On Black Holes in String Theory

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Black holes in GR

As is known from the 1970's, due to work by Bekenstein, Hawking and others, black holes in GR behave like thermodynamic objects. In particular, they have an entropy

$$S_{BH} = \frac{A}{4G_N}$$

where

A = area of the horizon;

G_N = Newton constant, related to Planck length l_P via $G_N \sim l_P^{d-1}$ (in $d+1$ dimensions).

For large BH's of size $r_0 \gg l_P$, the entropy is large,

$$S_{BH} \sim \left(\frac{r_0}{l_P}\right)^{d-1} \gg 1$$

This is presumably the first term in an expansion in powers of l_P/r_0 .

A natural question is what does the BH entropy S_{BH} mean.

It is believed that the answer is that in any theory of quantum gravity (that satisfies some, in general unspecified, conditions), a generic high energy state looks from afar like a black hole, and S_{BH} is the statistical entropy of such states.

In some cases, the details of this statement are understood. This is the case, in particular, in the context of AdS/CFT.

For example, in AdS_3/CFT_2 , the Brown-Henneaux central charge of the boundary CFT

$$c_{BH} = \frac{3R_{AdS}}{2l_P}$$

together with the assumption that this CFT is unitary and modular invariant (and the SL(2,R) invariant vacuum is a normalizable state in the theory) leads to the Cardy formula

$$S_C = 2\pi\sqrt{\frac{c}{6}ER}$$

where E is the energy and R the radius of the spatial circle.

The Cardy formula with $c = c_{BH}$ is the same as the entropy of a (BTZ) balck hole with mass M=E, Strominger (1997).

- The situation is similar in other AdS spacetimes.
- There is also a partial understanding in asymptotically linear dilaton backgrounds, but I will not discuss it here.
- In flat spacetime the situation is less clear. This will be the topic of interest in the rest of this talk.

Consider a $d+1$ dimensional Schwarzschild BH of size r_0:

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{d-1}^2$$

$$f(r) = 1 - \left(\frac{r_0}{r}\right)^{d-2}$$

Its mass is: $$M = \frac{(d-1)\omega_{d-1}}{16\pi G_N} r_0^{d-2}$$

Hawking temperature: $T = 1/\beta$, with $$\beta = \frac{4\pi r_0}{d-2}$$

For large M, the entropy grows like

$$S_{BH} \sim M^{\frac{d-1}{d-2}}$$

It is not understood from the statistical point of view.

One idea for gaining insight into this issue is to embed the problem in string theory, which we turn to next.

Black holes in string theory

In string theory, in addition to the Planck length l_P , there is another scale, l_s , which is much larger, $l_s \gg l_P$ at weak coupling. The description of the BH in terms of a metric is expected to be valid for $r_0 \gg l_s$. It should receive large corrections for $r_0 \sim l_s$. In that regime it should be thought of as a worldsheet CFT.

Physically, this has to do with the fact that as the size of the BH decreases, its Hawking temperature increases, and for $r_0 \sim l_s$ it approaches the Hagedorn temperature $T_H = 1/\beta_H$.

The density of states of strings at high energies $E \gg m_s$, is

$$S_{st} \sim \beta_H E$$

Thus, it is natural to ask what happens when the BH shrinks, and we approach the regime where its Hawking temperature approaches the Hagedorn temperature.

One may expect that in this regime, the system crosses over from the BH behavior, $S_{BH} \sim E^{\frac{d-1}{d-2}}$, to $S_{st} \sim \beta_H E$.

Horowitz and Polchinski (1996) showed that formally continuing the string and black hole results,

$$S_{BH} \sim S_{st}$$

at the energy for which the Hawking temperature $\beta \sim \beta_H$. A nice feature of this result is that the l.h.s. is thermodynamic in nature, while the r.h.s. has a statistical interpretation, in terms of counting states. In particular, one can hope to address questions like how the microstates differ from the BH background that describes them, i.e. what are stringy BH's made out of.

However, there are difficulties. One is that the BH background receives large corrections for $r_0 \sim l_s$, so one needs to understand the corresponding worldsheet theory. This will be our focus in the rest of this talk.

Another difficulty is that the correspondence energy , (the energy at which $S_{BH} \sim S_{st}$) is very large, $E_{cor} \sim m_s/g_s^2$, so extrapolating perturbative string results to it is unwarranted.

If we nevertheless formally continue, it seems that there is some tension: the BH is small, while the string is large. We will return to this issue later in the talk.

