A prescription for entanglement in field theory and gravity

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work with Anegawa and Iizuka,

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Entanglement in QFT

In QM entanglement entropy between A and B is defined by

$$\rho_A = \mathrm{Tr}_B\big(|\psi\rangle\langle\psi|\big)$$

$$S = -\mathrm{Tr}\big(\rho_A \log \rho_A\big)$$

For a QFT in its ground state and $A = \{x > 0\}$ things look simpler.

$$\langle f|\rho_A|i\rangle =$$

$$\rho_A = \exp\left(-2\pi H_R\right)$$

thermal w.r.t. rotation generator (Rindler Hamiltonian)

In terms of $\rho_R = \exp\left(-2\pi H_R\right)$ we can do

replica trick $S = - \left.\frac{d}{dn}\right|_{n=1} \mathrm{Tr}\left(\rho_R^n\right)$

cone trick $Z(\beta) = \mathrm{Tr}\, e^{-\beta H_R}$

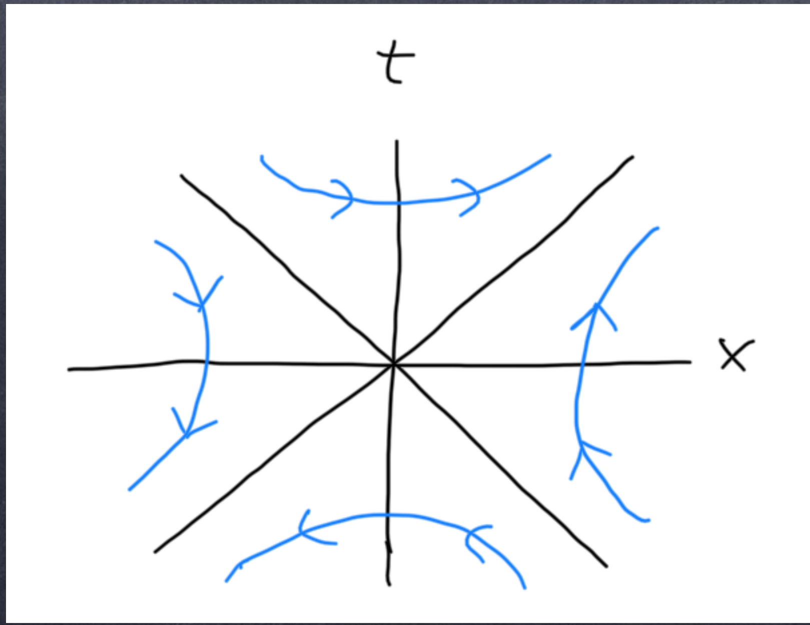
$$S = \left.(\beta\partial_\beta - 1)\right|_{\beta=2\pi} (-\log Z)$$

However the Hilbert space of a QFT isn't a tensor product and H_R isn't well-defined.

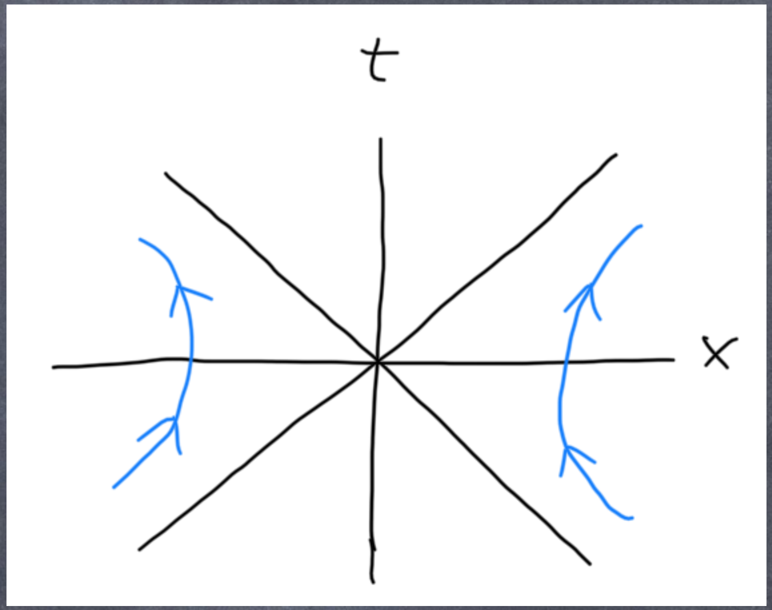
From Witten, 1803.04993:

> H_L and H_R have well-defined matrix elements $\langle\psi|H_L|\chi\rangle$ and $\langle\psi|H_R|\chi\rangle$ between suitable Hilbert space states χ and ψ, but if one tries to compute the norm of the state $H_L|\chi\rangle$ or $H_R|\chi\rangle$, one will find a universal ultraviolet divergence, near $x = 0$, independent of the choice of χ.

Curiously the combination $K = H_R - H_L$ is well-defined. It's the modular Hamiltonian or boost operator.



