# Lorentzian vacuum transitions

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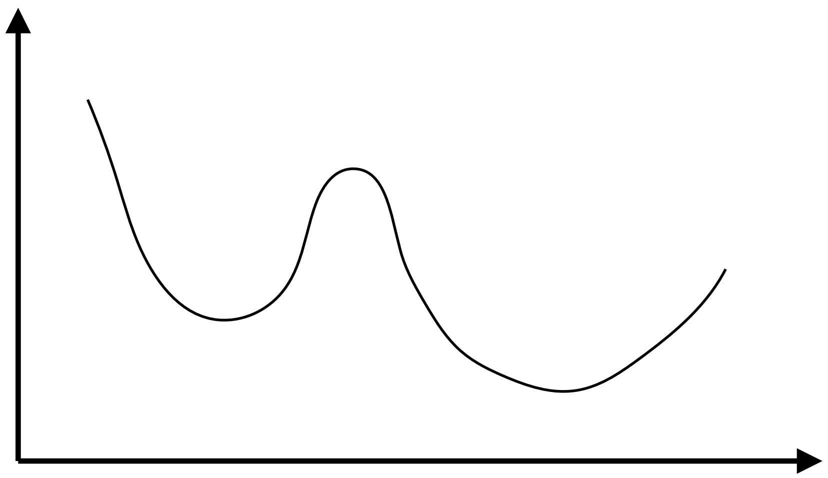
SC, S. de Alwis, F. Muia and F. Quevedo 2011.13936 (PRD)

# Summary

- Introduction
- Geometry after a transition
- WKB method
- Conclussions

# Transitions in QFT



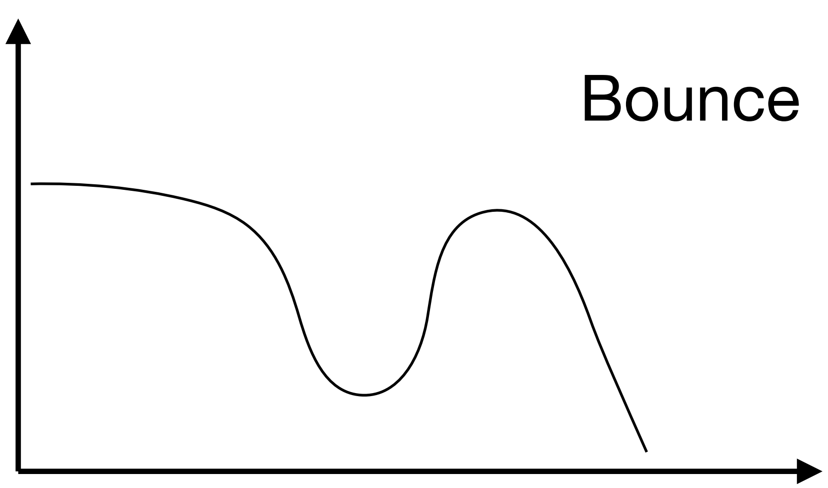

In analogy with QM let us look for bounces solutions

Transition rate is again given by

$$\Gamma/V = e^{-B}(1+\mathcal{O}(\hbar))$$

where $B = S_E(\phi_b) - S_E(\phi_F)$

$$S_E = \int d\tau d^3x \left[\frac{1}{2}(\partial_\tau\phi)^2 + \frac{1}{2}(\nabla\phi)^2 + V(\phi)\right]$$




Bounce $O(3,1) \to O(4)$

*[Coleman 1977, Coleman and Callan 1978 ]*

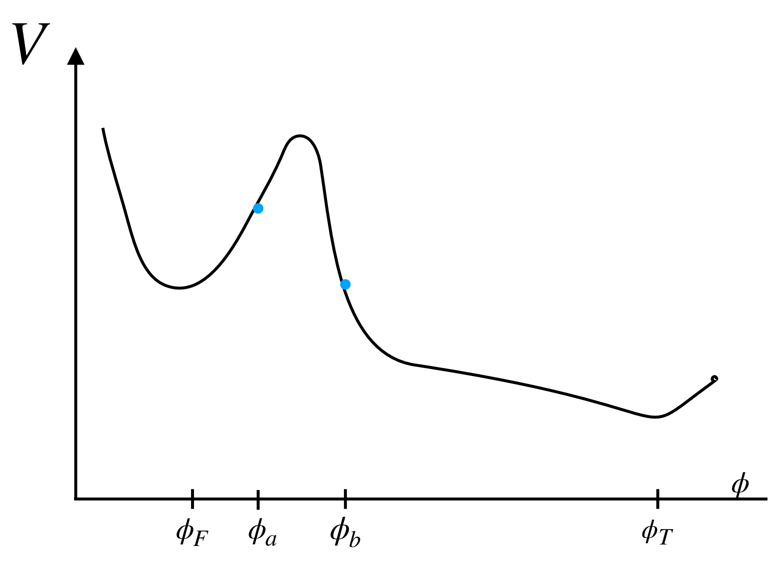
# Including gravity

- It is straightforwards to find an O(4) bounce with gravity

$$ds^2 = d\xi^2 + \rho(\xi)^2 d\Omega_3^2$$

Gravity is present through friction term.

It can make the transition more or less likely




In the thin wall limit

$$B = \frac{B_0}{(1 \pm (\rho_0/2\Lambda)^2)^2}$$

*[Coleman and de Luccia, 1978]*

# De Sitter transitions

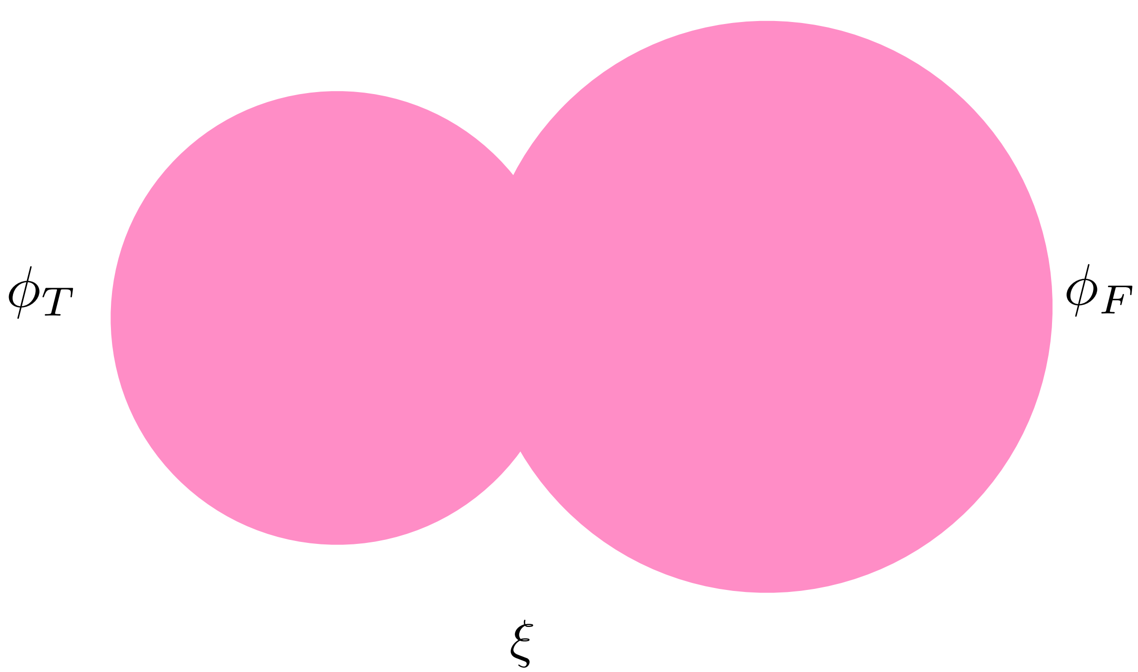
What is the geometry after the transition