Thus, we are led to the question what happens to the Schwarzschild geometry for $r_0 \sim l_s$. In general, the answer is not known, but it turns out that as $d \to \infty$ there are some simplifications. In particular, the geometry develops a region in which it looks like the two dimensional BH (Soda; Emparan et al; Chen, Maldacena)

$$ds^2 = k(d\rho^2 - \tanh^2\rho \; d\phi^2)$$

$$e^{-\Phi} = \cosh^2\rho$$

with

$$\cosh^2\rho = \left(\frac{r}{r_0}\right)^{d-2}, \qquad \phi = \frac{2\pi t}{\beta}, \qquad k = \left(\frac{2r_0}{d}\right)^2$$

A few things about this:

- The gravity analysis is only valid at large k, or $\frac{r_0}{d}$, but it can be continued to $k \sim 1$, by using the fact that this geometry corresponds to a solvable worldsheet theory, the coset SL(2,R)/U(1), CM (2021).

- The Hagedorn temperature corresponds to $k = 4$ (bosonic string), $k = 2$ (superstring).

- The SL(2,R)/U(1) BH is obtained after reducing on the sphere S^{d-1}, or equivalently integrating out the angular d.o.f. in the worldsheet theory.

Of course, large d is not really physical in string theory, so it would be nice to generalize this understanding to finite d. The Lorentzian problem has so far proven to be too hard, but progress has been made on the Euclidean version, to which we turn next.

One reason the Euclidean problem is simpler is that there is no singularity, so one can expect it to be more amenable to analysis.

Small Euclidean BH’s in string theory

We are interested in describing Euclidean Schwarzschild BH’s with Hawking temperature $\beta \sim \beta_H$ in classical string theory. This is a well defined worldsheet CFT problem. We have a line of worldsheet CFT’s labeled by β. We want to understand these CFT’s for $\beta \sim \beta_H$.

It is natural to ask whether there is an effective field theory that can be used for this purpose. For large BH’s that EFT is of course Einstein gravity, but that fails for small BH’s.

What can we do?

The Euclidean Schwarzschild geometry

$$ds^2 = f(r)d\tau^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{d-1}^2$$

$$f(r) = 1 - \left(\frac{r_0}{r}\right)^{d-2}$$

with Euclidean time identified as $\tau \sim \tau + \beta$, asymptotes at large r to $R^d \times S^1$. Since we will be interested in $\beta \sim l_s$, a low energy EFT that describes this background must be d dimensional.

The fields that should be included in the EFT are:

- The radion $\varphi(x)$ that parametrizes the local size of the S^1,

$$R(x) = Re^{\varphi(x)}$$

R is the asymptotic radius of the τ circle, so $\varphi \to 0$ at infinity.

- The string tachyon winding once around the Euclidean time circle at infinity, $\chi(x)$. This field becomes light near the Hagedorn temperature, and thus needs to be included in the low energy effective Lagrangian. Moreover, it is known to have a non-zero profile in the Euclidean BH CFT (even for large BH’s).

In principle, we should also include other light fields, like the dilaton and the metric on R^d, but their effects will turn out to be subleading.

Thus, we will start with an EFT for φ, χ, and will add additional fields as necessary.

Our first try for an effective action is:

$$I_d = \frac{\beta}{16\pi G_N} \int d^d x \left[(\nabla\varphi)^2 + |\nabla\chi|^2 + \left(m_\infty^2 + \frac{\kappa}{\alpha'}\varphi \right) |\chi|^2 \right]$$

We will refer to it below as the Horowitz – Polchinski (HP) effective action.

κ is a numerical constant, and m_∞ is the mass of the winding tachyon at infinity, where the radius of the circle it winds around is R , which is related to the inverse temperature, $\beta = 2\pi R$.

$$m_\infty^2 = \frac{R^2 - R_H^2}{\alpha'^2}$$

$$R_H^{\text{bosonic}} = 2l_s \ , \quad R_H^{\text{type II}} = \sqrt{2}l_s$$

We will use m_∞ to parametrize β, and take it to be small, $m_\infty \ll m_s$ to study the region $\beta \sim \beta_H$.

The HP action includes the leading terms in a certain expansion. To see what that expansion is, we write the e.o.m. of I_d:

$$\nabla^2\chi = (m_\infty^2 + \frac{\kappa}{\alpha'}\varphi)\chi \ ,$$
$$\nabla^2\varphi = \frac{\kappa}{2\alpha'}|\chi|^2 \ .$$

These equations have a scaling symmetry, under which:

$$[\chi] = [\varphi] = 2; \quad [\nabla] = [m_\infty] = 1$$

Under this symmetry, all terms in the effective Lagrangian have dimension six. All the terms we have neglected have higher dimensions. Thus, one would expect them to not be important for small m_∞. We will see later to what extent this expectation is realized.

The action I_d was written by Horowitz and Polchinski (1997) to describe gravitating strings. Our goal is to describe small Euclidean black holes, but it is not clear that the two are distinct objects. In any case, we are looking for spherically symmetric, normalizable solutions of the e.o.m. that behave at large r like:

$$\chi(r) \sim r^{-\frac{d-1}{2}} e^{-m_\infty r},$$
$$\varphi(r) \sim r^{-d+2}.$$

We next describe these solutions.

