Black hole signatures in holographic correlators

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The framework

I'll focus on the traditional $\text{AdS}_{d+1}/\text{CFT}_d$ setup in the regime where

- the central charge c is large $\Rightarrow$ the bulk is semiclassical
- the CFT is strongly coupled $\Rightarrow$ there is a large gap between the (super)gravity and string modes

I am particularly interested in the heavy sector $\frac{\langle O_H|\hat{\Delta}|O_H\rangle}{\langle O_H|O_H\rangle} \sim \mathcal{O}(c)$

For simplicity, I'll take $d = 2$ and work with type IIB on $\text{AdS}_3 \times S^3 \times T^4$

- The CFT enjoys an enhanced superconformal symmetry
- the supergravity description is easier than in the AdS_5 case
- the interesting questions about (large) black holes remain

Strominger Vafa 9601029

The general question: what can we learn about black holes (BHs) by probing heavy states (rather than the other way around)?

The key ingredients

The relevant CFT has a free locus (very much as $\mathcal{N}=4$ at $g_{YM}=0$). In this case it reduces to a collection of N quadruplets of free fields (four boson and four fermions) with a $\mathcal{S}_N$ permutation gauge symmetry

There is a 20-dimensional space of deformations preserving the $(4,4)$ superconformal symmetry. The $\frac{1}{2}$-BPS states are protected. When they are light ($\Delta \sim \mathcal{O}(c^0)$), they are in one-to-one correspondence with supergravity excitations of $\mathrm{AdS}_3 \times S^3 \times T^4$

We can construct a O_H by binding many light states. Example: if O_L is a light $\frac{1}{2}$-BPS state, consider $O_H \sim O_L^p$ with $\frac{p}{N} \sim 0.5$. Main goals:

- derive explicit expressions for a class of 4-point correlators $\langle \bar{O}_H(\infty) O_H(0) \bar{O}_L(1) O_L(z_c, \bar{z}_c) \rangle$ in the supergravity regime
- when the solution dual to O_H is approximately the BTZ black, how do the HHLL correlator compare with $\langle \bar{O}_L(1) O_L(z_c, \bar{z}_c) \rangle_{BTZ}$?

References

Background results

. . . , 1503.01463, 1607.03908, 1711.10474, . . . :
Superstrata geometries

1710.06820, 2007.12118, . . . :
AdS_3 heavy-light correlators

in (various) collaboration with: I. Bena, A. Bombini, A. Galliani, S. Giusto, M. Hughes, E. Martinec, E. Moscato, M. Shigemori, D. Turton, N. Warner

Work in progress with S. Giusto, C. Iossa
See also Bena, Heidmann, Monten, Warner 1905.05194

The plan

I'll introduce the superstrata and focus on a family of "scaling" solutions

The dual CFT description is a coherent state composed of a large number of "supergravitons" (i.e. light supersymmetric CFT operators)

Quadratic fluctuations around any such solution capture a Heavy-Light holographic correlator (HHLL)

$$\langle \bar{O}_L(1) O_L(z_c, \bar{z}_c) \rangle_{ds^2_H} \longleftrightarrow \langle \bar{O}_H(\infty) O_H(0) \bar{O}_L(1) O_L(z_c, \bar{z}_c) \rangle \equiv \mathcal{C}(z_c, \bar{z}_c)$$

Three regimes (I'll focus on the last two):

- in the light regime "$p \to 1$", one obtains the LLLL result. The first AdS_3 correlator was derived in this way! Giusto, RR, Wen 1812.06479
- $p/N \ll 1$ (but $p \sim N$) small BH limit Giusto, Hughes, RR, 2007.12118
- in the limit "$p/N \sim 1$" the geometry becomes that of BTZ. What happens to the HHLL correlator? Giusto, Iossa, RR, work in progress

The graviton gas: CFT side

If O_k is a anti-CPO of dimension k one can consider its descendants

1711.10474;1812.08761

$$O_{k,m,n,q} \equiv (J_0^+)^m (L_{-1})^n (G^{+1}_{-\frac{1}{2}} G^{+2}_{-\frac{1}{2}})^q \, O_k$$