The modular Hamiltonian isn't directly useful for evaluating entanglement, but it motivates us to consider



$$V = H_R + H_L$$

$$\exp\big(-2\pi V\big) = \rho_L \otimes \rho_R$$

$$\text{define } S = \tfrac{1}{2} \cdot (\text{von Neumann entropy of } \rho_L \otimes \rho_R)$$

This is only heuristic since V isn't well-defined (it puts a crease in the Cauchy surface).

To fix this we introduce a regulator $\epsilon \to 0$ and define

$$V = \int_{-\infty}^{\infty} dx\, |x|\, T^{00}$$

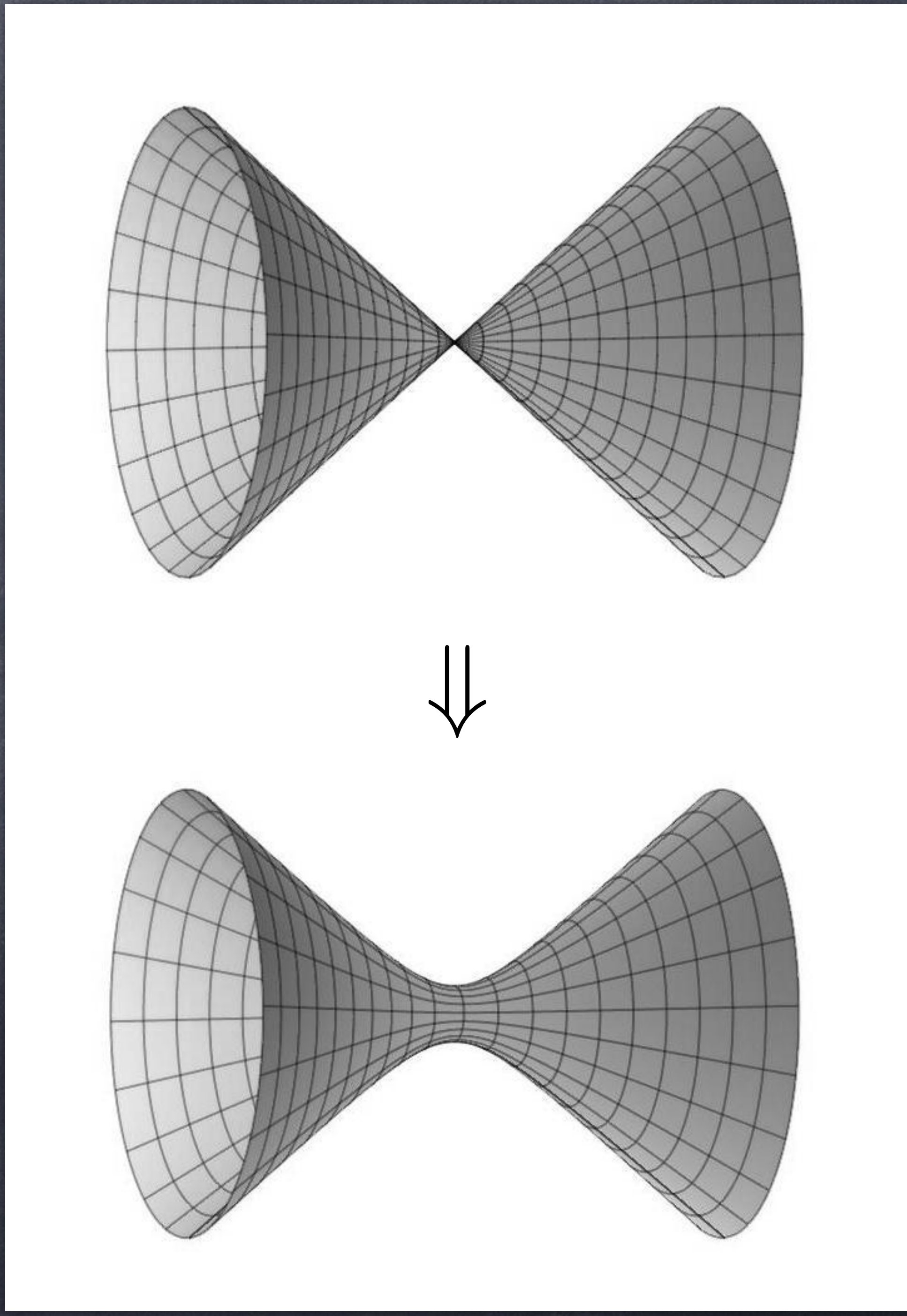
$$\Downarrow$$

$$V_\epsilon = \int_{-\infty}^{\infty} dx\, \sqrt{x^2 + \epsilon^2}\, T^{00}$$

$$Z(\beta) = \mathrm{Tr}\, e^{-\beta V}$$

$$\Downarrow$$

$$Z_\epsilon(\beta) = \mathrm{Tr}\, e^{-\beta V_\epsilon}$$



We define regulated entanglement entropy

$$S_\epsilon = \frac{1}{2}\left(\beta\partial_\beta - 1\right)\big|_{\beta=2\pi}\left(-\log Z_\epsilon\right)$$

Features

- A state-counting interpretation, with a density matrix $\exp(-\beta V_\epsilon)$ that is well-defined in the full Hilbert space.
- Corresponds to a smooth Euclidean geometry, with a freely-acting Killing vector like a standard thermal system.
- Manifestly gauge invariant (no need for b.c.'s or edge d.o.f. on entangling surface).