Horowitz - Polchinski solutions

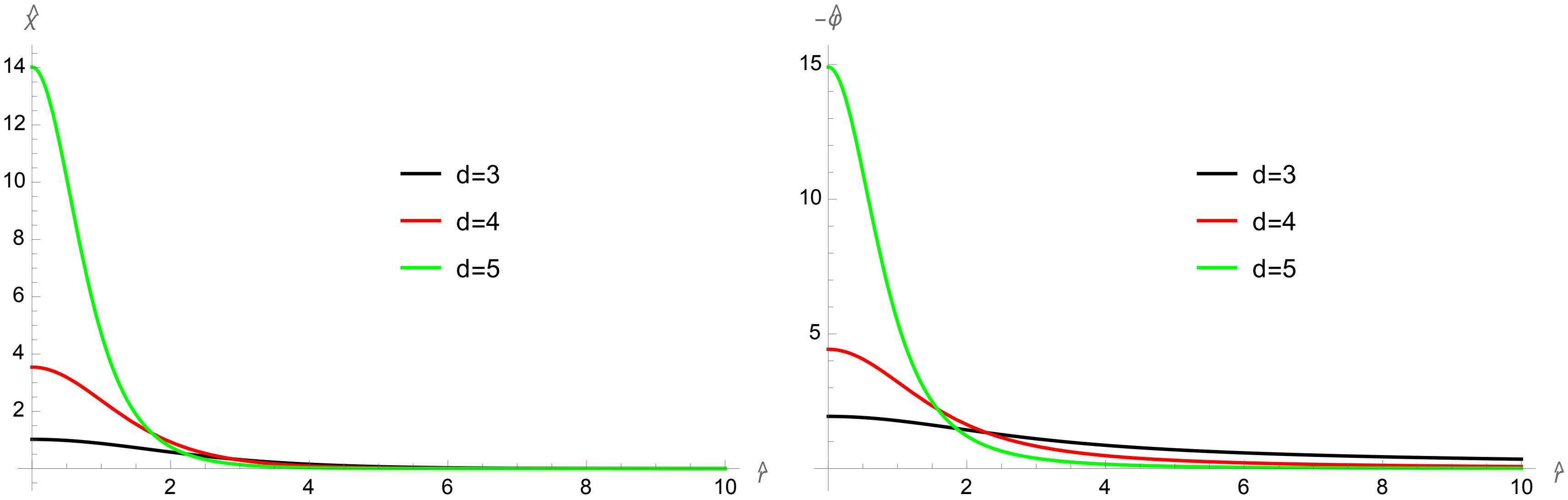
We can use the scaling symmetry mentioned above to define

$$x = \hat{x}/m_\infty \ ,$$
$$\chi(x) = \frac{\sqrt{2}\alpha'}{\kappa} m_\infty^2 \hat{\chi}(\hat{x}) \ ,$$
$$\varphi(x) = \frac{\alpha'}{\kappa} m_\infty^2 \hat{\varphi}(\hat{x}) \ .$$

In terms of these scaled parameters, the e.o.m take the form

$$\hat{\nabla}^2 \hat{\chi} = \hat{\chi}'' + \frac{d-1}{\hat{r}} \hat{\chi}' = (1+\hat{\varphi})\hat{\chi} \ ,$$
$$\hat{\nabla}^2 \hat{\varphi} = \hat{\varphi}'' + \frac{d-1}{\hat{r}} \hat{\varphi}' = \hat{\chi}^2 \ .$$

Thus, in the scaled parameters there is a unique solution for given d. There is no known analytic solution, but solving the equations numerically, one finds solutions that look like this:



Note that:

- φ is negative in these solutions. This just means that the Euclidean time circle shrinks as r decreases, as expected.

- The scaling symmetry implies that $\chi(0),\ \varphi(0) \sim m_\infty^2$. Thus, $\chi(r), \varphi(r) \to 0$ as $m_\infty \to 0$, i.e. as the Hawking temperature approaches the Hagedorn temperature.

