Eisenhart-Duval lift for minisuperspace quantum cosmology

Phys. Rev. **D104** (2021) 086001

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1. Introduction

Wheeler-DeWitt (WDW) equation

$$\left[\frac{1}{a^{s+1}}\frac{\partial}{\partial a}a^s\frac{\partial}{\partial a}-\frac{1}{a^3}\frac{\partial^2}{\partial\phi^2}+2U(a,\phi)\right]\Psi(a,\phi)=0$$

- There is a problem of factor ordering.
- It is difficult to define a positive−definite probability density.

➡ Dirac-square-root formulation

P.D.D'Eath, S.W.Hawking and O.Obregon (1993)

- probability density $||\Psi||^2$

Arbitrariness arises due to the presence of $U(a,\phi)$ in addition to S .

Our purpose

- We apply the method in Eisenhart-Duval lift to the simple models and extend the minisuperspace .

  It is possible to describe a system by geometric treatment even in the presence of the potential term.

- We introduce Dirac type WDW equation in term of the covariance of the extended minisuperspace.

  We obtain the fundamental solution to the Dirac type WDW equation in the extended minisuperspace of specific models.

- We derive a positive-definite probability density.

2. WDW equation

Einstein-Hilbert action + scalar field action

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2\kappa^2}R - \frac{1}{2}(\nabla\phi)^2 - V(\phi)\right]$$

$$\kappa^2 = 6$$

$$\phi = \phi(t)$$

FLRW metric

$$ds^2 = -N^2dt^2 + a^2\left[\frac{dr^2}{1-Kr^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\right]$$

$$\boxed{L = -\frac{1}{2N}a\dot{a}^2 + \frac{1}{2N}a^3\dot{\phi}^2 - NU(a,\phi)}$$

$$U(a,\phi) = a^3V(\phi) - \frac{1}{2}Ka$$

2. WDW equation

Quantize a and ϕ using the canonical quantization.

Lagrangian

$$L = -\frac{1}{2N} a\dot{a}^2 + \frac{1}{2N} a^3 \dot{\phi}^2 - NU(a, \phi)$$

Canonical conjugate momenta

$$\Pi_a = \frac{\partial L}{\partial \dot{a}} = -\frac{a\dot{a}}{N} \qquad \Pi_\phi = \frac{\partial L}{\partial \dot{\phi}} = \frac{a^3 \dot{\phi}}{N}$$

Hamiltonian

$$H = -\frac{1}{2}\frac{\Pi_a^2}{a} + \frac{1}{2}\frac{\Pi_\phi^2}{a^3} + U(a, \phi)$$

2. WDW equation

Replace the momenta with differential operators

$$\Pi_a \rightarrow -i\frac{\partial}{\partial a} \qquad \Pi_\phi \rightarrow -i\frac{\partial}{\partial \phi}$$

Hamiltonian constraint condition : $H\Psi = 0$

WDW equation

$$\left[\frac{1}{a^{s+1}}\frac{\partial}{\partial a}a^s\frac{\partial}{\partial a} - \frac{1}{a^3}\frac{\partial^2}{\partial \phi^2} + 2U(a,\phi)\right]\Psi(a,\phi) = 0$$

Arbitrariness arises due to the presence of $U(a,\phi)$

in addition to S when replacing it with the Dirac equation.

3. Eisenhart-Duval lift for minisuperspace

L. P. Eisenhart (1928)

C. Duval et al. (1985)

$$L = -\frac{1}{2N} a\dot{a}^2 + \frac{1}{2N} a^3 \dot{\phi}^2 - NU(a, \phi)$$

$$\Rightarrow \tilde{L} = -\frac{1}{2} a\dot{a}^2 + \frac{1}{2} a^3 \dot{\phi}^2 + \frac{1}{2} \frac{\dot{\chi}^2}{2U(a, \phi)} \qquad (N = 1)$$

$$= \frac{1}{2} \tilde{G}_{MN} \dot{X}^M \dot{X}^M$$

The introduction of χ allows us to treat it as a free system in a " curved space".