Is it equivalent to the replica trick?

- In some cases yes.
  - 2-D CFT
  - minimally-coupled scalar in 2-D
- In some cases no.
  - non-minimal scalar in 2-D
  - Maxwell field

It appears to be a prescription for evaluating "extractable entropy" - counting the effective number of Bell pairs split by the entangling surface - while suppressing other contributions.

Outline

1. 2D CFT on a circle

2. Free scalar in 2D
   (massive, with a non-minimal coupling)

3. Maxwell in 2D

4. Extension to spherical entangling surfaces in CFT

5. Log divergent terms for 4D Maxwell

6. Towards a bulk dual

2D CFT

We start with the modular Hamiltonian for a CFT on a circle, then flip time on one side and regulate.

$$K = \int_{-\pi}^{\pi} d\phi \, \sin\phi \, T^{00}$$

$$\Downarrow$$

$$V = \int_{-\pi}^{\pi} d\phi \, |\sin\phi| \, T^{00}$$

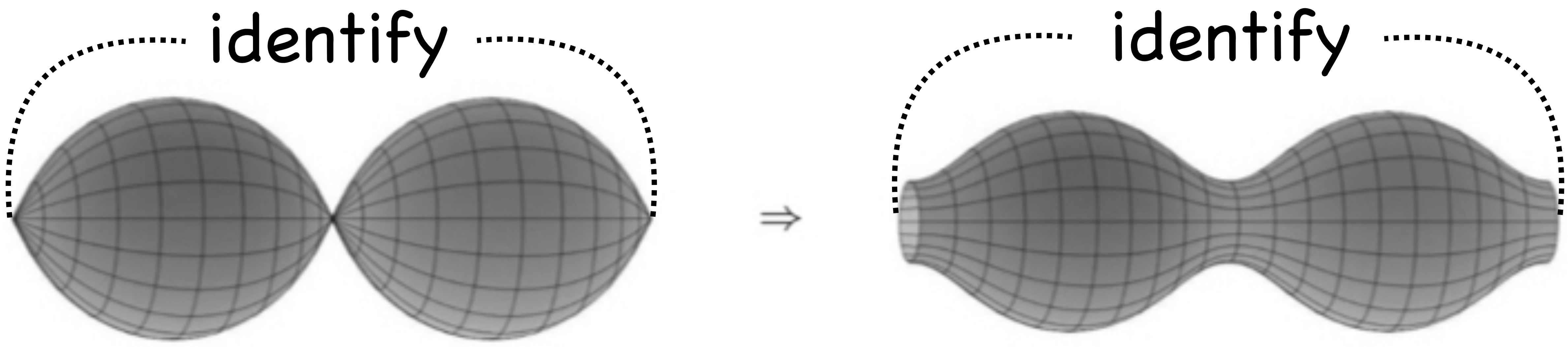
$$\Downarrow$$

$$V_\epsilon = \int_{-\pi}^{\pi} d\phi \, \sqrt{\sin^2\phi + \epsilon^2} \, T^{00}$$

The partition function $Z_\epsilon(\beta) = \mathrm{Tr}\, e^{-\beta V_\epsilon}$ corresponds to a Euclidean geometry

$$ds^2 = d\phi^2 + (\sin^2\phi + \epsilon^2)d\theta^2$$

$$\phi \approx \phi + 2\pi \qquad \theta \approx \theta + \beta$$



Two copies of off-shell de Sitter S^2, that touch at their common horizon S^0, smoothed to a torus T^2.

A change of variables makes the metric on T^2 conformally flat but changes the periodicity from 2π to $L_\epsilon = 4\log(1/\epsilon) + \text{finite}$.

From the partition function on a cylinder

$$-\frac{\log Z}{L_\epsilon} = -\frac{\pi c}{6\beta}$$

we recover the standard result

$$S = \frac{c}{3}\log\frac{1}{\epsilon}$$

Free scalar in 2D

Let's take a scalar in 2D with

$$S = \int d^2x \sqrt{g} \left(\frac{1}{2}(\partial\phi)^2 + \frac{1}{2}\xi R\phi^2 + \frac{1}{2}m^2\phi^2 \right)$$

The mass provides an IR cutoff so we can work in infinite volume. The non-minimal coupling doesn't affect the ground state wavefunction so it shouldn't affect the entropy.