Spectral flow maps $O_{k,m,n,q}$ to a D1-D5-P state with $h > \bar{h} = \frac{c}{24}$

By using $O_{k,m,n,q}$ (also of different types) we can build "semi-classical" multi-particle states characterised by the continuous parameters B_i

Kanitscheider, Skenderis, Taylor

$$|B_1, B_2, \ldots\rangle_{\rm NS} \sim \sum_{p_i} A^{N-p_\sigma} (B_1 O_{k_1,m_1,n_1,q_1})^{p_1} (B_2 O_{k_2,m_2,n_2,q_2})^{p_2} \ldots \, |0\rangle_{\rm NS}$$

$p_\sigma = \sum p_i$, $|A|^2 + \sum_i |B_i|^2 = N$. When $B_i^2 \sim N \gg 1$, these are coherent-like states as the sums over p_i-sum are peaked for $p_i \approx B_i^2/k_i$

What is the gravitational description of $|B_1, B_2, \ldots\rangle_{\rm NS}$?

The graviton gas: bulk side

AdS/CFT relates operators and sugra fields: $O_{k,m,n,q} \longleftrightarrow \phi_{k,m,n,q}$

At linear order in B_i, $|B_1,\ldots\rangle_{\rm NS}$ is a perturbation of the vacuum

$$|0\rangle_{\rm NS} + \sum_i B_i\, O_{k_i,m_i,n_i,q_i}\, |0\rangle_{\rm NS} \longleftrightarrow {\rm AdS}_3 \times S^3 + \sum_i B_i\, \phi_{k_i,m_i,n_i,q_i}$$

where ϕ_{k_i,m_i,n_i,q_i} solves the linearised sugra eqs. around $\mathrm{AdS}_3 \times S^3$

The "superstratum" approach provides an algorithm to extend the linear solutions to exact solutions valid for $B_i^2 \sim N$. The key points:

- The susy eqs. can be written in a "linear" form Bena, Giusto, Shigemori, Warner; 1306.1745
- The non-linear extension requires an ansatz: ambiguities are fixed by imposing regularity and input form the CFT 1503.01463;...; Heidmann, Warner
- Precision holography provides a posteriori checks of the non-linear completion and the holographic interpretation Kanitscheider, Skenderis, Taylor; 1507.00945; Giusto, Rawash, Turton

An interesting example

We can consider a state built with just one type of constituents $O_{1,0,\tilde{n},0}$
The heavy state and its (average) charges are

$$O_H^{(f)} = \sum_{p=0}^{N} (1-\eta^2)^{\frac{p}{2}}\, \eta^{N-p} \left(L_{-1}^{\tilde{n}} s_1^{(f)}\right)^p$$

single particle CPO with $h = \bar{h} = \frac{1}{2}$

$$\langle h_H \rangle = N\left(\tilde{n} + \frac{1}{2}\right)(1-\eta^2)\,, \quad \langle \bar{h}_H \rangle = \frac{N}{2}(1-\eta^2)\,, \quad \langle j_H \rangle = \langle \bar{j}_H \rangle = \frac{N}{2}(1-\eta^2)$$

The 6d geometry (Einstein frame) reads

$$ds_6^2 = \frac{\Lambda}{G} ds_3^2 + \Lambda d\theta^2 + \frac{\sin^2\theta}{\Lambda}\left(d\varphi_1 + (1-\eta^2)d\tau\right)^2 + \frac{G\cos^2\theta}{\Lambda}\left(d\varphi_2 + d\sigma - \frac{\eta^2}{G}(d\sigma + F(d\tau + d\sigma))\right)^2$$

$$ds_3^2 = G\,\frac{d\rho^2}{\rho^2+1} - \eta^2(\rho^2+\eta^2)\,d\tau^2 + \eta^2\rho^2\,d\sigma^2 + \eta^2\rho^2 F\,(d\tau + d\sigma)^2$$

$$G = 1 - \frac{1-\eta^2}{\rho^2+1}\left(\frac{\rho^2}{\rho^2+1}\right)^{\tilde{n}}\,,\ F = \frac{1-\eta^2}{\eta^2}\left[1 - \left(\frac{\rho^2}{\rho^2+1}\right)^{\tilde{n}}\right]\,,\ \Lambda = \left[1 - \frac{1-\eta^2}{\rho^2+1}\left(\frac{\rho^2}{\rho^2+1}\right)^{\tilde{n}}\sin^2\theta\right]^{\frac{1}{2}}$$