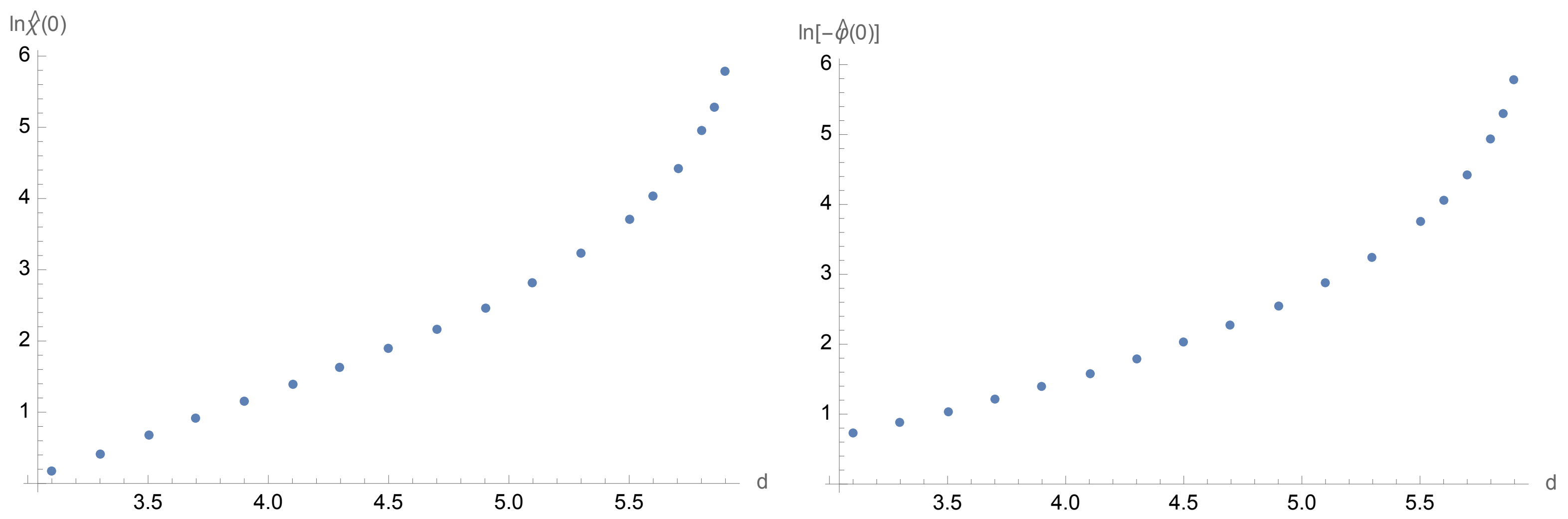
- We only have graphs for $d = 3, 4, 5$. This is because for $d \geq 6$ there are no solutions with the required boundary conditions. This is a curious feature; we will get back to it.

The absence of solutions for $d \geq 6$ seems problematic. The question what happens to Euclidean BH's as $\beta \to \beta_H$ clearly exists for all $d \geq 3$, so if the effective action I_d does not have suitable solutions, we presumably must conclude that such EBH's cannot be described by an effective action.

We will next see that the actual situation is more interesting.

The limit $d \to 6$

To see the origin of the problem for $d \geq 6$, it is convenient to treat d as a continuous variable, and examine the limit $d \to 6$. The numerics gives the following result for the scaled height of the solutions $\hat{\chi}(0), \hat{\varphi}(0)$:



It looks like these quantities grow without bound as $d \to 6$. One can actually show analytically that

$$\chi(0), -\varphi(0) \sim \frac{m_\infty^2}{6-d}$$

Thus, at fixed $d < 6$, as $m_\infty \to 0$, χ, $\varphi \sim m_\infty^2$, but at fixed m_∞, as $d \to 6$ they grow without bound. This seems to suggest that the small field approximation breaks down, and the EFT becomes unreliable.

We will next show that the actual situation is better.

To see what happens as $d \to 6$, we add the first subleading terms to I_d. These terms have dimension eight, and can be obtained by studying string scattering amplitudes. One finds:

$$I_d = \frac{\beta}{16\pi G_N} \int d^d x \left[(\nabla\varphi)^2 + |\nabla\chi|^2 + \left(m_\infty^2 + \frac{\kappa}{\alpha'}\varphi + \frac{\kappa}{\alpha'}\varphi^2 \right) |\chi|^2 + \frac{\kappa}{4\alpha'}|\chi|^4 \right]$$

The e.o.m. of φ, χ are now:

$$\nabla^2\chi = m_\infty^2\chi + \frac{\kappa}{\alpha'}\varphi\chi + \frac{\kappa}{2\alpha'}\chi^2\chi^* + \frac{\kappa}{\alpha'}\varphi^2\chi \ ,$$
$$\nabla^2\varphi = \frac{\kappa}{2\alpha'}|\chi|^2 + \frac{\kappa}{\alpha'}|\chi|^2\varphi \ .$$

We are looking for solutions of these equations for $\epsilon = 6 - d \ll 1$, $m_\infty \ll 1$. One can study this problem numerically (and we did), but there are a few analytic statements one can make as well.

- For $m_\infty = 0$, the e.o.m. are consistent with setting $\chi = -\sqrt{2}\varphi$. It turns out that the normalizable solutions satisfy this constraint. From the point of the EFT this looks accidental, but as we we will see next, it is a consequence of a symmetry.

