Probing the Big-Bang With Quantum Fields

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Uses earlier work with Henderson & Sloan (PRD & CQG).

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Preamble

• Space-like singularities are taken to be the absolute beginning or end of space-time. Geodesics of test particles end there. Tidal forces between them become infinite. But what if one uses quantum probes? It has been long argued that they may be tamer for physically more realistic probes. Examples:

Horowitz and Marolf (1995): In certain static space-times with time-like singularities: Dynamics of test quantum particles well-defined in some cases.)

Ishibashi & Hoyasa (1999); Stalker& Tahvildar-Zadeh (2004): Dynamics of classical fields well-defined across certain time-like singularities.

Hofmann and Schneider (2015): The Schwarzschild space-like singularity probed with test quantum fields. Found to be tame. But the arguments are formal; infinite number of degrees of freedom did not receive due care.

• Goal: Revisit the issue for the physically most important dynamical singularities with precision required to handle the infinite number of DOF of QFT carefully. Due to time limitation, this talk focuses on the Big Bang/Big Crunch singularities in the FLRW models. But the approach is more general.

• Main Question: Do test quantum fields $\hat{\phi}(x)$ and observables constructed from them such as $\langle\hat{\phi}(x)\,\hat{\phi}(x')\rangle$, $\langle\hat{\phi}^2(x)\rangle_{\rm ren}$, $\langle\hat{T}_{ab}(x)\rangle_{\rm ren}$ remain regular in the sense of QFT across the big bang? Surprisingly, the answer is in the affirmative!

Nature of Quantum Fields: Minkowski Space

• The Quantum Field in Minkowski space $(\mathring{M}, \mathring{g}_{ab})$ is an operator-valued (tempered) distribution (OVD) :

$$\hat{\phi}^\circ(x) = \int \frac{d^3k}{(2\pi)^3} \left[\hat{A}(\vec{k})\frac{e^{-i\omega\eta}}{\sqrt{2\omega}} + \hat{A}^\dagger(-\vec{k})\frac{e^{i\omega\eta}}{\sqrt{2\omega}}\right] e^{i\vec{k}\cdot\vec{x}} \quad \text{with } x \equiv (\vec{x}, \eta)$$

satisfying $(\mathring{\Box} - m^2)\hat{\phi}^\circ(x) = 0$. That is,

$\hat{\phi}^\circ(f) = \int_{\mathring{M}} d^4\mathring{V}\, \hat{\phi}^\circ(x) f(x)$ is a SA operator on the Fock space $\mathring{\mathcal{F}}$ satisfying $\int_{\mathring{M}} d^4\mathring{V}\, \hat{\phi}^\circ(x)(\Box - m^2) f(x) = 0$, for all test functions $f(x) \in \mathcal{S}$, the Schwartz space.

• The distributional character is not a mere technicality but is conceptually important. For example in

$$[\hat{\phi}^\circ(x),\, \hat{\phi}^\circ(x')] = i\hbar\,(G_{\rm ad} - G_{\rm ret})(x,\, x')\,\hat{I} \quad \text{and}$$

$$\langle \hat{\phi}^\circ(x)\, \hat{\phi}^\circ(x')\rangle_\circ = \frac{\hbar}{4\pi^2}\, \frac{1}{|\vec{x}-\vec{x}\,'|^2 - ((\eta-\eta\,') - i\epsilon)^2}$$

the right sides are genuine distributions; not functions. Meaning of $i\epsilon$: first integrate $\langle \hat{\phi}^\circ(x)\, \hat{\phi}^\circ(x')\rangle_\circ$ with $i\epsilon$ with the test functions and then take the limit. More importantly, products $\hat{\phi}^2(x)$ have to be regularized precisely because $\hat{\phi}(x)$ is an OVD. The textbook terminology of 'field operators' and '2-point functions' (and Dirac 'δ function') can be very misleading if taken literally.

FLRW space-times

• So the question is: Do quantum fields $\hat{\phi}(x)$ continue to be well-defined across the big-bang as OVDs ? For example, in 3-d, $1/r$ is singular as a function but C^∞ as a tempered distribution (satisfying the well-known equality $\vec{\nabla}^2(1/r) = 4\pi\delta^3(\vec{r})$).