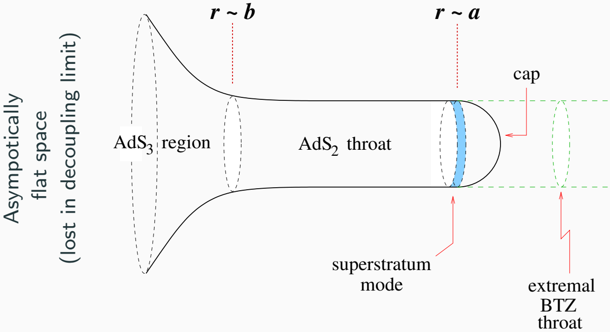
The BH "threshold" ($N\, n_P - j_R^2 > 0$ in the R-sector, $j_R = -\frac{N\eta^2}{2}$) implies

$$\eta^2 < 2\sqrt{\tilde{n}}\,(\sqrt{\tilde{n}+1} - \sqrt{\tilde{n}}) \equiv \eta_c^2(\tilde{n})$$

The BTZ limit

A cartoon of the dual sugra solution (“AdS_3” part) looks as follows

1607.03908



where $\eta^2 = \frac{a^2}{a^2+\frac{b^2}{2}} \equiv \frac{a^2}{a_0^2}$, $\rho = \frac{r}{a}$.

the BTZ is limit is $a \to 0$, r fixed ($\hat{r}^2 = \frac{r^2}{a_0^2} + \tilde{n}$ for the usual coordinates)

4-point correlators: generalities

So far we focused on constructing sugra solutions dual to heavy states

Most of the interesting dynamics is encoded in the perturbations around the geometries. The first step is to study the quadratic fluctuations

We consider perturbations O_L that are described by a scalar field in 6D

Technically, we need to derive the regular, non-normalisable solution that at the boundary ($\rho \to \infty$) scales as

$$z_c = e^{i(\tau+\sigma)} \qquad\qquad \text{vev of } O_L(z_c, \bar{z}_c)$$

$$\nwarrow \qquad\qquad\qquad \nearrow$$

$$\phi_\Delta(\rho; z_c, \bar{z}_c) \overset{\rho\to\infty}{\longrightarrow} \delta(z_c - 1)\,\rho^{\Delta-2} + b(z_c, \bar{z}_c)\,\rho^{-\Delta}$$

$$\searrow$$

$$\text{source for } \bar{O}_L(1)$$

A scalar probe

Consider a scalar probe $O = G^{-A}\tilde{G}^{-B}O_k^{(g)}$ (with $f \neq g$). The dual description is in terms of (the $(k-1)^{\mathrm{th}}$ S^3 harmonics) of a 6D scalar Φ_L

It satisfies $\Box_6\Phi_L = 0$ for any $\tilde{n}$, even when $\eta \neq 0$

By taking the S^3 decomposition (with k odd and $j = \bar{j} = 0$) and the Fourier transform in spacetime we have

$$\Phi_L = \psi(\rho,\tau,\sigma)Y_{k-1}(\theta,\varphi_1,\varphi_2)\,, \qquad \psi(\rho,\tau,\sigma) = \frac{1}{(2\pi)^2}\sum_{\ell\in Z}\int d\omega\, e^{i\omega\tau+i\ell\sigma}\, g(\omega,\ell)\,\psi(\rho)$$

$$\left[\Box_3 - \tfrac{\Delta(\Delta-2)}{G}\right]\psi(\rho,\tau,\sigma) = 0\,, \quad \Delta = h+\bar{h} = k+1\,, \quad m^2 = \Delta(\Delta-2)$$

$$\Downarrow$$

$$\psi''(\rho) + \tfrac{1+3\rho^2}{\rho(1+\rho^2)}\psi'(\rho) + \left\{\frac{\rho^2(\ell-\omega)\left[(\ell-\omega)\left(1-(1-\eta^2)\left(\frac{\rho^2}{1+\rho^2}\right)^{\tilde{n}}\right)-2\eta^2\ell\right]-\eta^4\ell^2}{\eta^4\rho^2(1+\rho^2)^2} - \frac{\Delta(\Delta-2)}{\rho^2+1}\right\}\psi(\rho) = 0$$

Tractable cases: $\tilde{n} = 0, 1, 2$.

