Simulating quantum cosmology on a quantum computer

Based on:

- G. Czelusta, J. M. *Quantum variational solving of the Wheeler-DeWitt equation, arXiv: 2111.03038 (2021)*
- D. Artigas, J. M. and C. Rovelli, *A minisuperspace model of compact phase space gravity*, Phys. Rev. D **100**, 043533 (2019)

See also:

- G. Czelusta, J. M. *Quantum simulations of a qubit of space*, Phys. Rev. D **103**, 046001 (2021)
- J. M. *Prelude to Simulations of Loop Quantum Gravity on Adiabatic Quantum Computers*, Front. Astron. Space Sci. **8, 95** (2021)

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The Hamiltonian of General Relativity (GR) is a sum of constraints:

$$H[N,\vec{N}] = S[N] + V[\vec{N}] = \int_\Sigma d^3x NC + \int_\Sigma d^3x \vec{N}\cdot\vec{C}$$

In the quantum case, the constraints are promoted to self-adjoint operators.

In homogeneous quantum cosmology only the scalar constraint $\hat{C}$ remains.

Action of the operator is defined on the kinematical Hilbert space $\mathcal{H}_{\rm kin}$.

Following the Dirac quantization of constrained systems, the physical Hilbert space $\mathcal{H}_{\rm phys}$ is constructed by solving the Wheeler-DeWitt (WDW) equation:

$$\hat{C}|\Psi\rangle \approx 0$$

so that $\ker\hat{C} = \mathcal{H}_{\rm phys} \subseteq \mathcal{H}_{\rm kin}$.

Solutions to the WDW equation are known for certain minisuperspace models.

New methods of solving the WDW equation for more complex configurations are worth seeking…

Outlook of our method of solving the WDW equation on a quantum computer:

1. Regularize theory to make the Hilbert space finite.

a) Replace the flat (affine) phase space for every classical degree of freedom with a sphere. The spherical phase space is a phase space of angular momentum (spin). The flat phase space case is recovered in the large spin limit.
b) Construct regularized quantum kinematics for the system under consideration.
c) Express the Hamiltonian constraint in terms of the spin variables.
d) Quantize the regularized Hamiltonian constraint.
e) Represent the spin operators in terms of qubits.

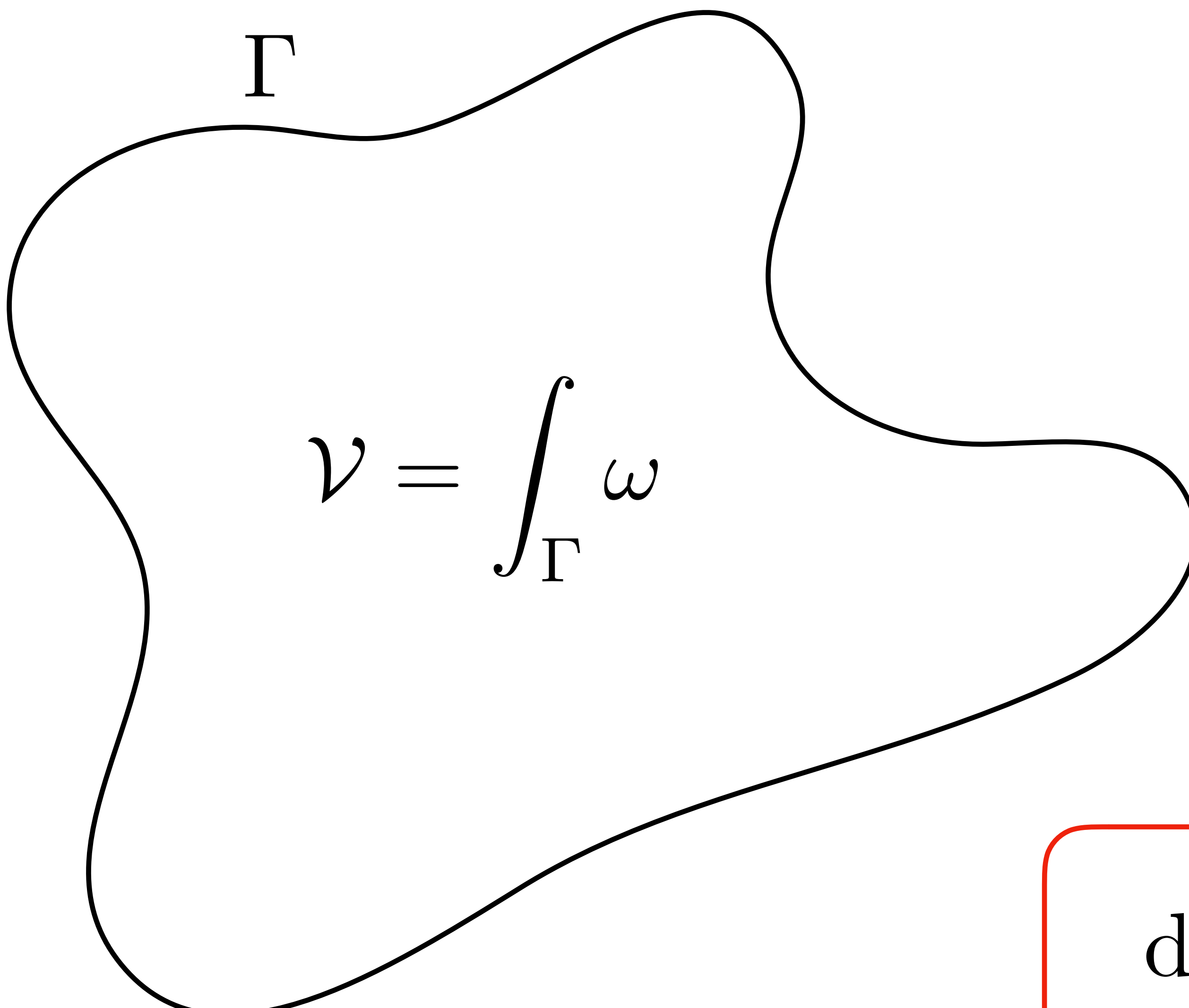
2. Apply Variational Quantum Eigensolver (VQE) to find the states minimizing „Hamiltonian” $\hat{C}^2$:

$$\hat{C}|\psi_0\rangle = 0 \Longleftrightarrow \langle\psi_0|\hat{C}^2|\psi_0\rangle = 0$$

3. Study the large spin limit to recover the affine case.

Compact phase space → Finite Hilbert space

Compact phase spaces arise in the semiclassical description of quantum systems with finite dimensional Hilbert spaces.