As we discussed before, the effective Lagrangians we are studying should be thought of as an approximation to a worldsheet CFT, that describes the small EBH. That CFT has in general a $U(1)_L \times U(1)_R$ symmetry, with the conserved charges being the left and right-moving momenta on the Euclidean time circle (or, equivalently, the conserved momentum and string winding on S^1).

In the EBH geometry, the circle is contractible, which means that the winding symmetry is spontaneously broken (a string winding around the circle can slip off the tip). This symmetry is also broken by the condensate of the winding tachyon χ. The two are closely related.

Thus, the EBH spacetime realizes the spontaneous symmetry breaking pattern

$$U(1)_L \times U(1)_R \rightarrow U(1)_{diag}$$

At the Hagedorn temperature, i.e. for $m_\infty = 0$, the asymptotic symmetry on $R^d \times S^1$ is enhanced to $SU(2)_L \times SU(2)_R$. This is directly related to the fact that the winding tachyon χ is exactly massless there. The charged $SU(2)$ currents are modes of χ, while the CSA generator is the zero mode of φ.

The relation $\chi = -\sqrt{2}\varphi$ is a reflection of the fact that the EBH exhibits the symmetry breaking pattern

$$SU(2)_L \times SU(2)_R \rightarrow SU(2)_{diag}$$

More precisely, it corresponds to the worldsheet Lagrangian

$$L = L_0 + \chi(r) J^a \bar{J}^a$$

where J^a ($\bar{J}^a$) are the left (right) moving $SU(2)$ currents on the worldsheet.

This is an interesting worldsheet theory. If χ is a constant coupling, this non-abelian Thirring model breaks conformal symmetry on the worldsheet, and also breaks $SU(2)_L \times SU(2)_R$ to the diagonal $SU(2)$. In our case, the coupling is itself a function of the radial direction on R^d, $\chi = \chi(r)$.

From the worldsheet point of view, the r dependence of χ is fixed by the requirement that the full theory remains conformal. In the regime of validity of the EFT we wrote, this is the same as the requirement that the equations of motion of the effective action are satisfied.

- One can show that solutions of the e.o.m. of the effective action, (χ_*, φ_*), satisfy the constraint

$$m_\infty^2 \int d^d x |\chi_*|^2 = \frac{1}{4} \int d^d x \left[(d-6)\frac{\kappa}{\alpha'}\varphi_*|\chi_*|^2 + (2d-8)\left(\frac{\kappa}{\alpha'}\varphi_*^2|\chi_*|^2 + \frac{\kappa}{4\alpha'}|\chi_*|^4 \right) \right]$$

We are interested in solutions for small $\epsilon = 6 - d$, m_∞. In this limit, we see that the dimension eight terms, that are naively suppressed, are actually important to keep, since the dimension six terms are suppressed by either ϵ or m_∞.

Note: terms with dimension > 8 can still be omitted in this limit, as in other examples of the ϵ expansion.

- To solve the e.o.m., we can perturb around $d = 6, m_\infty = 0$. At this point, the e.o.m. are

$$\nabla^2 \chi = \frac{\kappa}{\alpha'} \varphi \chi$$
$$\nabla^2 \varphi = \frac{\kappa}{2\alpha'} \chi^2$$

Here we assumed that $\varphi, \chi \ll 1$ to neglect the cubic terms in the e.o.m.

These equations have a one parameter line of solutions:

$$\chi(r) = \frac{\chi(0)}{\left(1 + \frac{\kappa}{24\sqrt{2}\alpha'}\chi(0) r^2\right)^2}$$

To solve the e.o.m. on p. 30, we plug this solution into the relation on p. 36. We find that:

$$m_\infty^2 = \frac{\sqrt{2}\kappa}{80\alpha'}(6-d)\chi(0) + \frac{3\kappa}{140\alpha'}\chi^2(0)$$

Recall that here $m_\infty, 6 - d = \epsilon$ are assumed to be small, and the solution gives the leading behavior of $\chi(0)$ in this limit.

We see that there are two different regions in parameter space, in which the behavior is different:

- For $m_\infty \ll \epsilon$ we have

$$\chi(0) \sim \frac{m_\infty^2}{\epsilon}$$

- For $\epsilon \ll m_\infty$ we have

$$\chi(0) \sim m_\infty$$

Thus, if we fix m_∞ and send $\epsilon \to 0$, we do not find that the EFT description breaks down, as indicated by the leading order effective action. Rather, we need to include the first subleading corrections to this action (but, importantly, not any higher order ones).

In particular, for $d = 6$ the modified effective action has a solution, while the HP one does not. To study this solution for small m_∞ , one can neglect the higher order corrections to the effective action.