The two charge case ($\tilde{n} = 0$)

The radial equation can be recast in the Schroedinger form

$$z = \frac{\rho^2}{1+\rho^2}\,, \quad \psi(\rho) = z^{-\frac{1}{2}}\,u(z) \quad \Rightarrow \quad \left(\partial_z^2 + V_n(z)\right)u(z) = 0$$

$$V_n(z) = \frac{x_0 + x_1 z + x_{\tilde{n}+1} z^{\tilde{n}+1}}{4\eta^4 z^2(1-z)} - \frac{\Delta(\Delta-2)}{4z(1-z)^2}$$

$$x_0 = \eta^4(1-\ell^2)\,,\; x_1 = (\eta^2(\ell-1)-(\ell-\omega))(\eta^2(\ell+1)-(\ell-\omega))\,,\; x_{\tilde{n}+1} = (\eta^2-1)(\ell-\omega)^2$$

When $\tilde{n} = 0$, we get the hypergeometric equation (as for AdS_3)

Impose the regularity conditions at $z = 0$ $\Rightarrow u_{reg}(z) \sim z^{\frac{1+|\ell|}{2}}\,{}_2F_1(\cdot,\cdot;\cdot;z)$

In the hypergeometric case we can use the known connection formulas

$$\psi_{reg}(\rho) = \mathcal{A}(\omega,\ell)\,\rho^{\Delta-2}\left(1+\mathcal{O}\left(\rho^{-2}\right)\right) + \mathcal{B}(\omega,\ell)\,\rho^{-\Delta}\left(1+\mathcal{O}\left(\rho^{-2}\right)\right) \Rightarrow \mathcal{C}(\omega,\ell) = \frac{\mathcal{B}(\omega,\ell)}{\mathcal{A}(\omega,\ell)}$$

$\mathcal{C}(\omega,\ell)$ has poles when $\mathcal{A} = 0$: at $\omega_n = \pm\frac{a}{a_0}\sqrt{(|\ell|+2n)^2 + \frac{b^2\ell^2}{2a^2}}$ 1710.06820

They are the (average) dimensions of the bound states : $O_H \partial^m \bar{\partial}^{\bar{m}} O_L$:

The ω_n's become dense as $a \to 0$, but are evenly space as $b \to 0$

The three charge case ($\tilde{n} > 0$)

One key feature missed by the previous calculation is the long AdS_2 throat that develops in the $a \to 0$ limit of the $\tilde{n} > 0$ case

Also this problem reduces to 3D, but we get an irregular singularity. Choosing $\tilde{n} = 1$, we get a reduced confluent Heun (triple pole at $z = \infty$)

$$V_1(z) = \frac{u - \frac{1}{2} + \alpha_1^2 + \alpha_0^2}{z(z-1)} + \frac{\frac{1}{4} - \alpha_1^2}{(1-z)^2} + \frac{\frac{1}{4} - \alpha_0^2}{z^2} - \frac{L^2}{4z}$$

$$\alpha_0 = \frac{|\ell|}{2}, \quad \alpha_1 = \frac{\Delta - 1}{2}, \quad L = \frac{i(\ell - \omega)\sqrt{1-\eta^2}}{\eta^2} = \frac{2iw\sqrt{1-\eta^2}}{\eta^2},$$

$$u = \frac{\ell^2(1-\eta^2) + \eta^2 - \omega^2}{4\eta^2} = \frac{(p+w)^2(1-\eta^2) + \eta^2 - (p-w)^2}{4\eta^2}.$$

w and p are

$$p = \frac{\ell + \omega}{2}, \quad w = \frac{\ell - \omega}{2}$$

The Heun equation appears in several other black hole related problems: Quasi Normal Modes, tidal response, thermal correlators. Recent progress exploiting the relation to Liouville CFT and its 4d AGT dual $\mathcal{N} = 2$ GT

Aminov, Grassi, Hatsuda; Bianchi, Consoli, Grillo, Morales; Bonelli, Iossa, Lichtig, Tanzini; Dodelson, Zhiboedov; ...