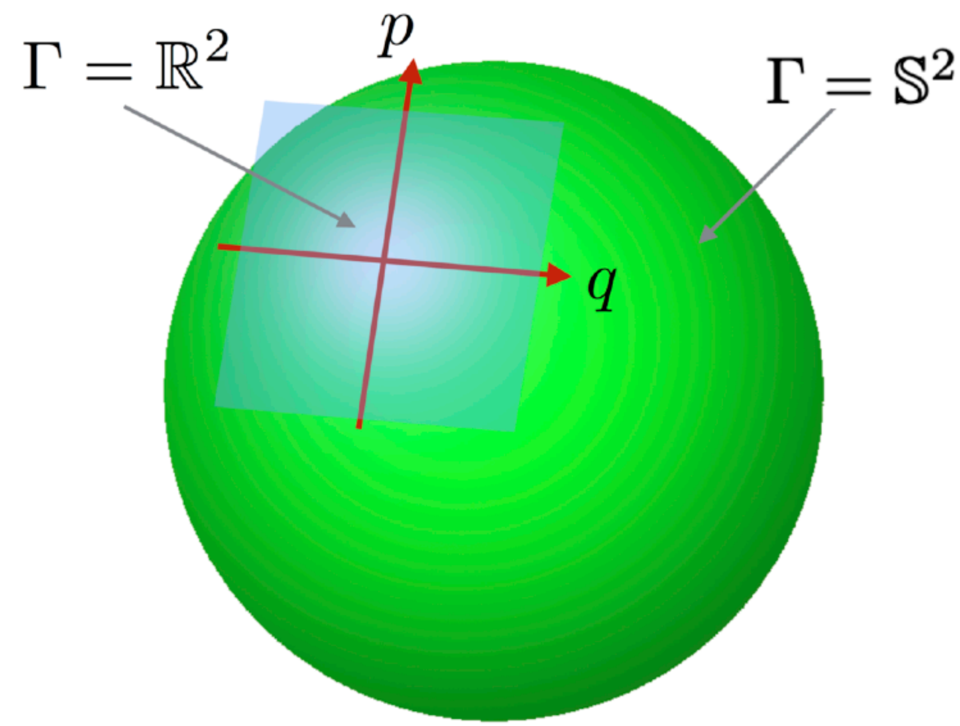
$$\dim \Gamma = 2m$$

$$\dim \mathcal{H} \sim \frac{\mathcal{V}}{(2\pi\hbar)^m}$$

$$\dim \mathcal{H} < \infty \Leftrightarrow \mathcal{V} < \infty$$

We are focused on the compact case rather on the phase space with boundaries.

Spherical phase space



The spherical space is consider here because of its relation to spin. It is also a non-trivial case since the 2-sphere is not a cotangent bundle, but is equipped with symplectic form and is a well defined symplectic manifold.

Symplectic form on 2-sphere:

$$\omega = S \sin\theta \, d\phi \wedge d\theta$$

Volume of phase space:

$$\mathcal{V} = \int_{\mathbb{S}^2} \omega = 4\pi S$$

Change of variables

$$\phi = \frac{p}{R_1} \in (-\pi, \pi],$$

$$\theta = \frac{\pi}{2} + \frac{q}{R_2} \in (0, \pi),$$

$$R_1 R_2 = S$$

Dimensional parameters

Symplectic form

$$\omega = \cos\left(\frac{q}{R_2}\right) dp \wedge dq,$$

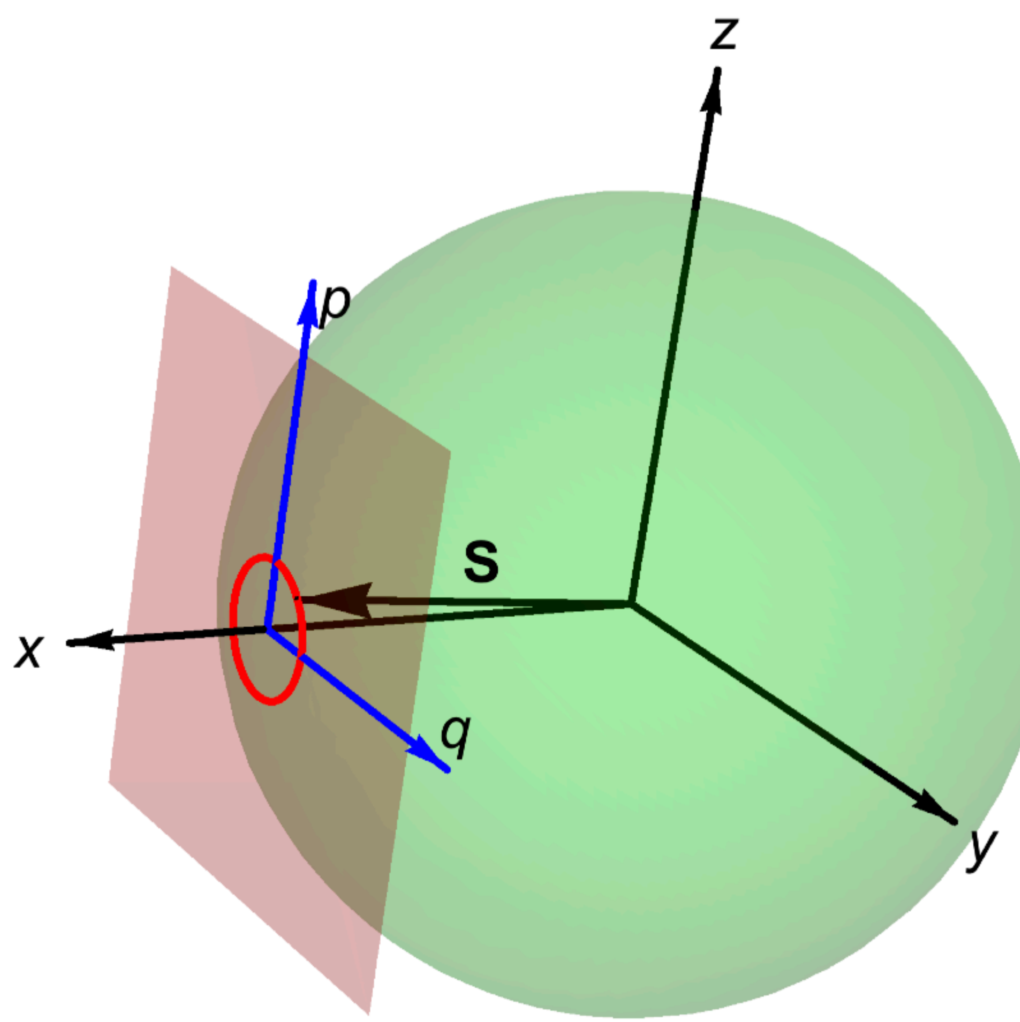
$$\{f, g\} = \frac{1}{\cos(q/R_2)} \left(\frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial q} \right)$$

One can introduce the vector

$$\vec{S} = (S_x, S_y, S_z)$$

with the following components:

$$S_x = S\cos\left(\frac{p}{R_1}\right)\cos\left(\frac{q}{R_2}\right),$$
$$S_y = S\sin\left(\frac{p}{R_1}\right)\cos\left(\frac{q}{R_2}\right),$$
$$S_z = -S\sin\left(\frac{q}{R_2}\right).$$



Poisson bracket: $\{f, g\} = \frac{1}{\cos(q/R_2)}\left(\frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial q}\right)$

Employing the above, the su(2) algebra is obtained: $\{S_i, S_j\} = \epsilon_{ijk} S_k$

The $\vec{S}$ is, therefore, the angular momentum (spin) vector with the norm S.