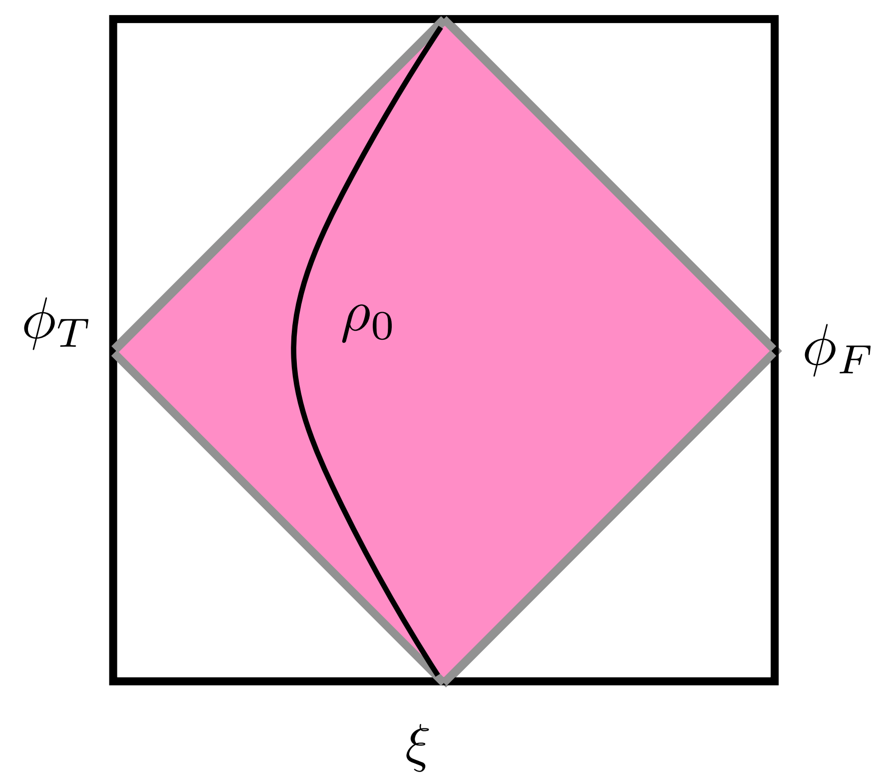
$$\psi \to \pi/2 + i\sigma$$

$O(4)$ bounce $\to O(3,1)$

$$ds^2 = d\xi^2 + \rho^2(\xi)(-d\sigma^2 + \cosh^2\sigma d\Omega_2^2)$$

$$ds^2 = d\xi^2 + \rho^2(\xi)(d\psi^2 + \sin^2\psi d\Omega_2^2)$$




*[Coleman and de Luccia 1981, Freivogel and Susskind 2002]*

# Open universes

What is the geometry after the transition

$O(4)$ bounce $\rightarrow O(3,1)$

$\psi \rightarrow \pi/2 + i\sigma$

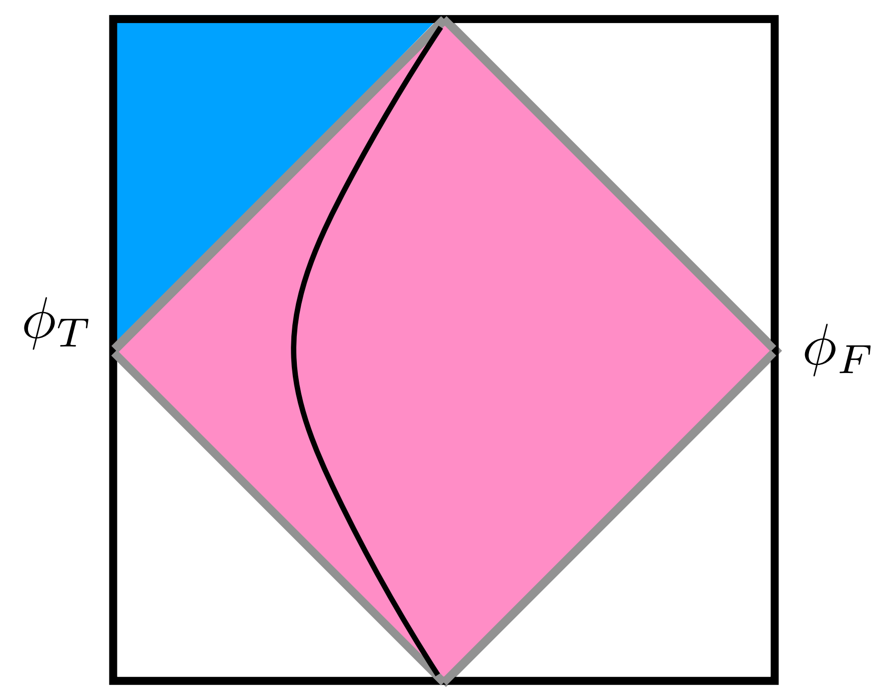
$$ds^2 = d\xi^2 + \rho^2(\xi)(-d\sigma^2 + \cosh^2\sigma d\Omega_2^2)$$