The aim: use this new technology to reassess the problem analysed by Bena, Heidmann, Monten, Warner 1905.05194

Heun's connection problem

The basic idea:

- A semiclassical limit ($c \to \infty$) of BPZ equations for certain 4-point Liouville correlators satisfy the Heun equation
- The CFT crossing symmetry relates the series expansion around different singular points. The connection formulae are given in terms of the (known) Liouville 3-point coupling and the Virasoro blocks
- In limits corresponding to weakly coupled $\mathcal{N}=2$ GT, the problem can be efficiently phrased in terms of the Nekrasov partition function

By using the $\mathcal{N}=2$ language we have

$$\mathcal{C}(\omega,\ell)=\frac{\Gamma\left(-2\alpha_1\right)\Gamma\left(\frac{1}{2}+\alpha_0+\alpha_1+\alpha\right)\Gamma\left(\frac{1}{2}+\alpha_0+\alpha_1-\alpha\right)}{\Gamma\left(2\alpha_1\right)\Gamma\left(\frac{1}{2}+\alpha_0-\alpha_1+\alpha\right)\Gamma\left(\frac{1}{2}+\alpha_0-\alpha_1-\alpha\right)}e^{-\partial_{\alpha_1}F}$$

where F is the so called NS prepotential and α is derived from u by inverting Matone's relation.

The small BH limit

The small BH limit corresponds to a weakly coupled $\mathcal{N} = 2$ GT

We can make everything explicit in an expansion for $(1-\eta) \ll 1$

We can derive the dimensions of the bound states : $O_H \partial^m \bar{\partial}^{\bar{m}} O_L$:

$$\omega^{\pm}_{n\ell} = \pm\left(|\ell| + 2n + \Delta\right) + \gamma^{(1)\pm}_{n\ell}(1-\eta^2) + \dots\,,$$

$$\gamma^{(1)+}_{n\ell\geq 0} = -\frac{(2n+\Delta)\left(2\ell^3 + 5\ell^2(2n+\Delta) + \ell(-2+5(2n+\Delta)^2) + (2n+\Delta)\left(6n^2 + 6n\Delta + (\Delta-1)(2\Delta+1)\right)\right)}{2(1+\ell+2n+\Delta)(\ell+2n+\Delta)(-1+\ell+2n+\Delta)}\,,$$

$$\gamma^{(1)+}_{n\ell\leq 0} = -\frac{(2\ell-2n-\Delta)\left(12n^3 + (3\ell-2\Delta-1)(1+\ell-\Delta)\Delta + 2n^2(9\Delta-8\ell) + 2n(-1+3\ell^2-8\ell\Delta+\Delta(5\Delta-1))\right)}{2(-1+\ell-2n-\Delta)(\ell-2n-\Delta)(1+\ell-2n-\Delta)}\,,$$

$$\gamma^{(1)-}_{n\ell<0} = -\gamma^{(1)+}_{n\ell>0}(-\ell)\,, \gamma^{(1)-}_{n\ell>0} = -\gamma^{(1)+}_{n\ell<0}(-\ell)$$

In the semiclassical limit ($\bar{m} = n\,,\ m = \ell + n$ large) we can compare with the results obtained from the phase shift

Karlsson, Kulaxizi, Ng, Parnachev, Tadic,,Zhiboedov

Giusto, Hughes, RR 2007.12118

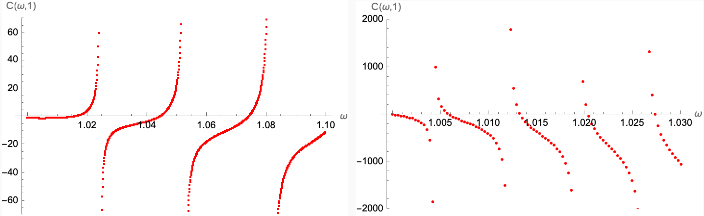
For instance $\gamma^{(1)+}_{n\ell\geq 0} \to -2m\bar{m}\frac{m^2+2m\bar{m}+3\bar{m}^2}{(m+\bar{m})^3}$

The ω_n's are (almost) evenly spaced in the regime $\eta \to 1$ (as for $\tilde{n} = 0$)

The large BH limit: numerical analysis

A different behaviour below the BH threshold ($\eta_c^2 \lesssim 0.828$)?