The symplectic for the system of *m* DOFs is:

$$\omega = \sum_{i=1}^{m} \omega_i = \sum_{i=1}^{m} dp_i \wedge dq_i$$

By performing compactification for every canonical pair the system's phase space changes as follows:

$$\boxed{\mathbb{R}^{2m} \to \mathbb{S}^{2m}}$$

In consequence, the regularized kinematical Hilbert space for the system is:

$$\boxed{\mathcal{H}_{\mathrm{kin}} = \mathcal{H}_s^{\otimes m}}$$

where $\mathcal{H}_s = \mathrm{span}\{s, -s\rangle, ..., |s, s\rangle\}$, so that:

$$\hat{S}^2 |s, s_z\rangle = s(s+1)|s, s_z\rangle$$

$$\hat{S}_z |s, s_z\rangle = s_z|s, s_z\rangle$$

Compactified flat de Sitter cosmology

Kinematics (symplectic form)

Affine

$$\omega = dp \wedge dq$$

2-sphere

$$\omega = \cos\left(\frac{q}{R_2}\right) dp \wedge dq,$$

Dynamics (scalar constraint)

$$C = q\left(-\frac{3}{4}\kappa p^2 + \frac{\Lambda}{\kappa}\right) \approx 0$$

$$C = \frac{S_3}{R_1}\left[\frac{3}{4}\kappa\frac{S_2^2}{R_2^2} - \frac{\Lambda}{\kappa}\right] \approx 0$$

$$p \to p_S := \frac{S_y}{R_2} = R_1 \sin\left(\frac{p}{R_1}\right)\cos\left(\frac{q}{R_2}\right),$$

$$q \to q_S := -\frac{S_z}{R_1} = R_2 \sin\left(\frac{q}{R_2}\right),$$

The procedure is ambiguous! We made the simplest choice.

Friedmann equation:

$$H^2 = \frac{\Lambda}{3}\left(\frac{\sin(q/R_2)}{q/R_2}\right)^2\left[\frac{\cos^2(q/R_2) - \delta}{\cos^2(q/R_2)}\right]$$

where $\delta := \frac{4}{3}\frac{\Lambda}{R_1^2\kappa^2} \in [0, 1]$

The affine case is recovered in the $R_1, R_2 \to \infty$ limit.

Polymer limit

In the $R_2 \to \infty$ limit the so-called polymerization of momentum is obtained.

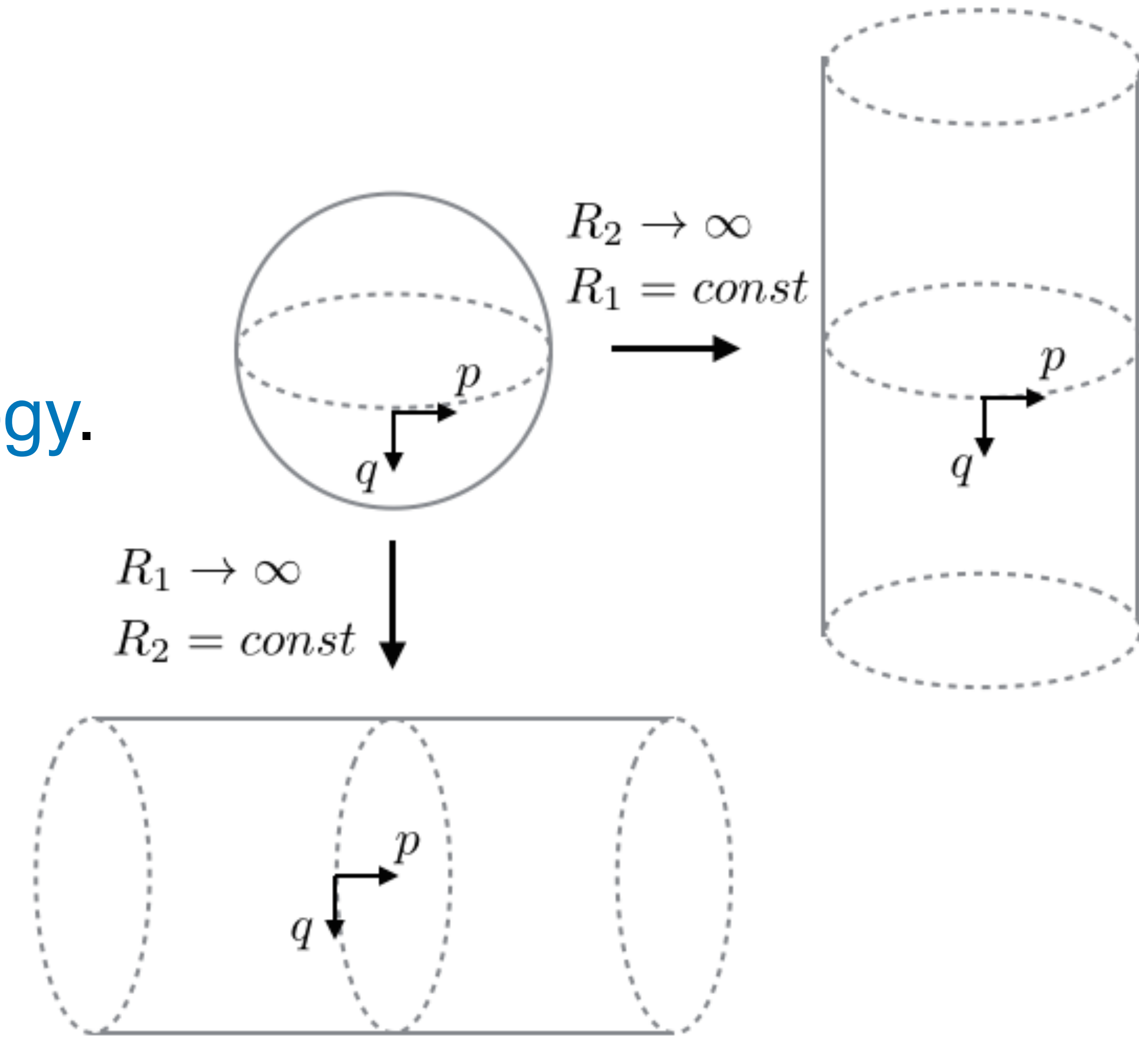
This is the case of loop quantum cosmology.