$X^M = (a, \phi, \chi)$: The component of the extended minisuperspace

$\tilde{G}_{MN} = \mathrm{diag.}(-a, a^3, [2U(a, \phi)]^{-1})$ The metric of the extended minisuperspace

3. Eisenhart-Duval lift for minisuperspace

Hamiltonian

$$\frac{1}{2}\tilde{G}^{MN}P_M P_N = 0 \quad \longleftarrow \quad \tilde{L} = \frac{1}{2}\tilde{G}_{MN}\dot{X}^M\dot{X}^N$$

$$G_{MN} = 2U(a,\phi)\tilde{G}_{MN} \quad \longleftarrow \quad \text{Conformal transformation}$$

$$= \mathrm{diag.}(-2U(a,\phi)a, 2U(a,\phi)a^3, 1)$$

Replace the Hamiltonian with differential operators in term of covariance

$$G^{MN}P_M P_N \to \frac{1}{\sqrt{-G}}\partial_M\sqrt{-G}G^{MN}\partial_N - \xi\mathcal{R}$$

3. Eisenhart-Duval lift for minisuperspace

WDW equation in the extended minisuperspace

$$\left[\frac{1}{\sqrt{-G}}\partial_M\sqrt{-G}G^{MN}\partial_N-\xi\mathcal{R}\right]\Psi=0$$

$$\left[\frac{1}{a^2}\frac{\partial}{\partial a}a\frac{\partial}{\partial a}-\frac{1}{a^3}\frac{\partial^2}{\partial\phi^2}-(2a^3V-Ka)\frac{\partial^2}{\partial\chi^2}+\underline{2\xi\mathcal{R}}\right]\Psi(a,\phi,\chi)=0$$

$$\mathcal{R}=\frac{2a^3[(V')^2-V''V]+Ka[V''-4V]}{(2a^3V-Ka)^2}$$

: the scalar curvature of the extended minisuperspace

➡ We consider two models with $\mathcal{R}=0$

3. Eisenhart-Duval lift for minisuperspace

$V(\phi) = 0$ C. Kiefer (1988)

$$\left[a\frac{\partial}{\partial a}a\frac{\partial}{\partial a} - \frac{\partial^2}{\partial \phi^2} + Ka^4\frac{\partial}{\partial \chi^2}\right]\Psi(a,\phi,\chi) = 0$$

Constraint condition : $-\frac{\partial^2}{\partial \chi^2}\Psi = p^2\Psi$

$$\Psi = \int d\nu \mathcal{A}(\nu)\psi_{\nu p}e^{ip\chi}$$

The fundamental solution :

$$\psi_{\nu p} = K_{i\nu/2}(\sqrt{K}|p|a^2/2)e^{i\nu(\phi-\phi_0)} \qquad (K>0)$$

$$\psi_{\nu p} = J_{\pm i\nu/2}(\sqrt{|K|}|p|a^2/2)e^{i\nu(\phi-\phi_0)} \qquad (K<0)$$

3. Eisenhart-Duval lift for minisuperspace

$$K = 0 \qquad V(\phi) = V_0 \exp \lambda\phi$$ A. A. Andrianov et al.(2018)

$$\left[a\frac{\partial}{\partial a} a \frac{\partial}{\partial a} - \frac{\partial^2}{\partial \phi^2} - 2a^6 V_0 \exp \lambda\phi \frac{\partial^2}{\partial \chi^2} \right] \Psi(a, \phi, \chi) = 0$$

Constraint condition : $-\dfrac{\partial^2}{\partial \chi^2}\Psi = p^2 \Psi$

$$C = 2V_0 \left(1 - \frac{\lambda^2}{36}\right)^{-1}$$

The fundamental solution :

$$\psi_{\nu p} = J_{\pm i\nu/3}(i\sqrt{C}|p|e^{3x}/3)e^{i\nu(y-y_0)} \qquad (C > 0)$$

$$\psi_{\nu p} = K_{i\nu/3}(i\sqrt{|C|}|p|e^{3x}/3)e^{i\nu(y-y_0)} \qquad (C < 0)$$