We looked at $\mathcal{C}(\omega, \ell)$ for $\ell = 1$, $\Delta = 2$ numerically



$\mathcal{C}(\omega, \ell)$ for (a) $\ell = 1$, $\Delta = 2$, $\eta = 0.1$ and (b) $\ell = 1$, $\Delta = 2$, $\eta = 0.01$.

We find poles on the real axis that become denser as $\eta \to 0$

This is different from the BTZ case where there no real poles

What is the relation between $\mathcal{C}(\omega, \ell)$ for $\eta \to 0$ and for BTZ?

The large BH limit: asymptotic analysis

We can use the Liouville CFT description to study the $\eta \to 0$ expansion, i.e. we have explicit perturbative expressions in $\frac{1}{L} \sim \frac{\eta^2}{iw}$ for

$$\mathcal{C}(\omega, \ell) = (iL)^{2\alpha_1} e^{-\partial_{\alpha_1} F_D} \frac{\Gamma(-2\alpha_1)}{\Gamma(2\alpha_1)} \frac{\frac{(4L)^{\frac{g}{2}} e^{L+\partial_g F_D}}{\Gamma\left(\frac{1-g}{2}-\alpha_1\right)} + \frac{(-4L)^{-\frac{g}{2}} e^{-L-\partial_g F_D}}{\Gamma\left(\frac{1+g}{2}-\alpha_1\right)}}{\frac{(4L)^{\frac{g}{2}} e^{L+\partial_g F_D}}{\Gamma\left(\frac{1-g}{2}+\alpha_1\right)} + \frac{(-4L)^{-\frac{g}{2}} e^{-L-\partial_g F_D}}{\Gamma\left(\frac{1+g}{2}+a_1\right)}}$$

$$u = \frac{gL}{2} + \frac{g^2+1}{8} - \frac{a_1^2}{2} + \frac{1}{4} - \alpha_0^2 + \frac{1}{2} L \partial_L F_D(\alpha_0, \alpha_1, g)$$

When ω (and thus $p = \frac{\ell+\omega}{2}$) has an imaginary part one of the terms in the denominator is small since $g \simeq -ip + \ldots \Rightarrow L^g \sim \eta^{-\mathrm{Im} p}$

If we neglect it, the real poles disappear and we get BTZ-like poles at

$$1 \pm g + 2\alpha_1 = -2n\,, \quad n \in \mathbb{Z}_{\geq 0}$$

so we have imaginary poles both in the upper and the lower half-plane

The large BH limit: WKB analysis

In order to connect the two pictures we considered a simple case in the semiclassical limit: $|\omega|$ large, $\Delta = \mathcal{O}(1)$ and $\ell = 0$

We follow closely Festuccia and Liu hep-th/0506202, 0811.1033 supplementing the WKB analysis with local solutions for $z \sim 0$ and $z \sim 1$

The real poles are (in terms of the Elliptic integral of the 1^{st} kind E)

$$\frac{\omega_n}{2n+\Delta} = \frac{\pi\eta^2}{2E(1-\eta^2)} = \begin{cases} 1 - \frac{3}{4}(1-\eta^2) + \dots \,, & \eta^2 \sim 1 \\ \frac{\pi\eta^2}{2} + \frac{\pi\eta^4}{8}\left(1 - \log\frac{\eta^2}{16}\right) + \dots \,, & \eta^2 \sim 0 \end{cases}$$

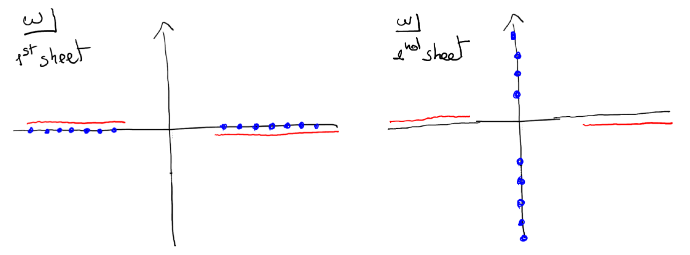
The $\eta \sim 1$ result matches the small BH limit and for $\eta \sim 0$ the poles become dense