The su(2) algebra reduces to the cylindrical case:

$$\left\{\sin\left(\frac{p}{R_1}\right), \cos\left(\frac{p}{R_1}\right)\right\} = 0,$$

$$\left\{q, R_1 \sin\left(\frac{p}{R_1}\right)\right\} = \cos\left(\frac{p}{R_1}\right),$$

$$\left\{q, \cos\left(\frac{p}{R_1}\right)\right\} = -\frac{1}{R_1}\sin\left(\frac{p}{R_1}\right),$$



The scalar constraint reduces to:

$$C = q\left(-\frac{3\kappa}{4}\frac{\sin^2(\lambda p)}{\lambda^2} + \frac{\Lambda}{\kappa}\right) \approx 0$$

where the polymerization scale is: $\lambda := \frac{1}{R_1}$

The Friedmann equation is:

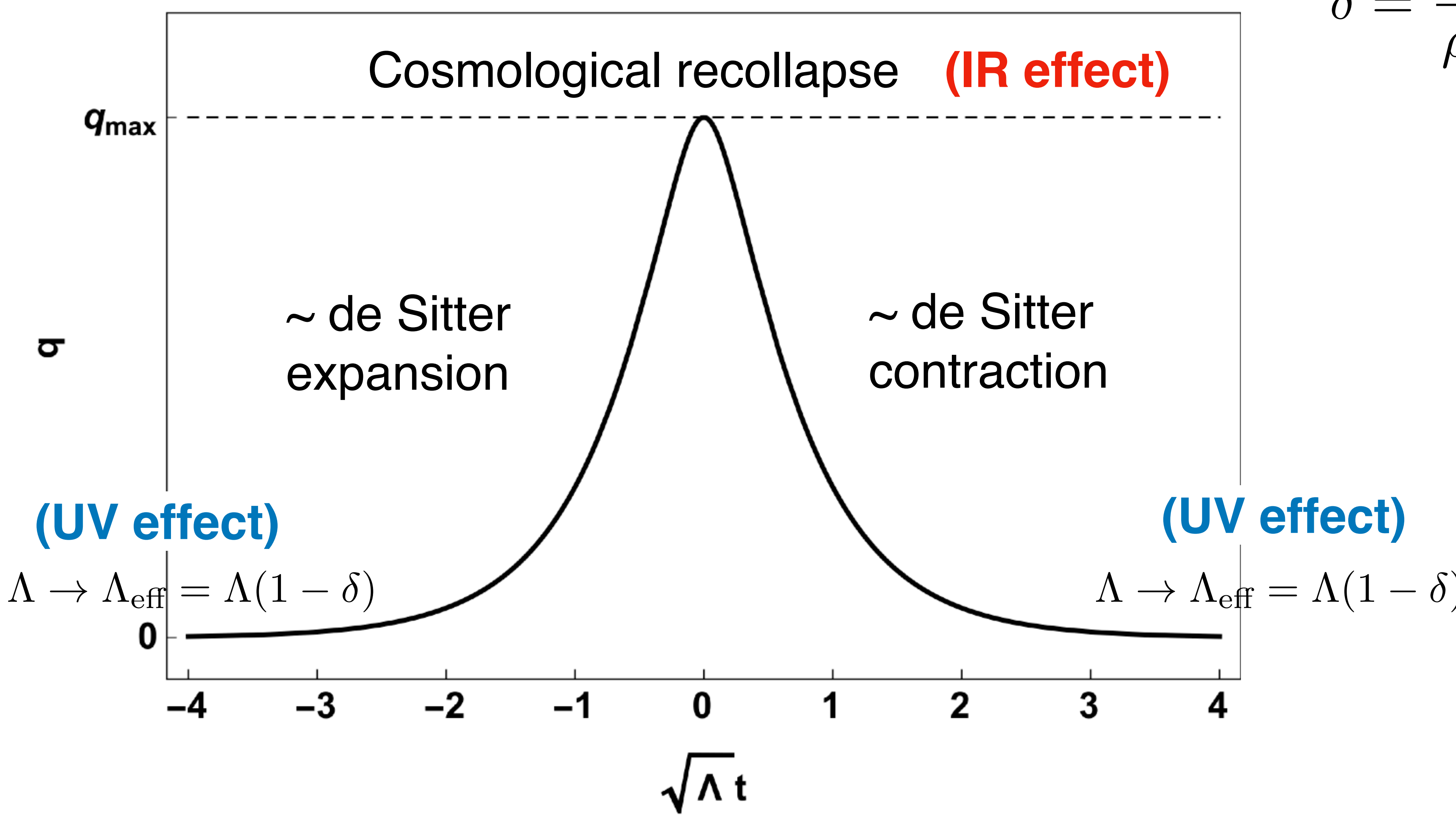
$$H^2 = \frac{\kappa}{3}\rho_\Lambda\left(1 - \frac{\rho_\Lambda}{\rho_c}\right)$$

where $\rho_c := \frac{3}{4}\frac{\kappa}{\lambda^2}$

Cosmological evolution

$$H^2 = \frac{\Lambda}{3}\left(\frac{\sin(q/R_2)}{q/R_2}\right)^2\left[\frac{\cos^2(q/R_2) - \delta}{\cos^2(q/R_2)}\right]$$

$$\delta = \frac{\rho_\Lambda}{\rho_c} \in [0,1]$$



$$H^2 = \frac{\Lambda_{\rm eff}}{3} - \frac{\Lambda}{9}(1+2\delta)\left(\frac{q}{R_2}\right)^2 + \mathcal{O}(q^4)$$

de Sitter expansion may terminate…

and the effective phantom can be a sign of it.

The leading correction

behaves as a phantom matter with the equation of state: $P_S = -3\rho_S$

The quantum constraint

Rescaled classical constraint:

$$C \to \frac{4S^2}{3\kappa R_1} C = S_3 S_2^2 - \delta S^2 S_3.$$

The symmetrized quantum scalar constraint:

$$\hat{C} = \frac{1}{3}\left(\hat{S}_z\hat{S}_y\hat{S}_y + \hat{S}_y\hat{S}_z\hat{S}_y + \hat{S}_y\hat{S}_y\hat{S}_z\right) - \delta \hat{S}^2 \hat{S}_z \approx 0$$

We have shown that the WDW has always solutions for any δ for the bosonic representations. For the fermionic representations the solutions do not exist, except some particular values of δ.

The simplest non-trivial case is s=1, for which:

$$\hat{C} = 2\left(\frac{1}{6} - \delta\right)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = 2\left(\frac{1}{6} - \delta\right)\hat{S}_z$$

The physical state, which satisfy the constraint, is:

$$|\Psi\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = |s = 1, s_z = 0\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$

The constraint squared in general case:

$$
\begin{aligned}
\hat{C}^2 = \frac{1}{64}\Bigg(& P_n\left(\sigma_y,\sigma_y,\sigma_y,\sigma_y,\sigma_z,\sigma_z\right)(1-\delta)^2 + P_n\left(\sigma_y,\sigma_y,\sigma_y,\sigma_y\right)(n-4)(1-\delta)^2 \\
& + P_n\left(\sigma_y,\sigma_y,\sigma_z,\sigma_z\right)\left(4(n-4)(1-\delta)^2 - 8\delta(1-\delta) - 6(n-4)\delta(1-\delta) - 12\delta^2 + 2\left(\frac{3n-2}{3} - \delta(5n-2)\right)(1-\delta)\right) \\
& + P_n\left(\sigma_y,\sigma_y\right)\left(4(n-2)(n-3)(1-\delta)^2 - 8(n-2)\delta^2 + 2(n-2)\left(\frac{3n-2}{3} - \delta(5n-2)\right)(1-\delta)\right) \\
& + P_n\left(\sigma_z,\sigma_z\right)\Bigg(2(n-2)(n-3)(1-\delta)^2 + 20(n-2)(n-3)\delta^2 + \left(\frac{3n-2}{3} - \delta(5n-2)\right)^2 \\
& + 4(n-2)\delta(1-\delta) - 6(n-2)\left(\frac{3n-2}{3} - \delta(5n-2)\right)\delta\Bigg) \\
& + \mathbb{I}^{\otimes n}\left(2n(n-1)(n-2)(1-\delta)^2 + 8n(n-1)(n-2)\delta^2 + n\left(\frac{3n-2}{3} - \delta(5n-2)\right)^2\right) \\
& + P_n\left(\sigma_x,\sigma_x\right)\left(4(n-2)(1-\delta)^2 + 4(n-2)(n-3)\delta^2 + 12(n-2)\delta(1-\delta) - 2(n-2)\left(\frac{3n-2}{3} - \delta(5n-2)\right)\delta\right) \\
& + P_n\left(\sigma_x,\sigma_x,\sigma_x,\sigma_x,\sigma_z,\sigma_z\right)\delta^2 + P_n\left(\sigma_x,\sigma_x,\sigma_x,\sigma_x\right)(n-4)\delta^2 \\
& + P_n\left(\sigma_x,\sigma_x,\sigma_z,\sigma_z\right)\left(10(n-4)\delta^2 - 8\delta(1-\delta) - 2\left(\frac{3n-2}{3} - \delta(5n-2)\right)\delta + 12\delta(1-\delta)\right) \\
& + P_n\left(\sigma_z,\sigma_z,\sigma_z,\sigma_z,\sigma_z,\sigma_z\right)\delta^2 - 2P_n\left(\sigma_x,\sigma_x,\sigma_y,\sigma_y,\sigma_z,\sigma_z\right)\delta(1-\delta) \\
& + P_n\left(\sigma_z,\sigma_z,\sigma_z,\sigma_z\right)\left(9(n-4)\delta^2 - 2\left(\frac{3n-2}{3} - \delta(5n-2)\right)\delta + 4\delta(1-\delta)\right) \\
& + P_n\left(\sigma_x,\sigma_x,\sigma_y,\sigma_y\right)\left(-2\delta(1-\delta)(n-4) + 2(1-\delta)^2 + 4\delta^2 + 8\delta(1-\delta)\right) \\
& - 2P_n\left(\sigma_y,\sigma_y,\sigma_z,\sigma_z,\sigma_z,\sigma_z\right)\delta(1-\delta) + 2P_n\left(\sigma_x,\sigma_x,\sigma_z,\sigma_z,\sigma_z,\sigma_z\right)\delta^2\Bigg).
\end{aligned}
$$

The action of the spin operators on n-qubit (spin-1/2) quantum register is given by:

$$\hat{S}_i = \frac{1}{2}\sum_{j=1}^{n} \mathbb{I}^1 \otimes \ldots \mathbb{I}^{j-1} \otimes \hat{\sigma}_i^j \otimes \mathbb{I}^{j+1} \otimes \ldots \mathbb{I}^n,$$

where n = 2s. We need n=2s qubits to represent spin s.

For convenience, we introduced:

$$P_n(\sigma_i) := \sum_j \mathbb{I}^1 \otimes \ldots \mathbb{I}^{j-1} \otimes \sigma_i^j \otimes \mathbb{I}^{j+1} \otimes \ldots \mathbb{I}^n,$$

$$P_n(\sigma_i, \sigma_j) := \sum_{k,l,k\neq l} \mathbb{I}^1 \otimes \ldots \mathbb{I}^{k-1} \otimes \sigma_i^k \otimes \mathbb{I}^{k+1} \otimes \ldots \mathbb{I}^{l-1} \otimes \sigma_j^l \otimes \mathbb{I}^{l+1} \otimes \ldots \mathbb{I}^n,$$

$$P_n(\sigma_i, \sigma_j, \sigma_p) := \sum_{k,l,q,k\neq l,k\neq q,l\neq q} \mathbb{I}^1 \otimes \ldots \sigma_i^k \otimes \ldots \sigma_j^l \otimes \ldots \sigma_p^q \otimes \ldots \mathbb{I}^n,$$

…

Because $\hat{C}^2$ is a unitary operator, the expectation values cannot be evaluated directly with the use of quantum computing. For this purpose the operator has to be expanded into unitarities (here, the Pauli matrices):

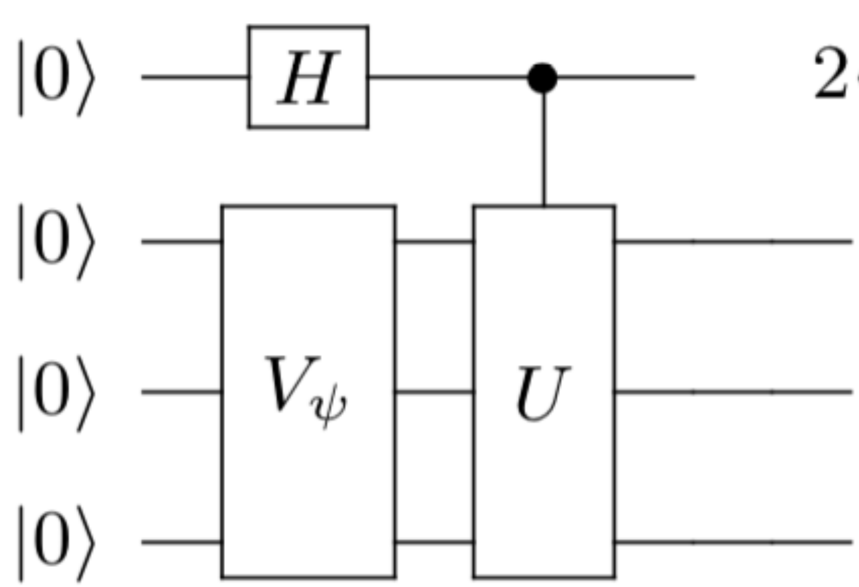
$$\hat{C} = \sum_j c_j \bigotimes_i \hat{\sigma}_{ij}^k$$

so that

$$\langle \hat{C} \rangle = \sum_j c_j \langle \bigotimes_i \hat{\sigma}_{ij}^k \rangle$$

Similarly for $\hat{C}^2$.

Every contributing expectation value has to be evaluated individually. The so-called Hadamard test can be used for this purpose:



$$2\langle \sigma_+ \rangle = \langle \hat{U} \rangle_\psi$$

The variational procedure

We iteratively search for the minimum of the cost function:

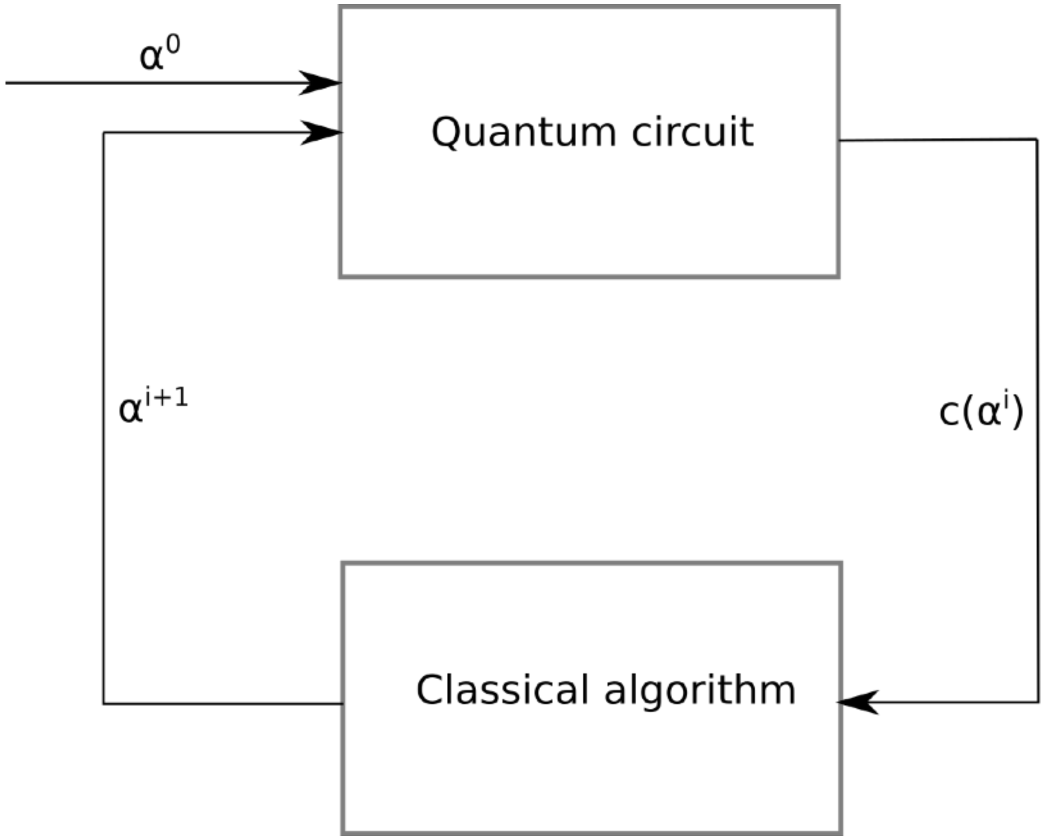
$$c(\alpha) = \frac{a}{\max|\lambda_i|^2}\langle\psi(\alpha)|\hat{C}^\dagger\hat{C}|\psi(\alpha)\rangle + b\left(1 - \frac{\langle\psi(\alpha)|\hat{\vec{S}}^2|\psi(\alpha)\rangle}{s(s+1)}\right)$$

where:

$$a, b \in (0,1)$$

$$a + b = 1$$

The second contribution fixes the spin-s subspace of a quantum register.



Acting iteratively we find:

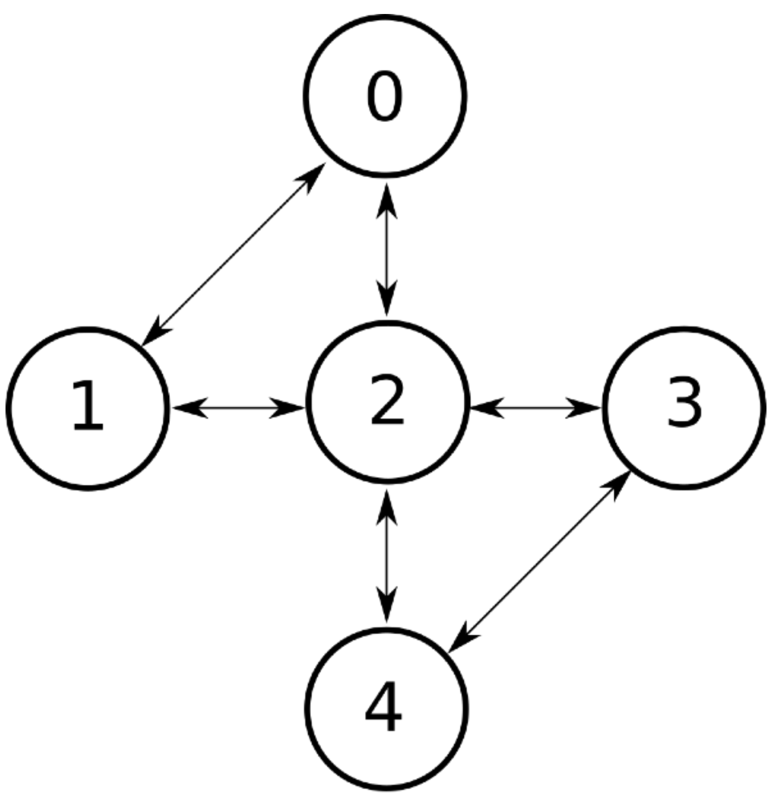
$$\alpha_{\min} := \text{argmin}_\alpha c(\alpha)$$

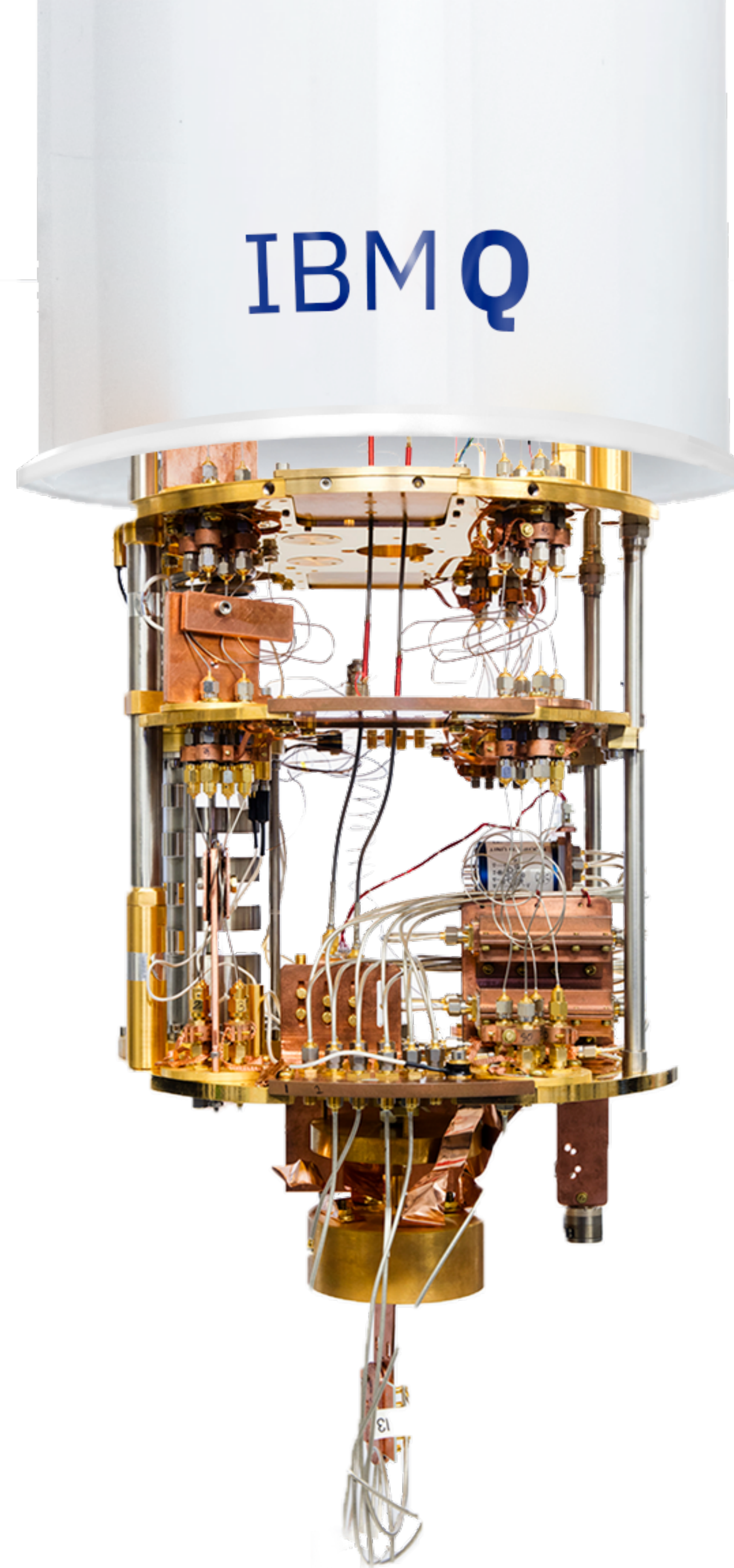
so that $|\psi_0\rangle = |\psi(\alpha_{\min})\rangle$