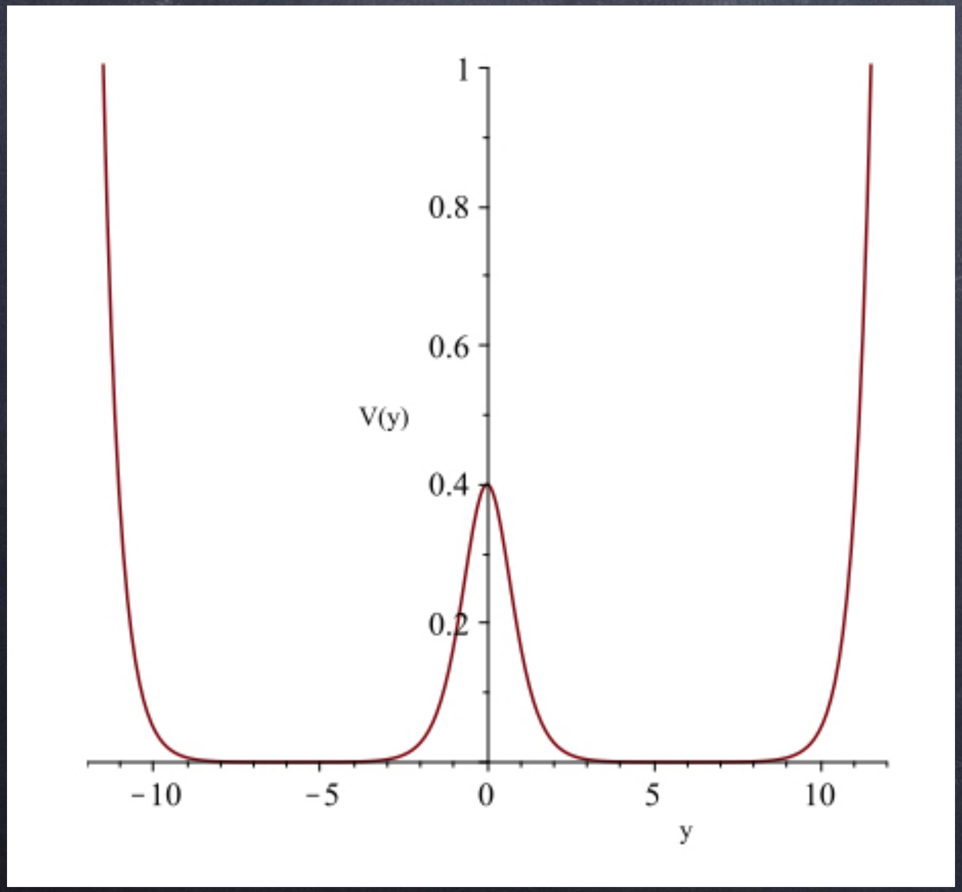
A puzzle: the non-minimal coupling affects the partition function on a cone at leading order.

In conformally flat coordinates (θ, y) the Klein-Gordon equation is

$$\left(-\partial_\theta^2 - \partial_y^2 + \mathcal{V}(y)\right)\phi = 0$$

There's an effective single-particle potential

$$\mathcal{V}(y) = m^2\epsilon^2\cosh^2 y - \frac{2\xi}{\cosh^2 y}$$



Ideal Bose gas in a potential. As $\epsilon \to 0$ the size of the box grows logarithmically. This growth is independent of ξ.

So the hourglass prescription gives the expected

$$S_\epsilon = \frac{1}{6} \log \frac{1}{m\epsilon} + \text{finite}$$

in contrast to the cone trick which gives

$$S_\epsilon = \left(\frac{1}{6} - \xi \right) \log \frac{1}{m\epsilon} + \text{finite}$$

I think this can be traced to the fact that the hourglass has a freely-acting Killing vector.

2D Maxwell

Take a Maxwell field in 1+1, on an interval $-L < x < L$, with charges at the ends of the interval.

No local d.o.f. so there shouldn't be any entropy.

A puzzle: the partition function of Maxwell on a cone is non-trivial.