• Recall that the Friedmann, Lemaître, Robertson, Walker (FLRW) space-time (M, g_{ab}) is conformally flat:

$$g_{ab}\mathrm{d}x^a\mathrm{d}x^b = a^2(\eta)\,\mathring{g}_{ab}\mathrm{d}x^a\mathrm{d}x^b \equiv a^2(\eta)\,(-\mathrm{d}\eta^2 + \mathrm{d}\vec{x}^2) \quad \text{with } a(\eta) = a_\beta\,\eta^\beta;\ \beta \geq 0.$$

$\eta > 0$ on M, and the big bang corresponds to $\eta = 0$. We can extend $a^2(\eta)$ and hence g_{ab} to the full Minkowski manifold $\mathring{M}$ with $\eta \in (-\infty, \infty)$ as a continuous tensor field (albeit degenerate at $\eta = 0$). Systematic Rationale:

• There is a generalization of the ADM framework for the initial value problem of full GR (based on 'connection dynamics' in which the metric arises as an emergent, secondary field), which is equivalent to the ADM framework when the 3-metric is non-degenerate, but which does not break down if it becomes degenerate. (AA, Henderson & Sloan). In FLRW (as well as Bianchi models and the Schwarzschild solution) it enables one to evolve across the singularity unambiguously (Koslowski, Mercati & Sloan; Mercati; AA & Valdes). (Recall: 'Hubble-normalized variables' used in cosmology.) In FLRW models, the extension yields just the simple prescription given above. As a tensor field, g_{ab} is C^0 at $\eta = 0$ & smooth if $\eta \neq 0$.

QFT in FLRW Space-times

- For definiteness, consider the massless scalar field: $\Box\hat{\phi} = 0$.

On M, $\phi^\circ = a(\eta)\phi(x)$ satisfies a simple equation with respect to the Minkowski metric $\mathring{g}_{ab}$ in presence of a 'universal' time dependent potential:

$$(\mathring{\Box} - V(\eta))\phi^\circ(x) = 0 \qquad \text{with} \qquad V(\eta) = \beta(\beta-1)/\eta^2.$$

Rigorous Result: Because of the form of the potential, one can introduce a canonical $\pm$ frequency decomposition (i.e. a canonical Kähler structure) on the space of classical solutions and write the general solution as

$$\phi^\circ(x) = \int \frac{\mathrm{d}^3k}{(2\pi)^3} \left[z(\vec{k})\mathring{e}(k,\eta) + z^\star(-\vec{k})\mathring{e}^\star(k,\eta)\right] e^{i\vec{k}\cdot\vec{x}}$$

where $\mathring{e}(k,\eta)$ are the positive-frequency modes and $z(\vec{k})$ are regular coefficients (in $\mathcal{S}$). Then the putative OVD on FLRW space-time is given by:

$$\hat{\phi}(x) = \frac{1}{a(\eta)}\hat{\phi}^\circ(x) = \frac{1}{a(\eta)} \int \frac{\mathrm{d}^3k}{(2\pi)^3} \left[\hat{A}(\vec{k})\mathring{e}(k,\eta) + \hat{A}^\dagger(-\vec{k})\mathring{e}^\star(k,\eta)\right] e^{i\vec{k}\cdot\vec{x}}.$$

- The mode functions $\mathring{e}(k,\eta)$ are explicitly known. Generically they diverge at $\eta = 0$. For example, for dust $(\beta = 2)$, they are $(e^{ik\eta}/\sqrt{2k})(1 - i/k\eta)$. As functions, they diverge at the big bang. And there is another $1/a(\eta)$ overall factor in $\hat{\phi}(x)$. The question is: Is $\hat{\phi}(x)$ nonetheless well-defined as an OVD across the big bang on full $\mathring{M}$?

Radiation-filled Universe

• Is $\hat{\phi}(f) = \int_{\mathring{M}} \mathrm{d}^4V \hat{\phi}(x) f(x)$ satisfying $\int_{\mathring{M}} \mathrm{d}^4V \hat{\phi}(x)(\Box f) = 0 \;\; \forall f \in \mathcal{S}$ well-defined on the extended space-time, i.e. for all $\eta \in (-\infty, \infty)$?

• Radiation-filled universe: $a(\eta) = a_1 \eta$ (i.e. $\beta = 1$) $\Rightarrow V(\eta) = 0$, whence, $\hat{\phi}^{\circ}(x) = a(\eta)\,\hat{\phi}(x)$ now satisfies $\mathring{\Box}\hat{\phi}^{\circ} = 0$ in Minkowski space! Hence mode functions $\mathring{e}(k, \eta)$ same as in Minkowski space; trivially regular for $\eta \in (-\infty, \infty)$. But the physical field on FLRW space-time is $\hat{\phi}(x) = a^{-1}(\eta)\,\hat{\phi}^{\circ}(x)$ and $a(\eta) = 0$ at $\eta = 0$! How could it then be regular on Minkowski Fock space?