Quantum chip

The expectation values are evaluated on a quantum processor.

Here, the IBM 5-qubit Yorktown superconducting quantum chip has been used. The connectivity of the quantum processor is the following:



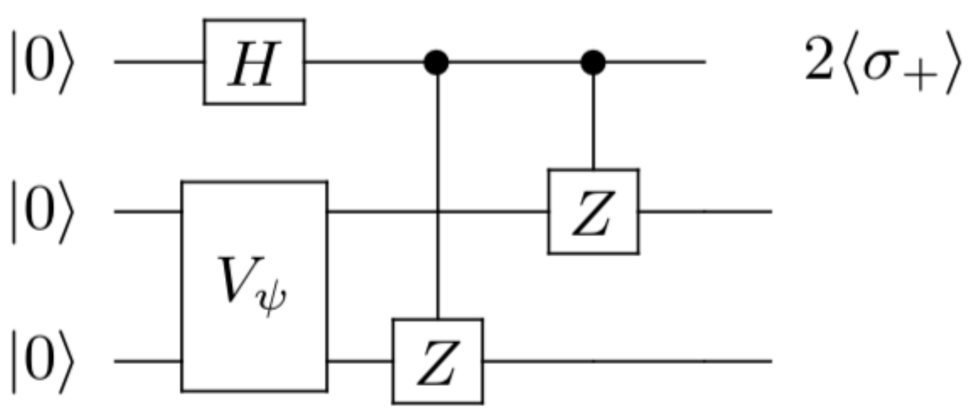


The s=1 case

Here:

$$\langle \hat{C}^2 \rangle = 2\left(\frac{1}{6} - \delta\right)^2 (1 + \langle \sigma_z \otimes \sigma_z \rangle)$$

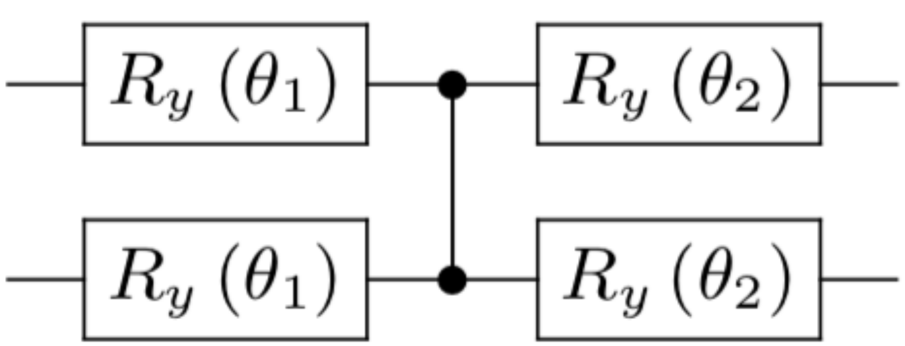
The Hadamard test becomes:



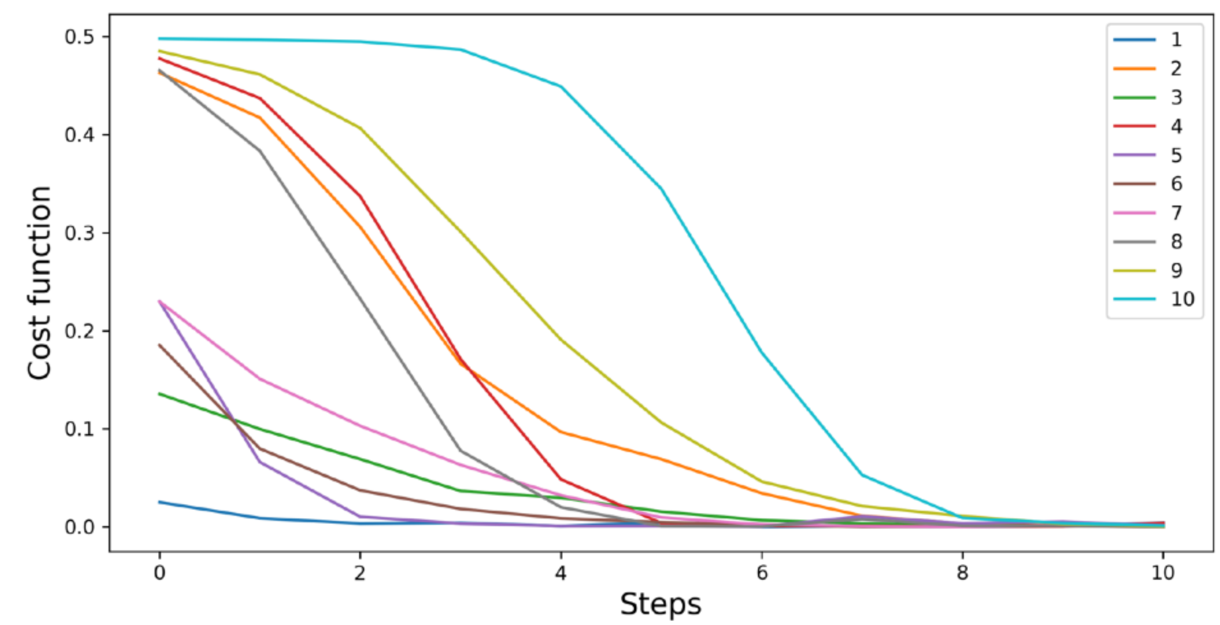
The ansatz for the state

$$|\psi(\alpha)\rangle = |\psi(\theta_1, \theta_2)\rangle = \hat{V}_\psi |00\rangle$$