To describe past the light cone

$\sigma \rightarrow i\pi/2 + \chi$ $\xi \rightarrow it$

$$ds^2 = -dt^2 + \rho^2(t)(d\chi^2 + \sinh^2\chi d\Omega_2^2)$$

**FRW with open slices**



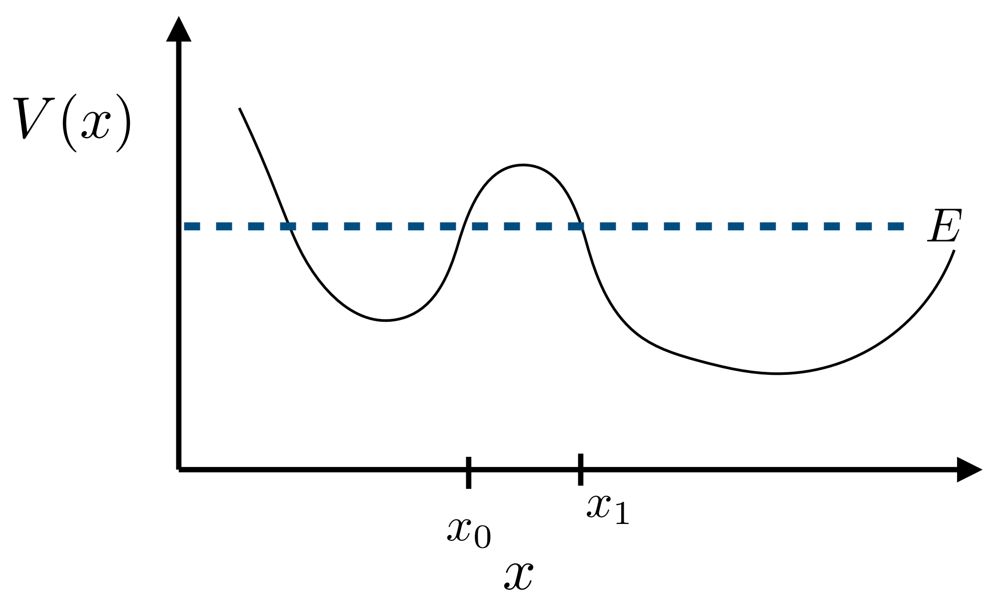

*[Coleman and de Luccia 1981, Freivogel and Susskind 2002]*

# CdL

- It reproduces the right decay rate as WKB and allows for a direct extension to QFT and gravity
- After bubble nucleation Euclidean continuation implies an open universe, but it is an educated guess
- Negative modes problem

*[Lavrelashvili, Rubakov and Tinyakov 1985]*
*[Sasaki and Tanaka 1992]*

# Vacuum transitions in QM




Transition rate is computed using WKB method

$$T \approx \frac{|\psi|^2_{\rm trans}}{|\psi|^2_{\rm refl}} \sim e^{-B/\hbar}$$

Where $$B = \int_{x_0}^{x_1} dx' \sqrt{2m(V(x') - E)}$$

Integral over the forbidden region

# WKB approach

- Is possible to generalise the quantum mechanics computation to gravity at least in the case of a brane.
- Instead of founding the instanton we would like to use a WKB approach
- Time shouldn't play any role $\mathcal{H}\Psi = 0$ **WdW equation**

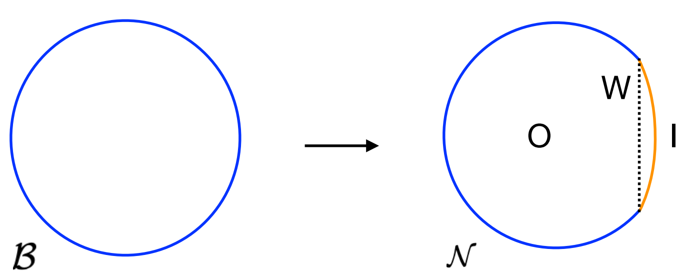
# Probabilities from WdW

Using the classical solution is possible to solve the WdW equation

$$\mathcal{H}\Psi = 0$$

$$\Psi = Ae^{iS} + Be^{-iS}$$

We need to keep both solutions




$$\mathcal{P}(\mathcal{B} \to \mathcal{N}) = \left| \frac{\Psi_{\mathcal{N}}}{\Psi_{\mathcal{B}}} \right|^2$$

$$\mathcal{P}(\mathcal{B} \to \mathcal{N}) \equiv \Gamma_{\mathcal{B} \to \mathcal{N}}$$

$$\mathcal{P}(\mathcal{B} \to \mathcal{N}) \simeq \exp\left[2\,\mathrm{Re}\left(I_{\mathrm{tot}}(\mathcal{N}) - I(\mathcal{B})\right)\right]$$

$$\mathcal{P}(A \to A/B \oplus \mathrm{W}) = \frac{|\Psi(A/B \oplus \mathrm{W})|^2}{|\Psi(A)|^2}$$

*[Hawking & Page 1984]*

*[de Alwis et al 2019]*

# WKB approach

If we consider a system of gravity + matter

$$\mathcal{H} = \frac{1}{2} G^{MN}(\Phi)\pi_M \pi_N + f(\Phi)$$

Field metric

We would like to find solutions $\psi = e^{\frac{i}{\hbar}S[\Phi]}$

$$S[\Phi] = S_0[\Phi] + \hbar S_1[\Phi] + \mathcal{O}(\hbar^2)$$

Hamiltonian constraint →

$$\frac{1}{2} G_{MN} \frac{\delta S_0}{\delta \Phi^n} \frac{\delta S_0}{\delta \Phi^m} + f[\Phi] = 0$$

+other equations

*See also Gervais and Sakita 1977, Bitar and Chang 1978*

# WKB approach

Introducing a set of integral curves (on a selected spatial slice)

$$c(s)\frac{d\Phi^N}{ds} = G^{MN}\frac{\delta S}{\delta \Phi^N}$$

$$S_0 = -2\int^s ds' c^{-1}(s') \int_X f[\Phi^s]$$

For instance, picking

$$\left(\frac{d\tau}{ds}\right)^2 = -2c^{-2}(s)\int_X f[\Phi] \longrightarrow S_0[\Phi] = \int^\tau d\tau' \sqrt{-2\int_X f[\Phi_{\tau'}]}$$

In QFT this reduces to the usual expression

$$S_0[\Phi] = \int^\tau d\tau' \sqrt{2(E - U(\phi(\tau'))}$$

# Minisuperspace

- Mini superspace approximation

$$ds^2 = -N^2(t)dt^2 + a^2(t)(dr^2 + \sin^2 r d\Omega_2^2)$$

Metric non positive definite

The Hamiltonian is

$$\mathcal{H} = N\left(-\frac{\pi_a^2}{12a} + \frac{\pi_\phi^2}{2a^3} - \underbrace{3a + a^3V(\phi)}_{f(a,\phi)}\right) \approx 0$$

By writing $\Psi \sim \exp(iS_0/\hbar)$

$$\frac{1}{2}G^{MN}\frac{\delta S_0}{\delta\Phi^M}\frac{\delta S_0}{\delta\Phi^N} + f[\Phi] = 0$$