Answer: The physical volume element is $\mathrm{d}^4V = a^4\,\mathrm{d}^4x$. Hence:

$$\hat{\phi}(f) = \int_{\mathring{M}} \mathrm{d}^4V\, \hat{\phi}(x) f(x) = \int_{\mathring{M}} \mathrm{d}^4x\, \hat{\phi}^{\circ}(x)\,(a^3(\eta) f(x))$$

and $a^3(\eta) f(x) \in \mathcal{S}$ if $f \in S$. Hence $\hat{\phi}(f)$ is in fact a well-defined operator on the Minkowski Fock space $\Rightarrow$ $\hat{\phi}(x)$ well defined OVD for all $\eta \in (-\infty, \infty)$!

Next, the expectation value of the product of fields

$$\langle \hat{\phi}(x)\,\hat{\phi}(x')\rangle = \frac{1}{a_1^2 \eta \eta'} \langle \hat{\phi}^{\circ}(x)\,\hat{\phi}^{\circ}(x')\rangle_{\circ} = \frac{1}{a_1^2 \eta\eta'}\,\frac{\hbar}{4\pi^2}\,\frac{1}{r^2 - (t - i\epsilon)^2}$$

(where $r = |\vec{x} - \vec{x}\,'|$ and $t = \eta - \eta'$) is also a well-defined bi-distribution because $\mathrm{d}^4V = a_1^4 \eta^4\,\mathrm{d}^4x$ and d^4x is well defined on all of $\mathring{M}$. Furthermore $\langle \hat{\phi}^2(x)\rangle_{\mathrm{ren}} = 0$ and $\langle \hat{T}_{ab}(x)\rangle_{\mathrm{ren}}$ is a well-defined, non-zero distribution on all of the extended manifold $\mathring{M}$. So the theory is not trivial.

Dust-filled Universe

• More interesting case: Dust-filled universe where $a(\eta) = a_2\eta^2$. Hence $(1/a(\eta))$ diverges faster at $\eta = 0$ and mode functions $(e^{ik\eta}/\sqrt{2k})(1 - i/k\eta)$ also diverge at the big bang (unlike in the radiation-filled case).

• The 1-particle Hilbert space is built out of solutions:

$$\phi(x) = \frac{1}{a(\eta)} \int \frac{\mathrm{d}^3k}{(2\pi)^3} \left[z(\vec{k})\mathring{e}(k,\eta) + z^\star(-\vec{k})\mathring{e}^\star(k,\eta)\right] e^{i\vec{k}\cdot\vec{x}} \text{ with } z(\vec{k}) \in \tilde{\mathcal{S}}$$

and they all diverge at $\eta = 0$. So how can there be a well-defined Fock space? There is, because the 1-particle norm $||\phi(x)||$ is perfectly finite (and non-zero) at $\eta = 0$ because the divergence in the value of $\phi(x)$ is precisely compensated by the vanishing of the 3-volume element there!

$$||\phi(x)||^2 = \frac{1}{\hbar} \int \frac{\mathrm{d}^3k}{(2\pi)^3} |z^\star(\vec{k})|^2.$$

This is analogous to the QM fact that while $\Psi(\vec{x}) := (1/r)e^{-\alpha r}$ is divergent as a function, it represents a well-defined state in $\mathcal{H} := L^2(\mathbb{R}^3)$, because the volume element d^3x goes as r^2.

• Since $\mathrm{d}^4V = a_2^4\eta^8\,\mathrm{d}^4x$, $\hat{\phi}(f)$ is a well-defined OVD. (However, there is an infrared subtlety (Ford and Parker (1977)). Already for $\eta > 0$, the action of $\hat{\phi}(x)$ is well-defined on a co-dimension 1 subspace $\mathcal{S}_1$ of $\mathcal{S}$ and there is a 1-parameter freedom in extending its action on full $\mathcal{S}$, representing an infrared cutoff ℓ. But this cutoff has nothing to do with the big bang. Once $\hat{\phi}(x)$ defined for $\eta > 0$, it continues to be well-defined for $\eta \leq 0$).