$d > 6$

Now that we found the solution for d=6, we can ask what happens for $d > 6$. As mentioned above, the HP action does not have solutions in this range, but our analysis suggests that the modified action does have such solutions for $d = 6 + \epsilon$.

Consider e.g. the case $m_\infty = 0$. The relation on p. 38 implies that there is a solution, with

$$\chi(0) = \frac{7\sqrt{2}}{12}\epsilon + O\left(\epsilon^2\right)$$

This is different from the case $d \leq 6$, where at $T = T_H$ the solution vanishes.

We checked the predictions of the above perturbative analysis by solving the modified HP e.o.m. numerically. An example of the results for $d = 6 + \epsilon$ is exhibited below:

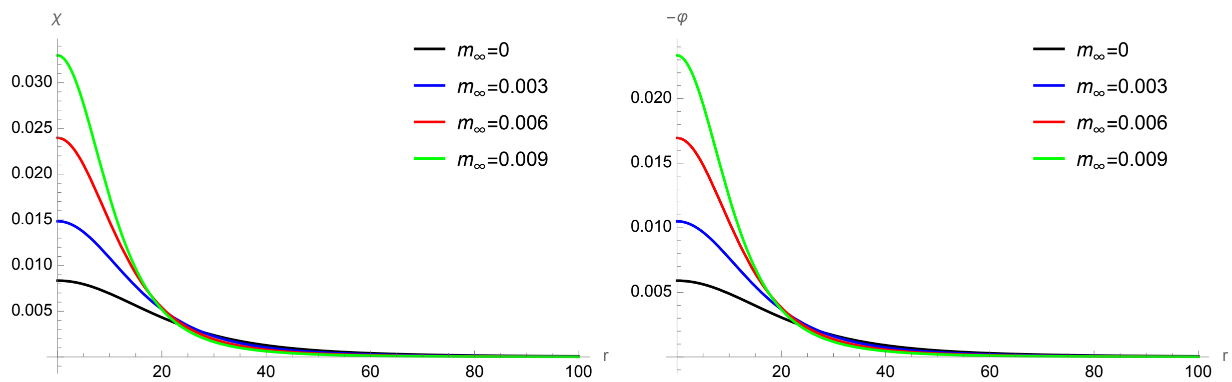
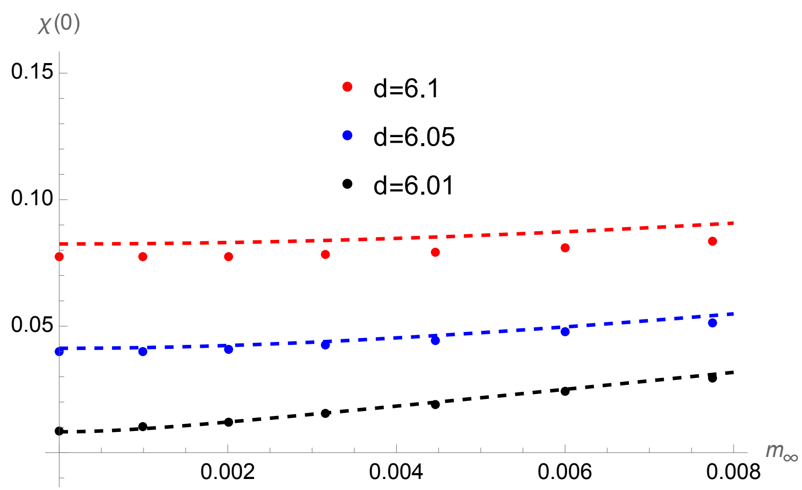


Figure 2: The profiles of χ and $-\varphi$ for $d = 6.01$.



For $m_\infty = 0$, we showed numerically that the solutions satisfy the constraint $\chi(r) = -\sqrt{2}\varphi(r)$:

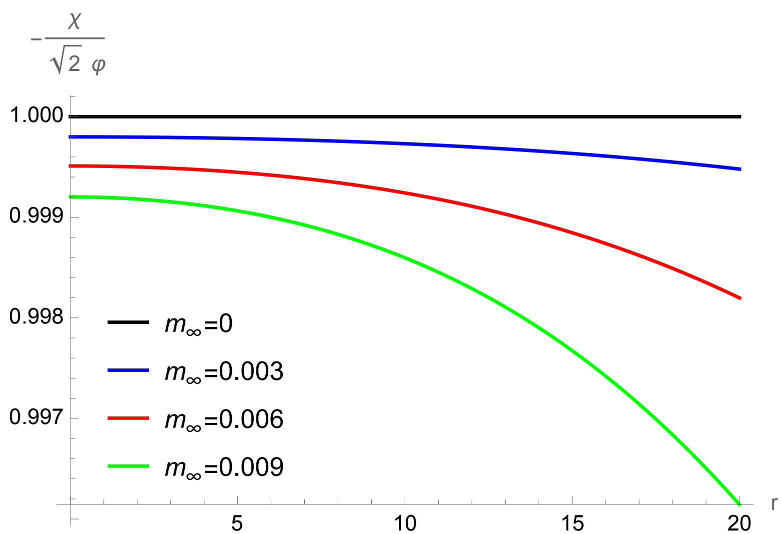


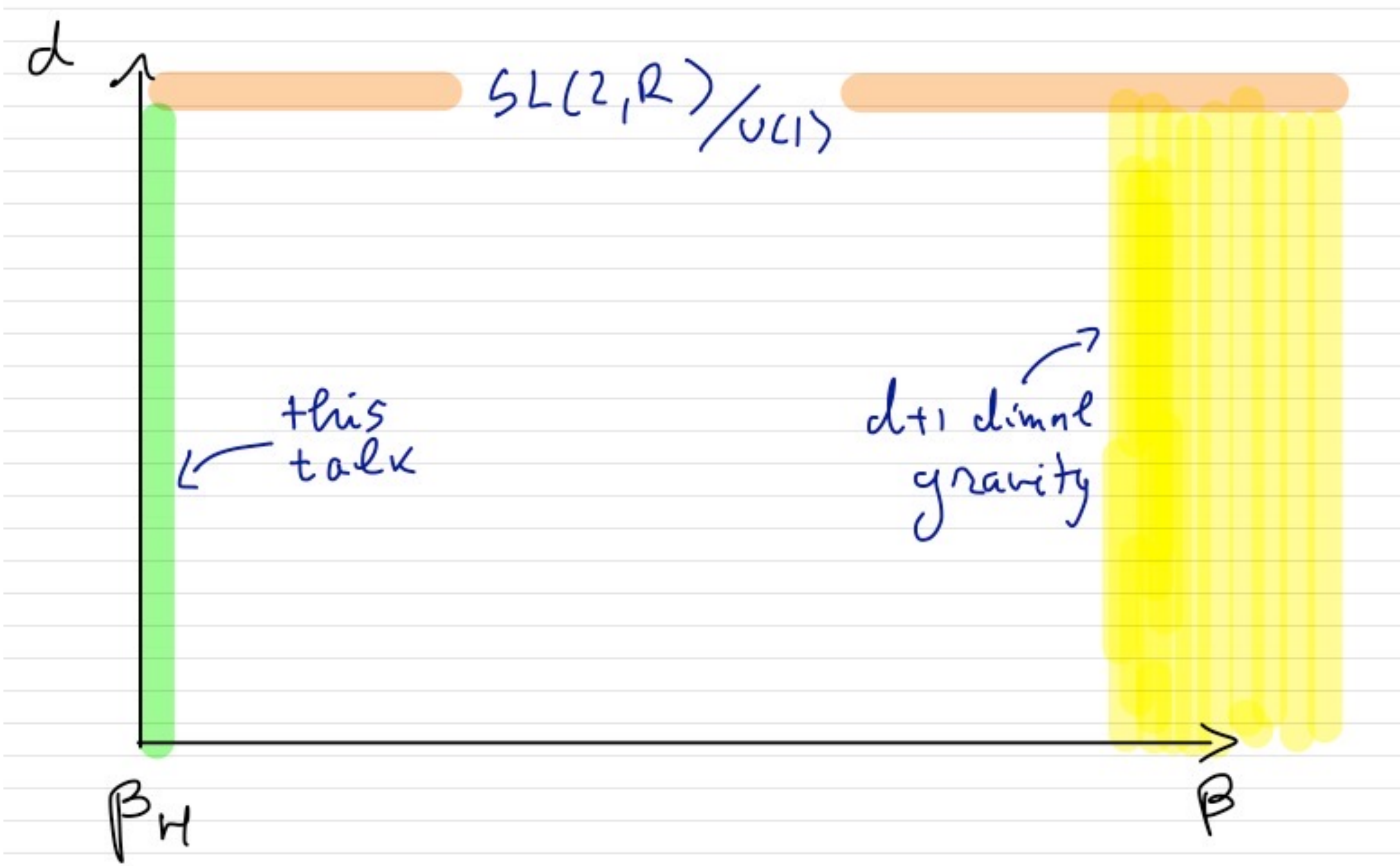
Figure 4: Plots of $-\frac{\chi(r)}{\sqrt{2}\varphi(r)}$ for different values of m_∞, at $d = 6.01$.

As discussed above, this is strong evidence for an enhanced SU(2) symmetry of the underlying worldsheet CFT.

Thus, we conclude that the modified HP effective action describes the small EBH for $d = 6 + \epsilon$. One can compute higher order corrections in ϵ by including higher order terms in the effective action.