4. Dirac type WDW equation

Dirac type equation in the extended minisuperspace

$$\gamma^A e_A^M D_M \Psi = 0$$

dreibein $\quad e_A^M = \mathrm{diag.}((2U)^{-1/2}a^{-1/2}, (2U)^{-1/2}a^{-3/2}, 1)$

gamma matrices $\quad \{\gamma^A, \gamma^B\} = -2\eta^{AB}$

covariant derivative $\quad D_M \equiv \partial_M + \frac{1}{4}\omega_{MAB}\Sigma^{AB}$

$$\Sigma^{AB} \equiv -\frac{1}{2}[\gamma^A, \gamma^B]$$

Spin connection $\quad \omega_{MAB} = \frac{1}{2}e_A^N(\partial_M e_{NB} - \Gamma^L_{MN} e_{LB}) - (A \leftrightarrow B)$

4. Dirac type WDW equation $V(\phi) = 0$

$$\gamma^A e_A^M D_M \Psi = 0$$

$$\left[\sigma^1 \left(a\frac{\partial}{\partial a} + 1 \right) + i\sigma^2 \frac{\partial}{\partial \phi} + i\sigma^3 \sqrt{-K} a^2 \frac{\partial}{\partial \chi}\right] \Psi = 0$$

By setting the wave function to $\Psi = \begin{pmatrix} \Psi_+ \\ \Psi_- \end{pmatrix} e^{ip\chi}$,

the equation is in matrix form:

$$\begin{pmatrix} -p\sqrt{-K}a^2 & a\frac{\partial}{\partial a} + 1 + \frac{\partial}{\partial \phi} \\ a\frac{\partial}{\partial a} + 1 - \frac{\partial}{\partial \phi} & p\sqrt{-K}a^2 \end{pmatrix} \begin{pmatrix} \Psi_+ \\ \Psi_- \end{pmatrix} = 0$$

4. Dirac type WDW equation $V(\phi) = 0$

$K > 0$

$$\Psi_{\pm,\nu p} = \frac{1}{\sqrt{2}} e^{\pm i\frac{\pi}{4}} K_{\frac{i\nu}{2}\mp\frac{1}{2}}(\sqrt{K}|p|a^2/2) e^{i\nu(\phi-\phi_0)}$$

$K < 0$

$$\Psi_{\pm,\nu p} = \frac{1}{\sqrt{2}} J_{\frac{i\nu}{2}\mp\frac{1}{2}}(\sqrt{|K|}|p|a^2/2) e^{i\nu(\phi-\phi_0)}$$

$$\Psi_{\pm,\nu p} = \pm\frac{1}{\sqrt{2}} J_{-\frac{i\nu}{2}\pm\frac{1}{2}}(\sqrt{|K|}|p|a^2/2) e^{i\nu(\phi-\phi_0)}$$

4. Dirac type WDW equation $K=0 \quad V(\phi)=V_0 \exp \lambda\phi$

$$\gamma^A e_A^M D_M \Psi = 0$$

$$\left[\sigma^1\left(a\frac{\partial}{\partial a}+\frac{3}{2}\right)+i\sigma^2\left(\frac{\partial}{\partial \phi}+\frac{\lambda}{4}\right)+i\sigma^3\sqrt{2V_0 e^{\lambda\phi}}a^3\frac{\partial}{\partial \chi}\right]\Psi=0$$

By setting the wave function to $\Psi = \begin{pmatrix} \Psi_+ \\ \Psi_- \end{pmatrix} e^{ip\chi}$,

the equation is in matrix form.