Dust filled Universe: 'BKL Behavior'

- The expectation value of the product of fields is given by

$$\langle\hat\phi(x)\,\hat\phi(x')\rangle = \frac{\hbar}{4\pi^2}\,\frac{1}{a_2^2\eta^2\eta'^2}\Big[\frac{1}{(r^2-(t-i\epsilon)^2)} + \frac{1}{2\eta\eta'}[2(1-\gamma)+\ln\frac{r^2-(t-i\epsilon)^2}{\ell^2}\Big].$$

It is a well-defined bi-distribution, i.e., $\int_{\mathring{M}} \mathrm{d}^4V\,\mathrm{d}^4V'\,\langle\hat\phi(x)\,\hat\phi(x')\rangle\,f_1(x)f_2(x)$ is well-defined because $\mathrm{d}^4V = a_2^4\eta^8\mathrm{d}^4x$, and d^4x is well-defined on all of $\mathring{M}$.

For space-like and time-like separated points, one interprets $\langle\hat\phi(x)\,\hat\phi(x')\rangle$ as as a correlation function. In Minkowski space, correlations decay as $1/\mathrm{Dist}^2$ for both space-like and time-like separations.

- Now, there is an interesting space vs time asymmetry as one approaches the singularity: Consider points that are space-like or time-like separated by a fixed proper (geodesic) distance D. As one approaches the big bang, but space-like correlations dominate over but time-like ones: $\lim_{\eta_\circ\to 0}\frac{\langle\hat\phi(\vec x,\eta_o)\,\hat\phi(\vec x\,',\eta_\circ)\rangle}{\hat\phi(\vec x_\circ,\eta_o)\hat\phi(\vec x_\circ,\eta)} = \infty$ (as $2D/a_2\eta^3$). Strong correlations $\sim$ smaller variations $\Rightarrow$ smaller derivatives. Therefore, "time derivatives dominate over space-derivatives" as in the well-known BKL behavior of GR. But one has to keep in mind that conceptually these are quite different statements: this behavior refers to test quantum fields on a given FLRW background while the BKL behavior refers to the gravitational field itself.

Dust filled Universe: Operator-Products

• $\hat{\phi}(x)$ is a 'dimension 1' OVD, while $\langle\hat{\phi}^2(x)\rangle_{\rm ren}$ is a 'dimenmsion 2' OVD and $\langle\hat{T}_{ab}(x)\rangle_{\rm ren}$ a 'dimension 4' OVD. So, a priori the fact that $\hat{\phi}(x)$ is well-behaved across the big bang does not mean that these operator-products would be well-defined. Are they?

• Old works (Bunch, Davies, ...) imply $\langle\hat{\phi}^2(x)\rangle_{\rm ren} = \frac{\hbar\mathcal{R}}{288\pi^2}\,(5 - 2\ln\frac{2\mathcal{R}}{3a_2\ell^3\mu^3})$. At the big bang, $\mathcal{R} \sim 1/\eta^6$ is divergent as a function but a C^∞ tempered distribution: $\mathrm{d}^4V = a_2^4\eta^8\mathrm{d}^4x$ and $\eta^8\langle\hat{\phi}^2(x)\rangle_{\rm ren}(x)$ is in fact a C^2 function! Unlike in the radiation-filled case, it does not vanish because $R \neq 0$.

• Old works also provide the expression of $\langle\hat{T}_{ab}(x)\rangle_{\rm ren}$. Being a 'dimension 4' OVD, it involves products and second derivatives of curvature tensors. The explicit expression is long but has the simple form $\langle\hat{T}_{ab}(x)\rangle_{\rm ren} = T_1(\eta)\nabla_a\eta\nabla_b\eta + T_2(\eta)\mathring{g}_{ab}$, where the most divergent term in T_1 and T_2 go as $(\eta^{-8}\ln|\eta|)$. Now, $\mathrm{d}^4V \sim \eta^8$ and $\eta^8 T_1 \sim \eta^8 T_2 \sim \ln|\eta|$, which is a locally integrable function and hence in particular, a C^∞ tempered distribution!

• Summary: Dynamics of $\hat{\phi}$ is much more non-trivial in the dust-filled case: It represents the generic case where the scalar curvature does not vanish. Still, $\hat{\phi}$ is a well-defined OVD in every sense one asks in QFT in CST!

Summary and Generalizations

• Summary: There is a long history of probing classical GR singularities with classical fields and quantum particles. But most analyses were for (conformally) static space-times with time-like singularities.

• Here we considered time-dependent space-times with space-like singularities, which are also physically far more interesting. But time-dependence forces one to consider quantum fields as probes. Somewhat surprisingly, the big bang and big crunch singularities are remarkably tamer when probed with observables associated with quantum fields, when one keeps in mind that these are OVDs.