Hamilton Jacobi equation

Defining $C(s)\frac{d\Phi^N}{ds} = G^{MN}\frac{\delta S_0}{\delta\Phi^M}$

$$S_0[\Phi] = -2\int^s ds' C^{-1}(s') f[\Phi(s')]$$

*[SC et al 2020]*

# Hartle-Hawking

- For constant potential

$$S_0[\Phi] = -12\pi^2 \int_0^s ds' C(s)^{-1} \left(-a + a^3 \frac{V_0}{3}\right)$$

From the constraints

$$-6a\left(\frac{da}{ds}\right)^2 = -2C^{-2}(s)(-3a + a^3 V_0)$$

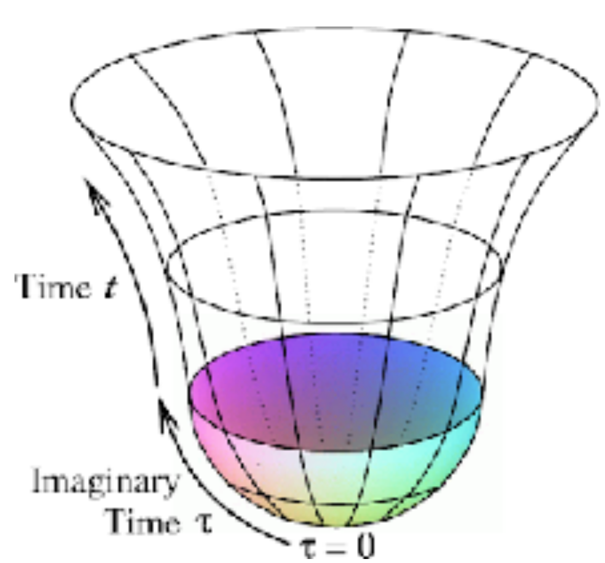
$$a = \sqrt{\frac{3}{V}} \sin\left(\sqrt{\frac{V}{3}}\tau\right) \quad \text{for } a^3V < 3a$$

$$a = \sqrt{\frac{3}{V}} \cosh\left(\sqrt{\frac{V}{3}}t\right) \quad \text{for } a^3V > 3a$$

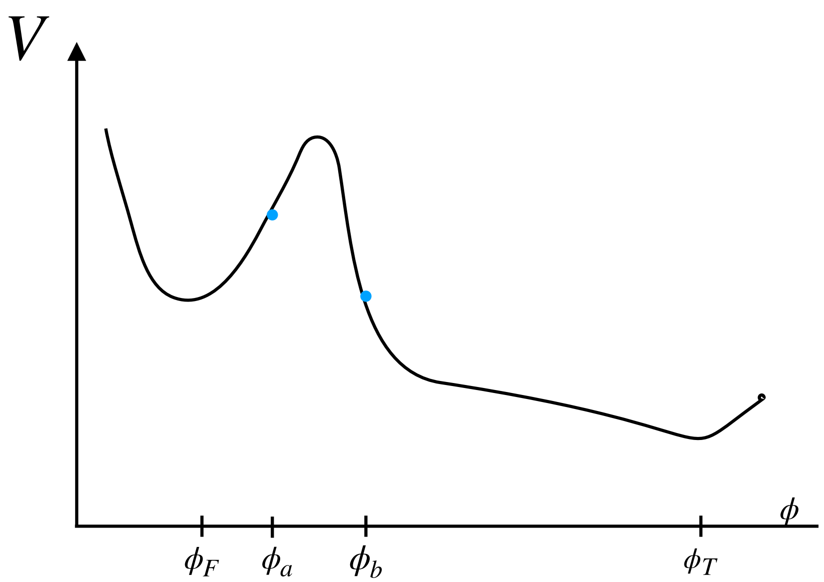
$$|\Psi|^2 \sim e^{\pm\frac{12\pi^2}{V}}$$