$$\begin{pmatrix} -pa^3\sqrt{2V_0 e^{\lambda\phi}} & a\frac{\partial}{\partial a}+\frac{3}{2}+\frac{\partial}{\partial \phi}+\frac{\lambda}{4} \\ a\frac{\partial}{\partial a}+\frac{3}{2}-\frac{\partial}{\partial \phi}-\frac{\lambda}{4} & pa^3\sqrt{2V_0 e^{\lambda\phi}} \end{pmatrix}\begin{pmatrix} \Psi_+ \\ \Psi_- \end{pmatrix}=0$$

4. Dirac type WDW equation $K = 0 \quad V(\phi) = V_0 \exp \lambda\phi$

$$C > 0 \qquad C = 2V_0 \left(1 - \frac{\lambda^2}{36}\right)^{-1}$$

$$\Psi_{\pm,\nu p} = \frac{1}{\sqrt{2}}\sqrt{1 \pm \frac{\lambda}{6}} J_{\frac{i\nu}{3} \mp \frac{1}{2}}(\sqrt{C}|p|e^{3x}/3)e^{i\nu(y-y_0)}$$

$$\Psi_{\pm,\nu p} = \pm\frac{1}{\sqrt{2}}\sqrt{1 \pm \frac{\lambda}{6}} J_{-\frac{i\nu}{3} \pm \frac{1}{2}}(\sqrt{C}|p|e^{3x}/3)e^{i\nu(y-y_0)}$$

$$C < 0$$

$$\Psi_{\pm,\nu p} = \frac{1}{\sqrt{2}} e^{\pm i\frac{\pi}{4}}\sqrt{1 \pm \frac{\lambda}{6}} K_{\frac{i\nu}{3} \mp \frac{1}{2}}(\sqrt{|C|}|p|e^{3x}/3)e^{i\nu(y-y_0)}$$

4. Dirac type WDW equation

- the conservation law

$$\partial_M(\sqrt{-G}\bar{\Psi}\hat{\gamma}^M\Psi) = 0$$

- the probability density

$$\propto \ \sqrt{|2U|}a^{3/2}||\Psi||^2 = \sqrt{|2U|}a^{3/2}(|\Psi_+|^2 + |\Psi_-|^2)$$

$\Rightarrow$ positive-definite

4. Dirac type WDW equation

The modified Bessel function in the Klein-Gordon type

$$K_{i\frac{\nu}{2}}\left(\frac{y}{2}\right) \sim \sqrt{4\pi}e^{-\frac{\nu\pi}{4}}(\nu^2 - y^2)^{-\frac{1}{4}} \sin\left(\frac{\pi}{4} - \frac{1}{2}\sqrt{\nu^2 - y^2} + \frac{\nu}{2}\cosh^{-1}\frac{\nu}{y}\right)$$

The modified Bessel in the function Dirac type

$$K_{i\frac{\nu}{2}\pm\frac{1}{2}} \sim \sqrt{4\pi}e^{-\frac{\nu\pi}{4}\pm i\frac{\pi}{4}}(\nu^2 - y^2)^{-\frac{1}{4}}\left[\sqrt{\frac{\nu+y}{2y}}\sin\left(\frac{\pi}{4} - \frac{1}{2}\sqrt{\nu^2 - y^2} + \frac{\nu}{2}\cosh^{-1}\frac{\nu}{y}\right)\right.$$
$$\left.\mp i\sqrt{\frac{\nu-y}{2y}}\cos\left(\frac{\pi}{4} - \frac{1}{2}\sqrt{\nu^2 - y^2} + \frac{\nu}{2}\cosh^{-1}\frac{\nu}{y}\right)\right]$$

➡ These wave functions have a common wave packet solution.

5. Summary and Prospects

Summary

- We have applied the method in Eisenhart-Duval lift to a simple model and extend the minisuperspace.

  We have formulated Dirac type WDW equation in an extended minisuperspace.

  ⟹ The probability density is positive definite.

Prospects

- Applying the technique to the general cosmological models.

- Third quantization and global property of extended minisuperspace.