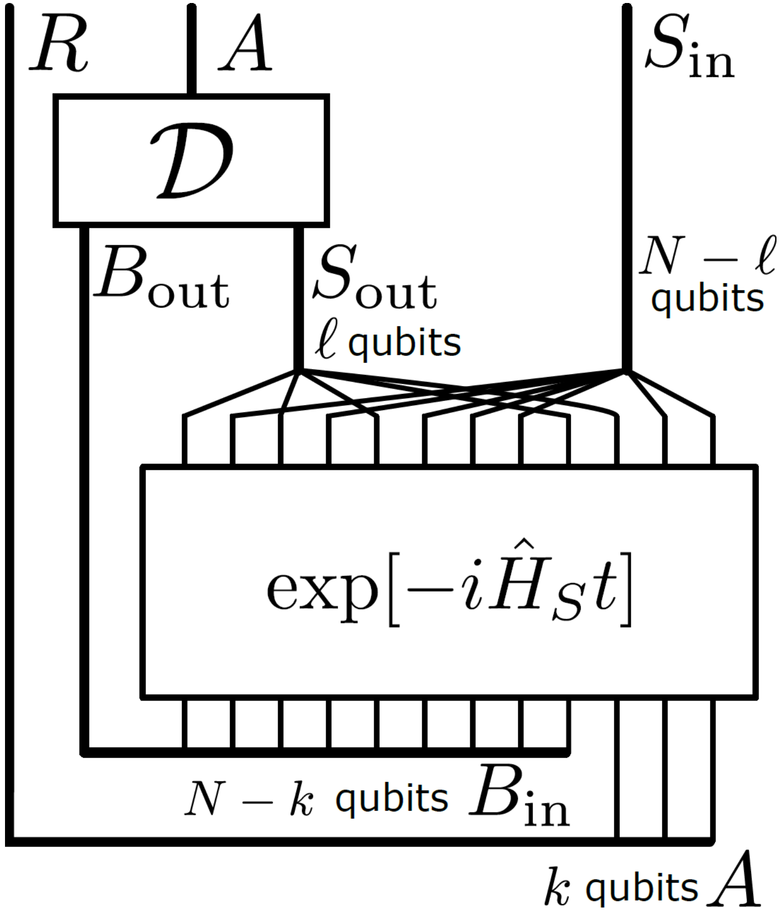
$$h_j \in [-W, W]$$



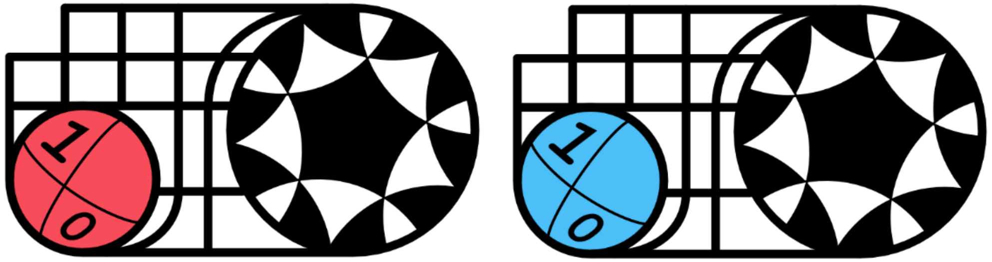
- Sample-averaged error stabilizes after $t \sim 10$
- **The Haar value is not reached**
- Error increases monotonically as a function of W

Masaki Tezuka: Hayden-Preskill Recovery in Sachdev-Ye-Kitaev type models and spin chains [Yoshifumi Nakata and MT: arXiv:2303.02010]

Summary



$$\overline{\Delta}_{\hat{H}}(t,\beta) \equiv \min\left\{1, \sqrt{\left\| \rho_{S_{\rm in}R} - \rho_{S_{\rm in}} \otimes \frac{I_R}{d_R} \right\|_1}\right\}$$



- Studied quantum error correction by scrambling Hamiltonian dynamics
- **SYK & sparse version** with coupling = ± 1 [2208.12098]: Haar-like scrambling properties at short time if spectrum is random matrix-like
- **SYK4+2**: suffers from wavefunction localization in Fock space even if still chaotic; plateau for intermediate regime (partial decoupling [1903.05796, 2007.00895])
- **Spin chains**: no Haar-like exponential decay of error as ℓ is increased, even in chaotic region