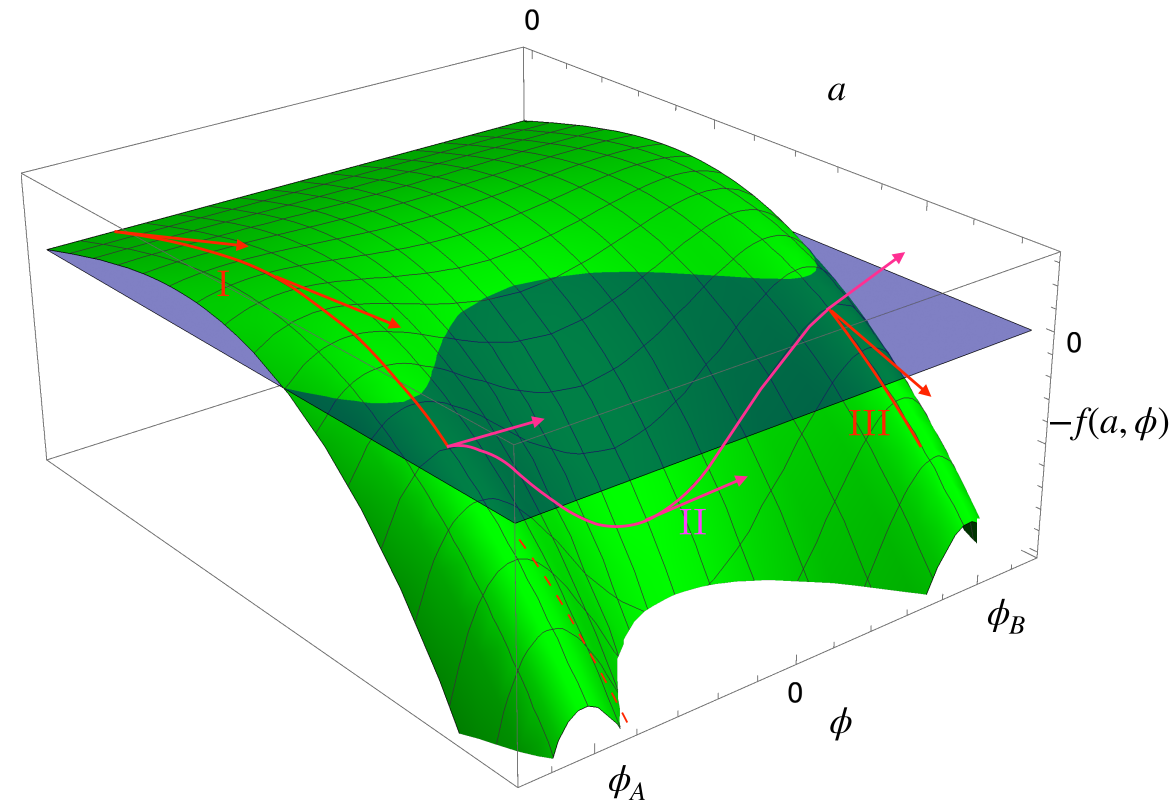
$+$ Hartle Hawking

$-$ Vilenkin/Tunneling



# Quantum transitions




$I$

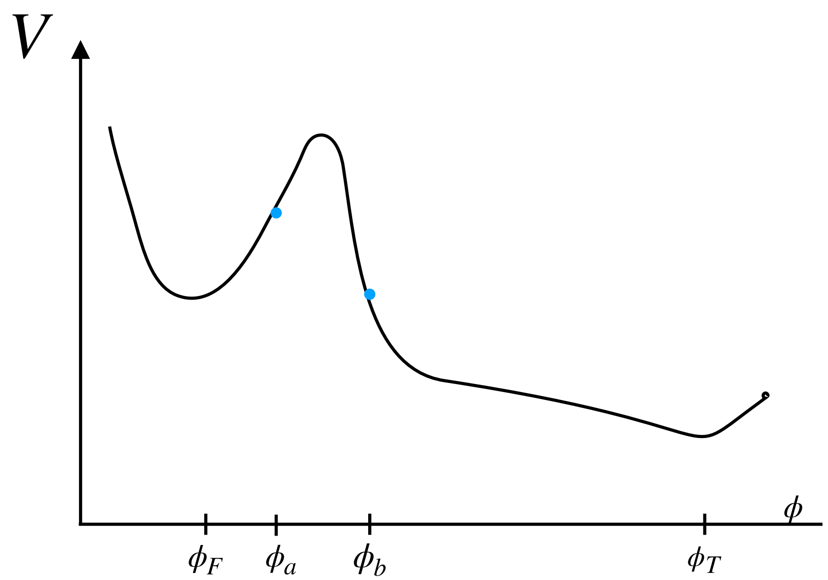
$$\pm\frac{B}{2} = 12\pi^2\left\{\frac{1}{V_B}\left[\left(1-(\bar{a}-\delta a)^2\,\frac{V_B}{3}\right)^{3/2}-1\right]-\frac{1}{V_A}\left[\left(1-(\bar{a}-\delta a)^2\,\frac{V_A}{3}\right)^{3/2}-1\right]\right\}$$

$$+\quad 2\pi^2\bar{a}^3T\,. \qquad (4.22)$$

$III$

$II$

# CDL




If we define the probability

$$P(A \to B) = \left| \frac{\Psi(a_0, \phi_B; a_{\max}, \phi_A)}{\Psi(a_0, \phi_A; a_{\max}, \phi_A)} \right|^2 \equiv e^{-B}$$

$$\pm \frac{B}{2} = 12\pi^2 \left\{ \frac{1}{V_B} \left[ \left( 1 - (\bar{a} - \delta a)^2 \frac{V_B}{3} \right)^{3/2} - 1 \right] - \frac{1}{V_A} \left[ \left( 1 - (\bar{a} - \delta a)^2 \frac{V_A}{3} \right)^{3/2} - 1 \right] \right\} + 2\pi^2 \bar{a}^3 T \, .$$

**Thin wall approximation**

$$B = \pm 8\pi^2 \left[ \frac{\left\{ \left( H_A^2 - H_B^2 \right)^2 + T^2 \left( H_A^2 + H_B^2 \right) \right\} \bar{a}}{4 T H_A^2 H_B^2} - \frac{1}{2} \left( H_B^{-2} - H_A^{-2} \right) \right]$$

# CdL?

- If we add an scalar fields we can also obtain CdL

$$B = \pm 8\pi^2 \left[ \frac{\left\{ \left(H_A^2 - H_B^2\right)^2 + T^2 \left(H_A^2 + H_B^2\right) \right\} \bar{a}}{4T H_A^2 H_B^2} - \frac{1}{2}\left(H_B^{-2} - H_A^{-2}\right) \right]$$

- This approach works because CdL instanton is basically an Euclidean minisuperspace computation

- After nucleation the minisuperspace computation is not valid

CdL boundary conditions are not consistent
with a closed universe

# CdL?

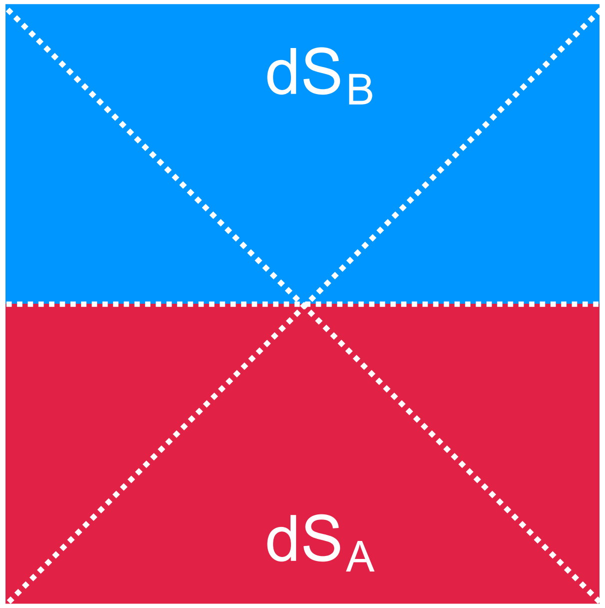
- If we add an scalar fields we can also obtain CdL

$$B = \pm 8\pi^2 \left[ \frac{\left\{ \left(H_A^2 - H_B^2\right)^2 + T^2\left(H_A^2 + H_B^2\right) \right\} \bar{a}}{4T H_A^2 H_B^2} - \frac{1}{2}\left(H_B^{-2} - H_A^{-2}\right) \right]$$