On the hourglass it becomes trivial. There's a background electric field

$$E_x = q$$

which means

$$V = \int_{-L}^{L} dx\,|x|\,T^{00} = \frac{1}{2}L^2 q^2$$

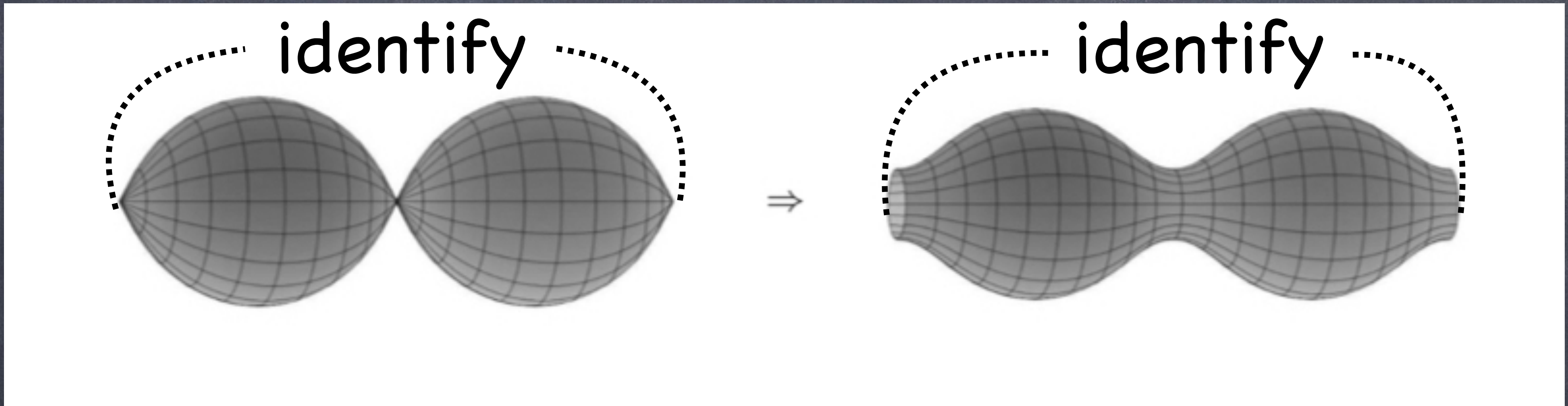
There's nothing to quantize and no need for a UV regulator.