Classical fields $\phi(x)$ that define 1-particle states do diverge at the big bang singularity. But their norm in the 1-particle Hilbert space is finite because the shrinking of the volume element exactly compensates for this divergence. Similar to the fact that the wave function $\Psi(\vec{r}) = (1/r)e^{-r/r_o}$ diverges at the origin but is a well-defined element of the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^3)$ in QM. Similarly, the mode functions that enter the expansion of $\hat{\phi}(x)$ diverge but it is a well-defined OVD: smeared operators $\hat{\phi}(f)$ are well defined. $\langle\hat{\phi}(x)\,\hat{\phi}(x')\rangle$ –and even $\langle\hat{\phi}^2(x)\rangle_{\rm ren}$ and $\langle\hat{T}_{ab}(x)\rangle_{\rm ren}$– are well defined tempered distributions, just as they are in Minkowski space.

• Generalization: The main result on tame behavior of linear, test quantum fields extend to other FLRW models with $\beta > 0$. I used Radiation and dust filled cases because the mode functions are sufficiently simple to display explicit results. Extension has also been made to closed and open $K = \pm 1$ cases (in preparartion).

Generalizations: Contd

• Higher spins: Since FLRW space-times are conformally flat, quantum (as well as classical) Maxwell fields are trivially regular across the big bang and big crunch. Results on the massless scalar field imply that tameness persists also for spin 2 (i.e. linearized gravitational) fields.

• What about black hole singularities? (Work in Progress)

The 2-D CGHS BH: (spherically symmetric reduction of a string inspired theory): In this case, classically, we have an analytic solution for a scalar field collapse. For test quantum fields, singularity seems completely tame.

The Schwarzschild singularity: One can focus on the 'interior' region inside the horizon (Kontowski-Sachs metric). We have analytic expressions of mode functions as infinite convergent series. But do not have a theorem of uniqueness of (the Kähler structure, or) the positive and negative frequency decomposition. If one uses an 'obvious notion', one can construct a Fock representation on which smeared field operators are well defined. Work in progress to find the notions that correspond to the Unruh and Hartle-Hawking vacua of the (right) asymptotic region and compute $\langle \hat{\phi}(x)\, \hat{\phi}(x')\rangle$, $\langle \hat{\phi}^2(x)\rangle_{\rm ren}$, $\langle \hat{T}_{ab}(x)\rangle_{\rm ren}$ in these states at the singularity. Good control on calculations. Vaidya collpase also within reach.

Generic BH singularities: Expected to be null (Cauchy horizon instability) rather than space-like. Much more difficult. But there is no 'in principle obstacle'.

Broader Perspective

• Key conceptual and mathematical point of this analysis is: Quantum fields $\hat{\phi}(x)$ are operator-valued distributions not operators; and it is important to keep in mind the distributional nature of associated observables such as $\langle\hat{\phi}^2(x)\rangle_{\rm ren}$, $\langle\hat{T}_{ab}(x)\rangle_{\rm ren}$.

• In particular:
(i) Every locally integrable function $f(\eta)$ (such as $\ln|\eta|$) is a tempered distribution: $\mathcal{S} \ni t(\eta) \to \int_{\mathbb{R}} {\rm d}\eta\, t(\eta) \ln|\eta|$ is a continuous map from $\mathcal{S}$ to $\mathbb{R}$; and, (ii) Every tempered distribution is infinitely differentiable. Hence, $1/x^m$ is a tempered distribution.
This is why even when the expectation values $\langle\hat{\phi}(x)\,\hat{\phi}(x\,')\rangle$, $\langle\hat{\phi}^2(x)\rangle_{\rm ren}$, $\langle\hat{T}_{ab}(x)\rangle_{\rm ren}$ diverge as functions, they can be well-defined tempered distributions. Recall: Even in Minkowski space-time, observables of quantum fields are tempered distributions, not functions. Cannot ask them to be better at singularities!!

• These results provide hints for a full quantum gravity theory: to obtain a self-consistent theory that allows matter and geometry to interact quantum mechanically, geometry should also have a distributional character at the micro level. This feature arises in diverse approaches to quantum gravity where, in the Planck regime, excitations of quantum geometry have support in 2 (space-time) dimensions (see, e.g., Carlip's 2009 short review). A concrete example is provided by the distributional nature of quantum geometry in loop quantum gravity and spinfoams. Therefore, these investigations open up the possibility of bridging QFT in CST with full quantum gravity.