- Nevertheless there are another allowed decays

$$B = 24\pi^2 \left\{ \mp \frac{1}{V_B} \pm \frac{1}{V_A} \right\}$$

Transitions between two de Sitter

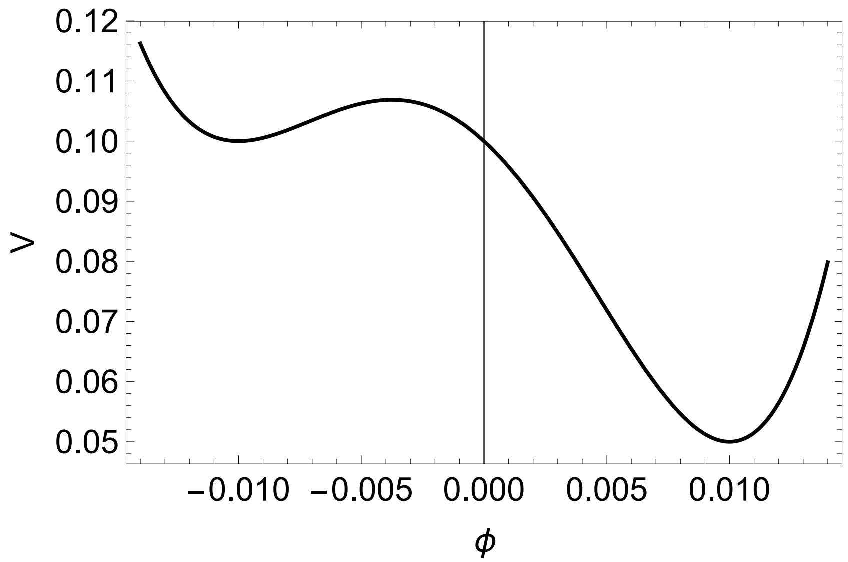


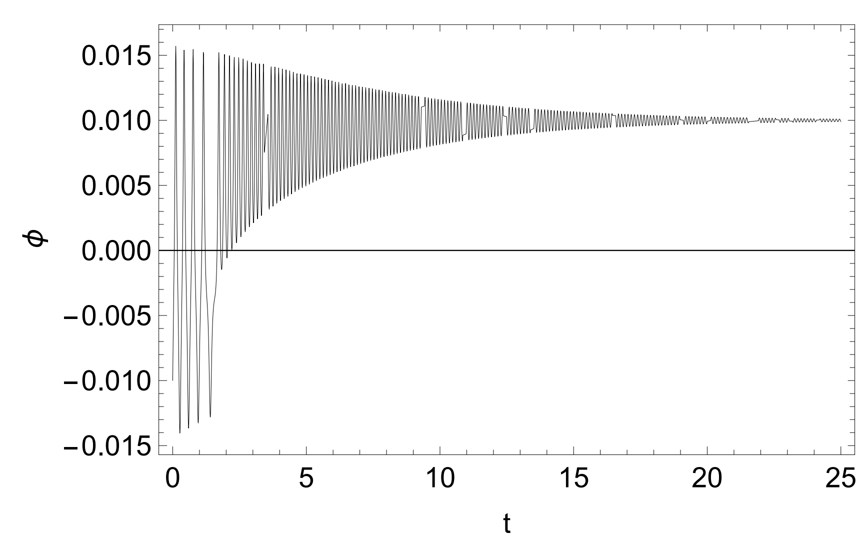

*[SC, de Alwis, Muia and Quevedo 2020]*

# Other solutions

- There are other solutions in the case of a potential

In the case there is an initial kinetic energy the field can move classically between vacua

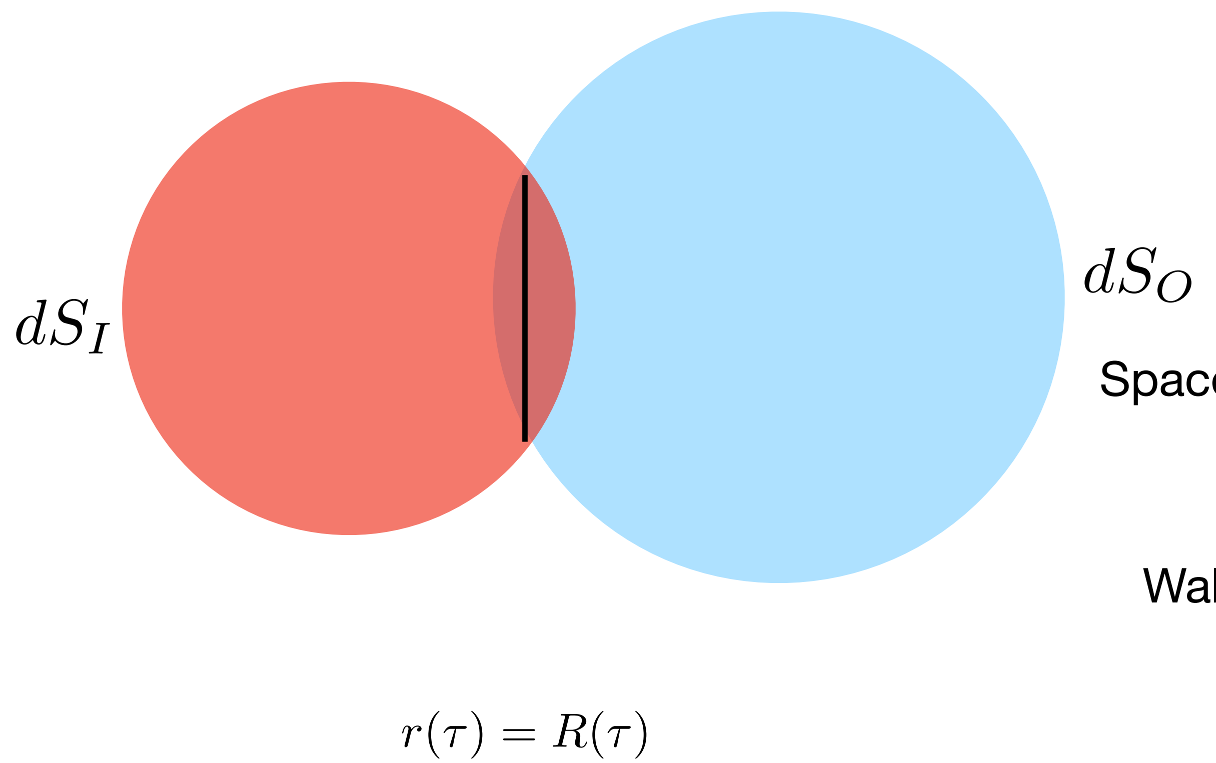



*[SC, de Alwis, Muia and Quevedo 2020]*

# Beyond minisuperspace

- First let us study the motion of a brane between two dS




Imposing junction conditions at the wall we can obtain equations of motion for the brane

$$\Delta K_{ab} - \Delta K h_{ab} = -\kappa S_{ab}$$

Spacetime metric

$$ds^2 = -(1 - H_{i,o}^2 r^2)dt^2 + \frac{dr^2}{1 - H_{i,o}^2 r^2} + r^2 d\Omega^2$$