As ϵ increases, we expect $\chi(0)$ to increase as well. When it becomes of order one, which happens for integer $d > 6$, $\chi(0)$ presumably becomes of order one, and one needs to solve the full worldsheet CFT. E.g. for $m_\infty = 0$, i.e. at the Hagedorn temperature, one needs to solve the non-abelian Thirring model with an r – dependent coupling described on p. 34.

We arrive at the following qualitative picture:



One interesting check of this picture is the overlap of the green and orange lines at large d . Our picture predicts that at $T = T_H$ the EBH background is described by a worldsheet theory with an enhanced $SU(2)$ symmetry for all d. Thus, the large d theory must have such a symmetry as well. The $SL(2,R)/U(1)$ CFT at level k=2 (bosonic) or 4 (supersymmetric) is indeed known to have such a symmetry.

In fact, the $SL(2,R)/U(1)$ at level k=2 (4) has a description as a non-abelian Thirring with a coupling that depends on the radial direction, so it's natural that the two are related as indicated in the figure on the previous page.

Open string analog

CMW pointed out that there is an interesting analog of the EBH problem in D-brane physics. It involves two D-branes extended in R^d and separated by a distance L in a transverse direction. Callan and Maldacena found a solution of the DBI e.o.m. for this system, that describes the branes connected by a throat. Their analysis is valid for $L \gg l_s$. In this limit, the width of the throat is $\sim L$.

One can think of the CM solution as an analog of the large EBH, with L playing the role of β.

Just like the EBH gives at large d the SL(2,R)/U(1) two dimensional BH, the CM solution gives at large d the hairpin brane of Zamolodchikov et al.

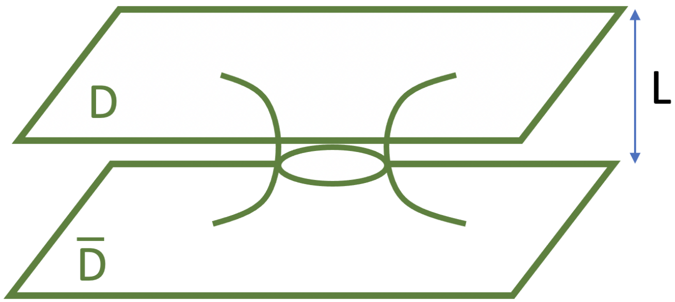
Similarly to the BH analysis, one can ask what happens when we decrease L, and in particular approach the point where a tachyon stretched between the branes becomes massless, $L \to L_c$.

It turns out that in this regime the system is descrbed by the same effective action that we studied, with

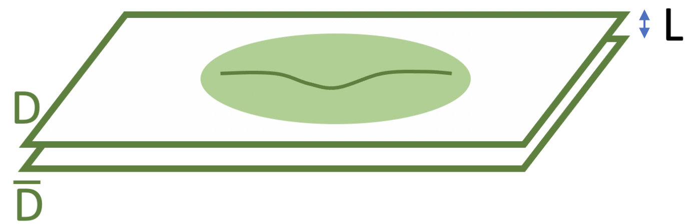
$$m_\infty^2 = \frac{L^2 - L_c^2}{(2\pi\alpha')^2} \ .$$

Thus, the solutions have the same properties!

For $L \gg L_c$ the brane configuration looks like



For $L \simeq L_c$:



Varying L, or m_∞ , continuously interpolates between the two, despite the fact that they look topologically distinct!

For $d > 6,\ m_\infty = 0$, the solutions again have the property that $\chi = -\sqrt{2}\varphi$. In the closed string case, this was a consequence of an enhanced SU(2) symmetry. Is this the case here as well?

Answer: yes!

The worldsheet theory describing the branes is defined on the upper half plane, with the boundary interaction

$$\delta L = \chi_{\rm op}(r) J^i \sigma^i$$

The analog of the localized Thirring model is a localized Kondo system.

Discussion

The main qualitative conclusion of this work is that the hypothesis that continuing EBH's to $T \simeq T_H$ gives worldsheet theories that are well described by a HP-type effective Lagrangian is consistent with all the test we subjected it to.

It leads to a picture according to which small EBH's are not really small. They have a long range condensate of the winding tachyon χ, whose range goes to infinity as $T \to T_H$.

We have identified the worldsheet theory that governs the EBH at $T = T_H$.

There are many things left to do.

In the Euclidean case, it would be nice to solve the localized non-abelian Thirring and Kondo CFT's that we were led to for describing the critical closed and open string systems (with $m_\infty = 0$).

A particularly interesting question concerns the Lorentzian analogs of the systems we described. It has been proposed that the winding tachyon condensate corresponds in this case to a kind of stringy corona surrounding the horizon of the BH, but this has not yet been made precise. It might be easier to understand it first in the open case (as in cigar vs hairpin).

Other questions include:

- Why is $d = 6$ special?

- CMW?

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