$$Z = e^{-\beta V} = e^{-\beta L^2 q^2/2}$$

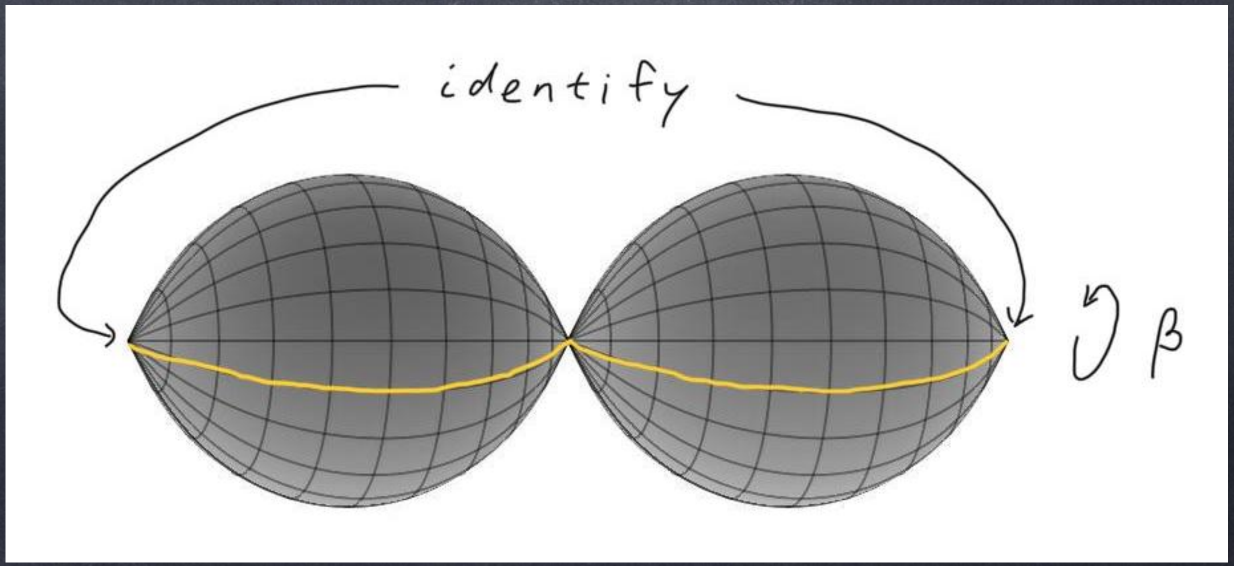
$$S = 0$$

Spherical entangling surfaces in CFT

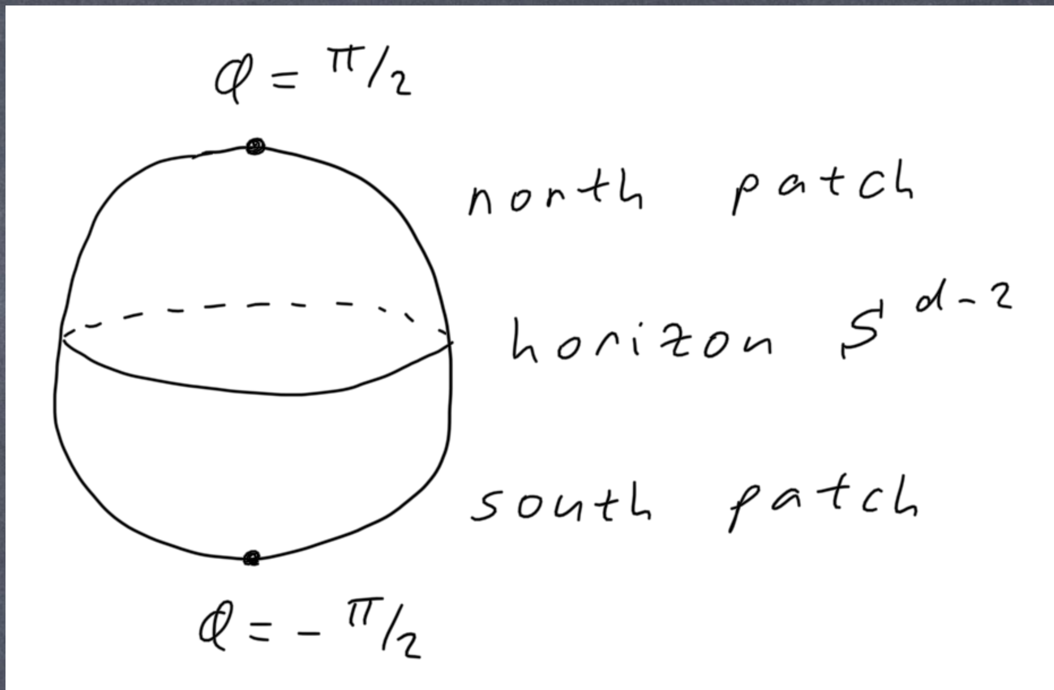
Recall that in 2D, for a CFT on a circle, we had two de Sitter static patches that touched at their common horizon.



It helps to think about a time slice of the singular geometry.



In higher dimensions, gluing the t = 0 slice of two static patches together gives a sphere S^{d-1}. Their common horizon S^{d-2} is the equator of the sphere.



The proper temperature varies over the sphere.

$$ds^2 = R^2 \sin^2 \phi \, d\theta^2 + R^2 d\Omega_{d-1}^2$$
$$\theta \approx \theta + \beta$$

Lowest at the poles, diverges at the equator.

We regulate by introducing a limiting temperature.

$$ds^2 = (R^2 \sin^2 \phi + \epsilon^2) d\theta^2 + R^2 d\Omega_{d-1}^2$$

It's convenient to make a conformal transformation so the proper temperature is constant. The spatial geometry becomes two copies of hyperbolic space, cut off at large radius and glued together.

$$ds^2_{\text{optical}} = d\theta^2 + d\rho^2 + \sinh^2 \rho d\Omega_{d-2}^2$$

$$\theta \approx \theta + \beta$$

$$\text{cut off and glue at } \rho \sim \log R/\epsilon$$

Log terms in 4D Maxwell

A thermal partition function on hyperbolic space gives the regulated entanglement entropy. Divergences arise from the infinite volume as $\epsilon \to 0$.

Dowker

David & Mukherjee

How does this go for 4D Maxwell?

We're particularly interested in the (universal) log divergence, which comes from

$$\mathrm{vol}(\mathcal{H}^3) = \frac{\#}{\epsilon^2} - 2\pi \log \frac{R}{\epsilon} + \text{finite}$$

The spin-1 heat kernel on $S^1 \times \mathcal{H}^3$ is known.

Giombi, Maloney, Yin
David, Gaberdiel, Gopakumar

It leads to

$$S = \# \frac{R^2}{\epsilon^2} - \frac{16}{45} \log \frac{R}{\epsilon} + \text{finite}$$

Dowker
Casini et al.
Soni & Trivedi
David & Mukherjee

The coefficient of the log divergence agrees with Soni & Trivedi, who subtracted the classical correlations required by the Gauss constraint from the replica result to obtain "extractable entropy".