When $\mathrm{Im}(\omega) \neq 0$ we have to keep track of the (single) inversion point $z_t = \frac{1}{1-\eta^2}$. The Stokes lines determine which coefficient of the WKB solution jumps when moving from $z \sim 0$ to $z \sim 1$ (if any)

The analytic structure in Fourier space

We can follow the semiclassical correlator in the upper half plane (at large ω). The goal is to understand the analiticity structure

The most natural interpretation of our results is that $\mathcal{C}(\omega, 0)$ has a square-root branch as sketched in the following picture



The exact correlator has just real poles (in blue, left fig.); in the semiclassical limit the poles merge into cuts (red lines) which connect to a second sheet with imaginary poles (in blue, right fig.) in the BH regime

Conclusions

We studied HHLL holographic correlators involving a tractable, but interesting class heavy states

- We obtained explicit results on the bound states energies using recent progress on the Heun connection problem
- For $\eta \ll 1$ the poles merge into cuts and a second sheet appears
  reminiscent of Dodelson, Zhiboedov 2204.09749
- In the large BH regime, on the second sheet we obtain the imaginary poles appropriate for describing the Wightman BTZ correlator

Open questions:

- Study the configuration space correlators — work in progress
- What is the role of quantum corrections in the "BH"-like limit?
  Lin, Maldacena, Rozenberg, Shan
- Are there CFT constraints on HHLL correlators with pure states that have an interesting spacetime interpretation?

Extra slides

An example: $k = 1$, $n \geq 1$, $m = q = 0$

It is convenient to parametrise the 6D metric as

$$ds_6^2 = -\frac{2}{\sqrt{\mathcal{P}}}(dv+\beta)\left[du+\omega+\frac{\mathcal{F}}{2}(dv+\beta)\right]+\sqrt{\mathcal{P}}\,ds_4^2\,, \quad \mathcal{P} = Z_1 Z_2 - Z_4^2$$

and all remaining fields have analogous parametrisations

In the case $k = 1$, $n \geq 1$, $m = q = 0$ (Details are not important)

$$ds_4^2 = \frac{\Sigma\,dr^2}{r^2+a^2}+\Sigma\,d\theta^2+(r^2+a^2)\sin^2\theta\,d\phi^2+r^2\cos^2\theta\,d\psi^2 \qquad \beta = 2^{-1/2}\,a^2 R_y\,\Sigma^{-1}\,(\sin^2\theta\,d\phi-\cos^2\theta\,d\psi) \qquad \Sigma \equiv r^2+a^2\cos^2\theta$$

$$\hat{v}_{k,m,n} \equiv \sqrt{2}\,R_y^{-1}\,(m+n)\,v+(k-m)\phi-m\psi \qquad \Delta_{k,m,n} \equiv a^k\,r^n(r^2+a^2)^{-(k+n)/2}\cos^m\theta\,\sin^{k-m}\theta$$

$$Z_1 = \frac{Q_1}{\Sigma}+\frac{R_y^2}{2Q_5}b_{k,m,n}^2\frac{\Delta_{2k,2m,2n}}{\Sigma}\cos\hat{v}_{2k,2m,2n} \qquad Z_2 = \frac{Q_5}{\Sigma}\,, \quad Z_4 = b_{k,m,n}R_y\frac{\Delta_{k,m,n}}{\Sigma}\cos\hat{v}_{k,m,n}$$

$$\mathcal{F}_{1,0,n} = -a^{-2}\left(1-r^{2n}(r^2+a^2)^{-n}\right) \qquad \omega_{1,0,n} = 2^{-1/2}\,R_y\,\Sigma^{-1}\left(1-r^{2n}(r^2+a^2)^{-n}\right)\sin^2\theta\,d\phi$$

This class of solutions is fully regular if $a^2+\frac{b_{1,0,n}^2}{2} = \frac{Q_1 Q_5}{R_y^2}$. For an appropriate choice of parameters it's described by the picture for (e1)