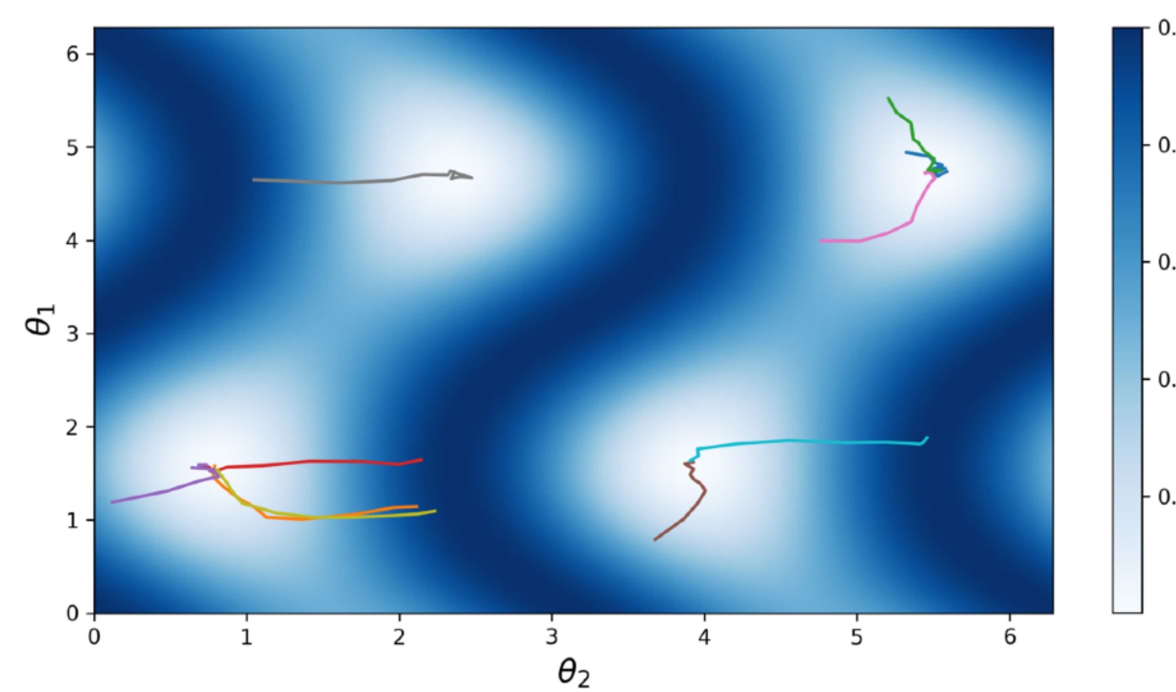
is



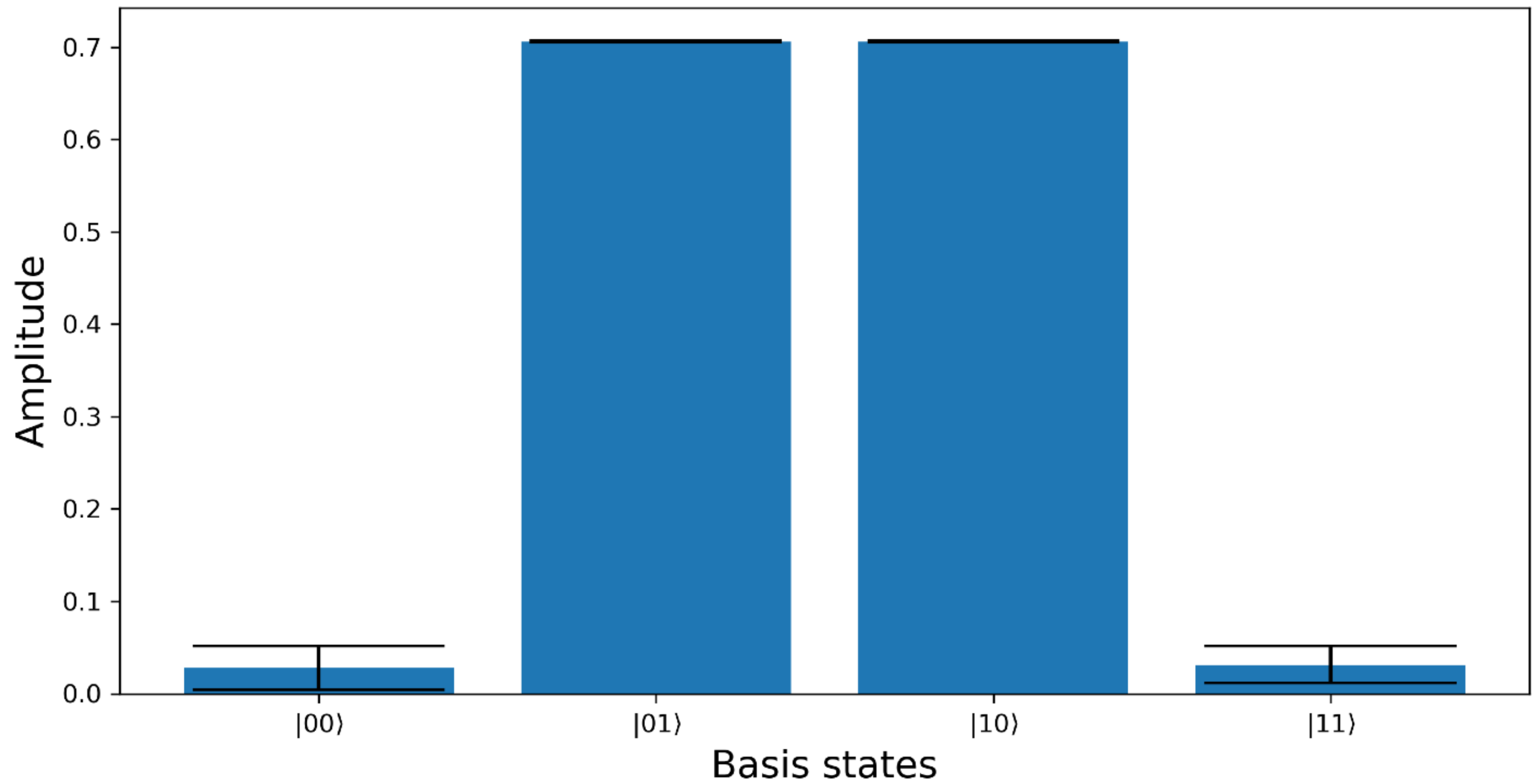
Cost function during minimalization for 10 runs, with randomly initialized parameters:



The cost function landscape:



Averaged (over 10 runs) amplitudes of the final state:



In the simulations, 1024 shots for each circuit have been made.

The theoretical prediction: $$|\Psi\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$

The quantum fidelity of the found state is: $$0.997 \pm 0.003$$

Summary

- A method of solving WDW on quantum computers exists.
- The method employs the compact phase space regularization.
- The method generates physical states on a quantum register. The states can be further used to e.g. evaluate transition amplitudes.
- Degeneration of the kernel can be extracted employing the Gram-Schmidt procedure. In has been tested for the s=2 case.
- The method is not effective (compared to the classical methods) for small number of degrees of freedom (small m).
- The method becomes theoretically advantageous over classical methods (exponential speedup) for large number of interacting degrees of freedom (large m).
- The method is not yet useful because of limitations of the existing quantum resources.
- The approach opens an opportunity to investigate models of quantum cosmology and quantum gravity in the lab.

Thank you!

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