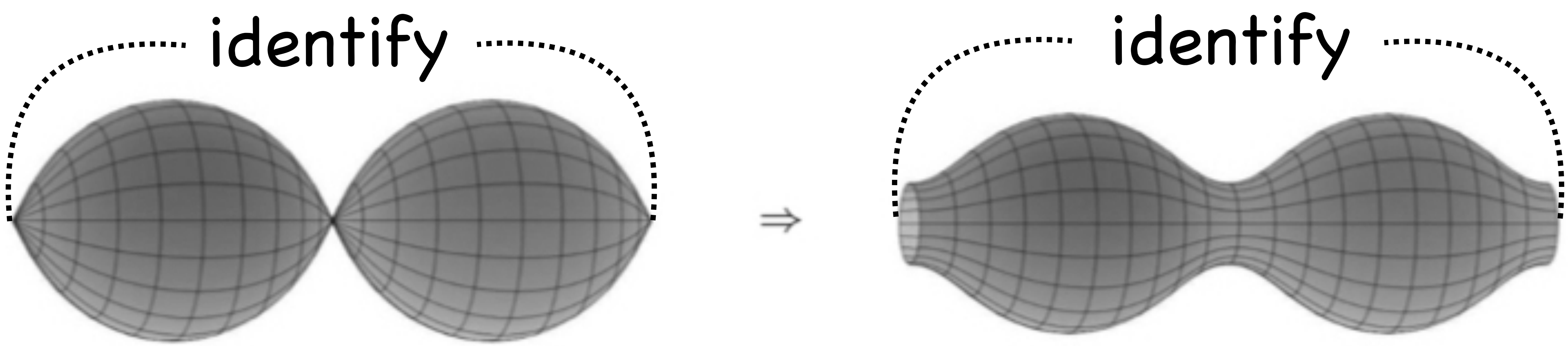
Towards a bulk dual

Suppose we apply the hourglass prescription to a boundary CFT. What does it correspond to in the bulk?

Start in D = 2, with boundary metric

$$ds^2 = d\phi^2 + (\sin^2\phi + \epsilon^2)d\theta^2$$

$$\phi \approx \phi + 2\pi \qquad \theta \approx \theta + \beta$$



A change of coordinates makes the boundary metric conformally flat.

$$ds^2 = \left(\sin^2\phi + \epsilon^2\right)\left(dy^2 + d\theta^2\right)$$
$$y \approx y + L \qquad \theta \approx \theta + \beta$$

The boundary is conformal to a torus with modulus

$$\operatorname{Im}\tau = \frac{L}{\beta} = \frac{4}{\beta}\log\frac{1}{\epsilon} + \text{finite}$$

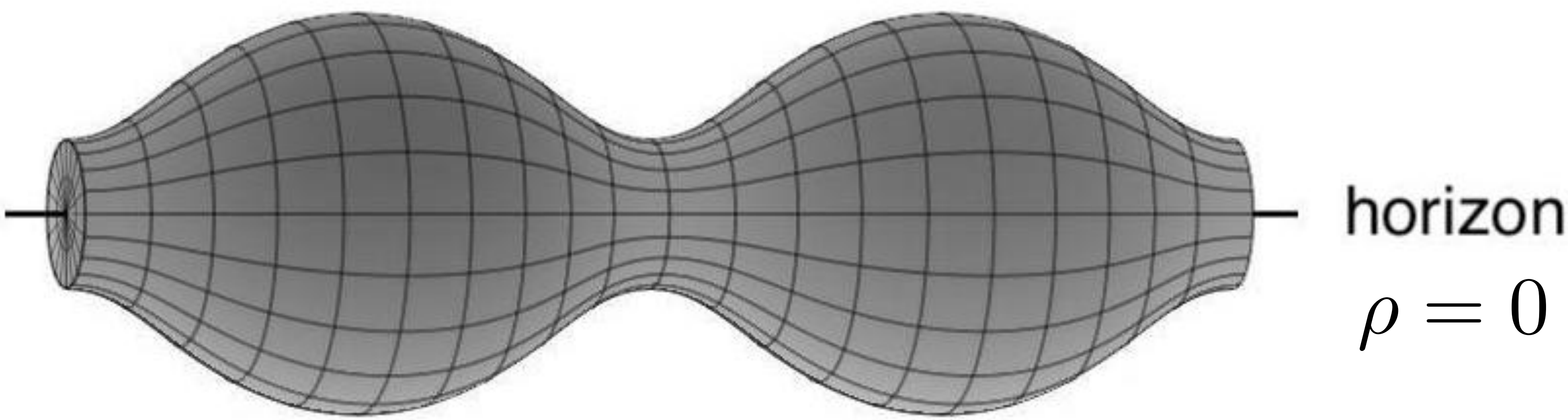
As $\epsilon \to 0$ the torus becomes long and thin.

Bulk geometry?

Can fill in the torus with a Euclidean BTZ black hole, also known as $\mathcal{M}_{0,1}$.

$$ds^2 = d\rho^2 + \sinh^2\rho\, d\theta^2 + \cosh^2\rho\, dy^2$$

$0 \leq \rho < \infty$	$\theta \approx \theta + 2\pi$	$y \approx y + 2\pi \mathrm{Im}\,\tau$
radial	Euclidean time (contractible)	non-contractible



Other choices are possible, corresponding to different choices for the contractible cycle.