Wall metric

$$ds^2 = -d\tau^2 + R^2(\tau) d\Omega^2$$

*[Blau, Guendelman and Guth 1987]*

# Classical solutions

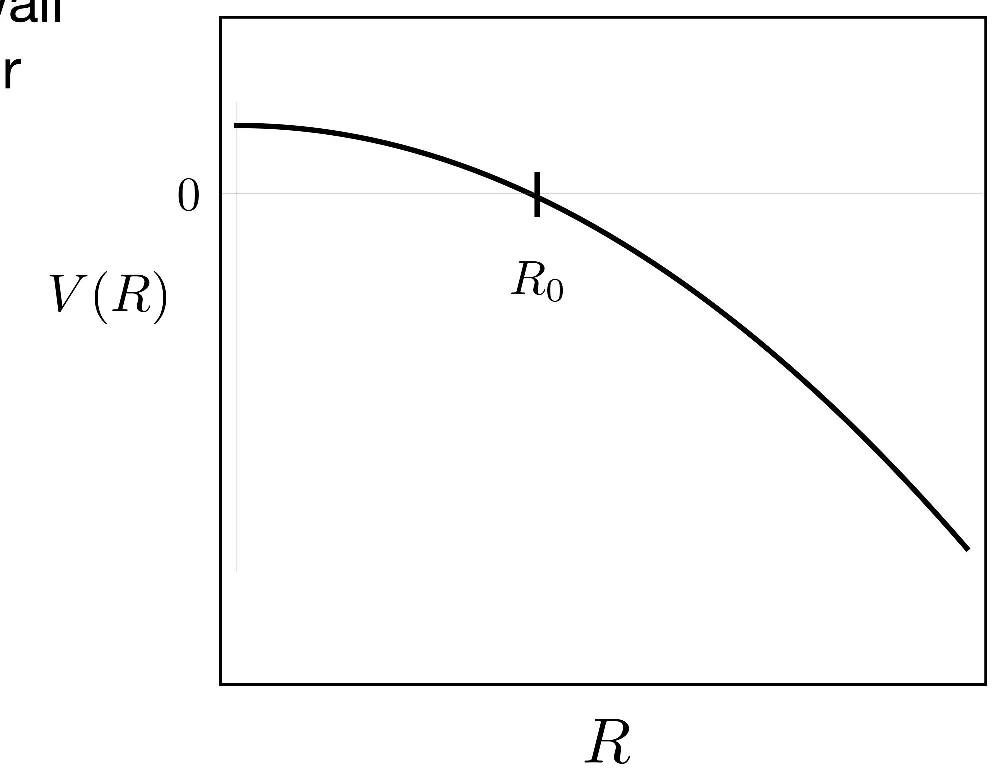
- First let us study the motion of a brane between two dS

Imposing junction conditions at the wall we can obtain equations of motion for the brane

$$\Delta K_{ab} - \Delta K h_{ab} = -\kappa S_{ab}$$

$$\left(\frac{dR}{d\tau}\right)^2 = 1 - \frac{R^2}{R_0^2}$$

Classically bubble has minimum radius $R_0$ from where it starts expanding



# Hamiltonian analysis

In order to include the wall we use the following metric

$$ds^2 = -N_t^2(t,r)dt^2 + L^2(t,r)(dr + N_r dt)^2 + R^2(t,r)d\Omega_2^2 \qquad SO(3)$$

$$\pi_L = \frac{R}{G}\left[\frac{R'^2}{L^2} - A_\alpha\right]^{1/2}, \qquad \alpha = O, I,\ \eta = \pm 1$$

$$A_\alpha = 1 - H_\alpha^2 R^2, \qquad H_\alpha = \frac{8\pi G}{3}\Lambda_\alpha$$

Wall Tension

$$S_{\rm tot} = S_{\rm EH} + S_K + S_{\rm mat} + S_W$$

Gravity

Wall

Solution to constraints

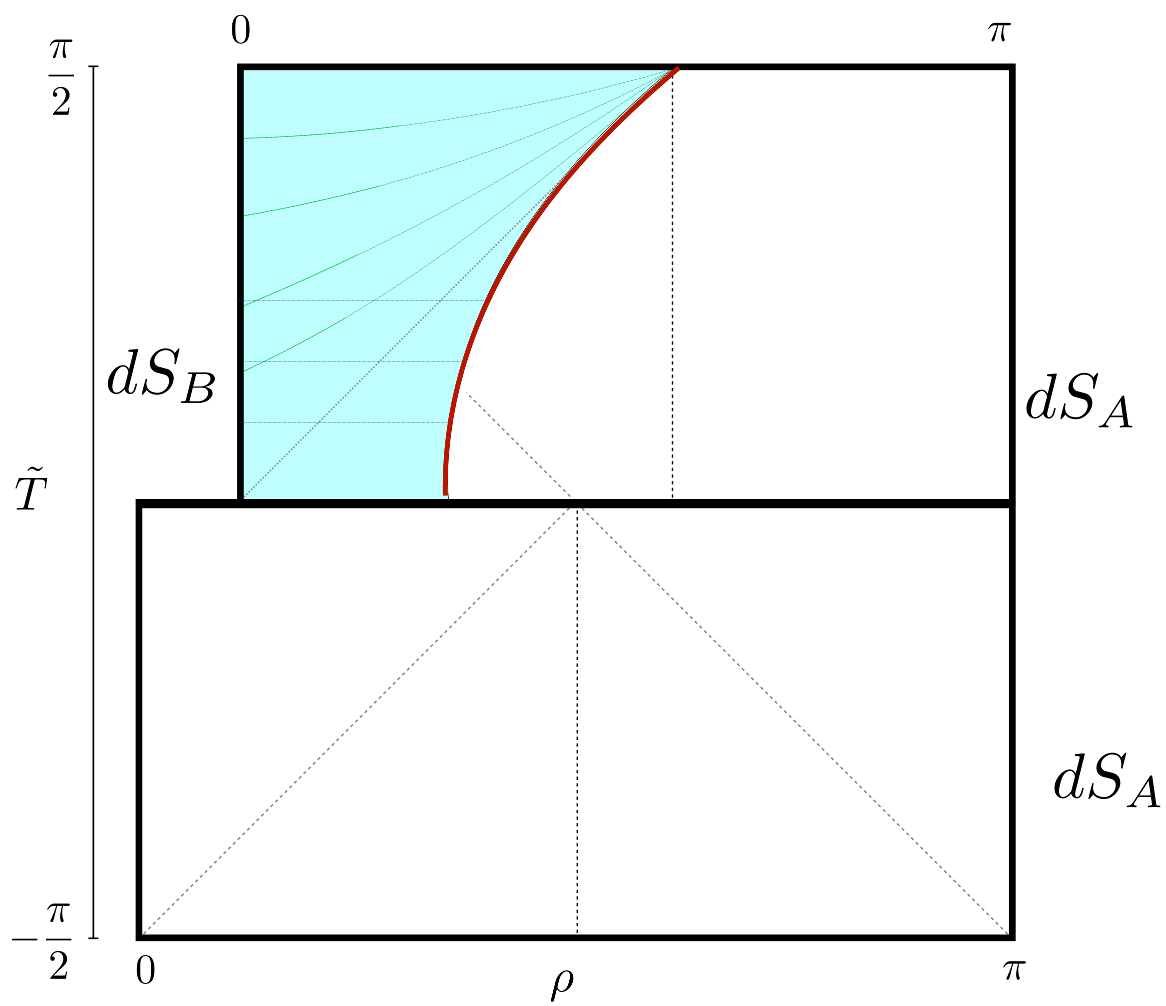
$$ds_\alpha^2 = -A_\alpha(R)\, d\tau^2 + A_\alpha^{-1}(R)\, dR^2 + R^2\, d\Omega_2^2$$