But $\mathcal{M}_{0,1}$ dominates as $\epsilon \to 0$, with action

$$I(\tau) = -4\pi k \mathrm{Im}\,\tau \qquad k = c/24$$

Then $Z_{\mathrm{bulk}} = Z_{\mathrm{CFT}}$ leads to the expected

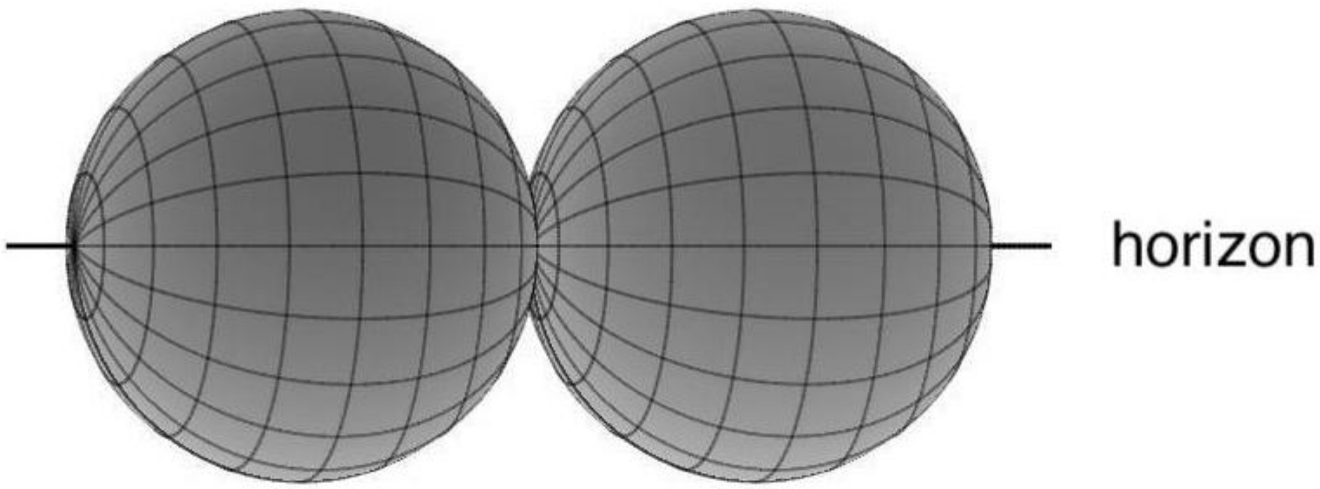
$$S = \frac{c}{3} \log \frac{1}{\epsilon}$$

What about higher dimensions? We know the boundary geometry.

$$ds^2 = d\phi^2 + \left(\sin^2\phi + \epsilon^2\right) d\theta^2 + \cos^2\phi \, d\Omega_{d-2}^2$$

$$-\pi/2 < \phi < \pi/2 \qquad\qquad \theta \approx \theta + \beta$$

Harder to find the right bulk. As $\epsilon \to 0$ the leading divergence comes from the regulated area of an AdS-Rindler horizon.



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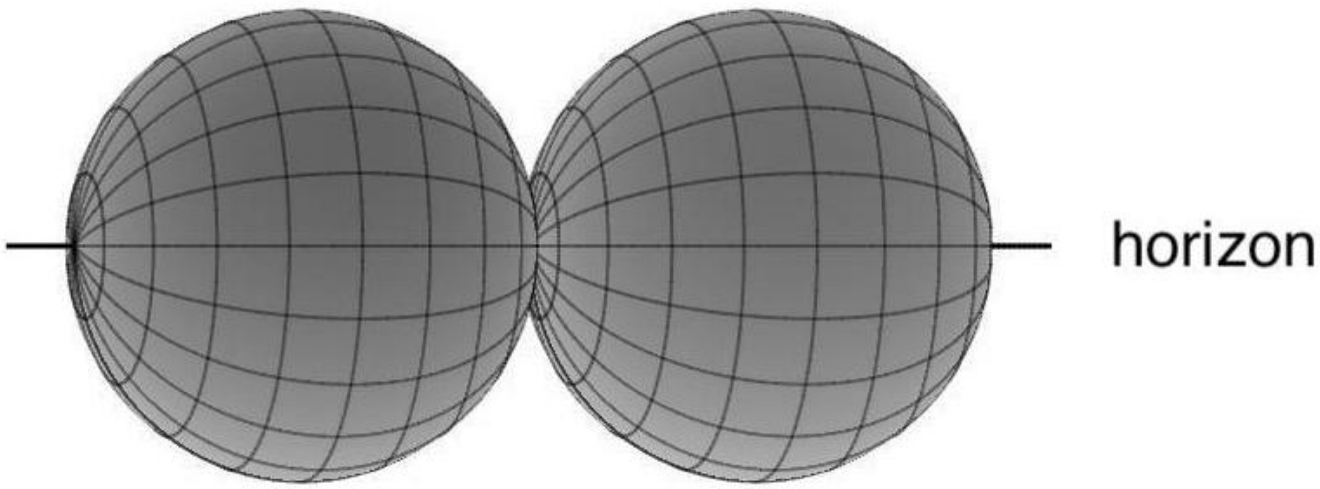

What about subleading logs?

Conclusions

The hourglass prescription provides a direct Euclidean technique for calculating extractable entropy - counting the number of Bell pairs split by an entangling surface.

Extensions to other theories, other dimensions?

What about non-conformal theories or non-spherical entangling surfaces?

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Thank you!