**Solution to constrains**
**+**
**Junction conditions**

→ $\psi$

*[Fishler, Morgan & Polchinksi 1989,1990]*

# dS to dS




$$\mathcal{P}(\mathrm{dS} \to \mathrm{dS}/\mathrm{dS} \oplus \mathrm{W}) = \frac{|\Psi(\mathrm{dS}/\mathrm{dS} \oplus \mathrm{W})|^2}{|\Psi(\mathrm{dS})|^2}$$

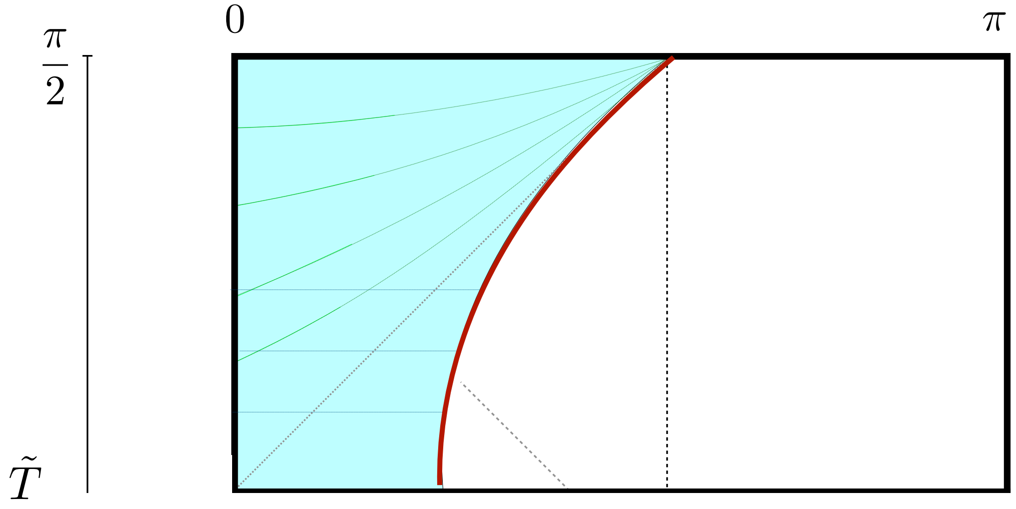
$$= \exp\left(-\frac{B_0}{(1+(HR_0/2)^2)^2}\right)$$

Same as CDL!

Spacetime tunnels from one
de Sitter to two
de Sitter separated by a wall

*[Fishler, Morgan, Polchinsky 91,
Brown and Teitelboim '88,
Bachlechner 2017,
de Alwis et al 2019]*

# Wall dynamics




**By solving the junction conditions**

In global coordinates

$$ds^2 = \frac{1}{H^2 \cos^2 T} \left(-dT^2 + d\rho^2 + \sin^2 \rho d\Omega^2\right)$$

Trajectory of the wall is given by

$$\cos(\rho) = \sqrt{1 - H^2 R_0^2} \cos T$$

Same as CDL!

Eq of an hyperboloid with SO(3,1) symmetry

*[SC et al 2020]*

- Curvature of spacetime was not given as a solution of the WdW equation. Computation only assumed SO(3) symmetry

- Using an open slicing is possible to find a homogenous constant time foliation

- There are more general cases when open slicing might not be possible.

- This approach (FMP) has not been generalised to include scalar fields.

# Conclussions

- There are important aspects of vacuum decay with gravity that needs more scrutiny

- CdL simple picture is very intuitive but it might not be the end of the game

- Is it possible to distinguish a closed from an open universe?

  [Aavis et al 2019]

- There are concrete applications for this, eg quantum criticality to explain the mass of the Higgs *[Khoury et al 2020, Giudice et al 2021]*

# Negative modes

Including subheading terms wave function is

$$\Psi[\Phi_s] = \frac{1}{P_0} \sqrt[4]{G_s} \sqrt{\det \left[ \frac{\delta^2 S_0}{\delta \Phi^A \delta \alpha^{\bar{A}}} \right]_s} e^{\frac{i}{\hbar} S_0[\Phi s]} \Psi[\Phi_0] \,,$$

$$S_0[\Phi_s] = \int_o^s ds' \sqrt{-2 \int_X f[\Phi_{s'}] ds'} + \text{constant} \,,$$

- Formula is a generalisation of QM and refactor is given by the VanVleck determinant
- Decay rate is not necessarily related to the negative modes

# Going beyond minisuperspace

- It is possible to add perturbations

$$\Psi = \psi(a)\chi(a,\phi)$$

$$\frac{i}{6a}\frac{\delta S_0}{\delta a}\frac{\delta \chi}{\delta a} + i\frac{\delta^2 S_0}{\delta a^2}\chi + \mathcal{H}_2\chi = 0$$

Schrodinger equation

$$\left(\frac{\partial}{\partial \tau} + \mathcal{H}_2\right)\chi = 0$$ Under the barrier

Using the classical solutions is possible to reintroduce time

$$\left(i\frac{\partial}{\partial t} - \mathcal{H}_2\right)\chi = 0$$ Over the barrier

# Going beyond minisuperspace

- It is possible to add perturbations

$$\Psi = \psi(a)\chi(a,\phi)$$

This naturally selects
the Bunch-Davies vacuum

$$\chi = \exp\left( i \sum_{l,m} \frac{\sec\eta}{2H^2} \frac{l(l+2)}{i(1+l)\cos\eta - \sin\eta} \varphi_{l,m}\varphi_{l,m} \right)$$

Writing $\varphi(x) = \sum_{l,\mathbf{m}} \varphi_l(t) Y_{l\mathbf{m}}(\Omega)$

$$\langle \varphi^2 \rangle \sim \frac{H^2}{l(l+1)(l+2)}$$

# Observable effects

- Inflation washes out the curvature
- The density perturbations depend on the curvature,

$$P(q) = A_s q^{(n_s-1)-3}\left(1 - \frac{19}{8}\frac{k}{q^2}\right) + \mathcal{O}(\Omega_k^2))$$

# Observable effects

- Inflation washes out the curvature
- The density perturbations depend on the curvature,

$$P(q) = A_s q^{(n_s-1)-3}\left(1 - \frac{19}{8}\frac{k}{q^2}\right) + \mathcal{O}(\Omega_k^2))$$

- This might effect the low l CMB modes

$$l(l+1)C_l = \frac{4\pi}{25}A_s\left(1 - \frac{19}{8}\frac{kr_L^2}{3(l-1)(l+2)}\right)$$