Tunneling wave function of the universe

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Quantum cosmology

$\psi(g,\phi)$ – wave function of the universe

$\mathcal{H}\,\psi = 0$ – Wheeler-DeWitt equation

In ordinary QM the boundary conditions for ψ are determined by the physical setup external to the system.

But there is nothing external to the universe. ➡ The b.c. for ψ should be postulated as an independent physical law.

The b.c. should determine ψ uniquely.

Path integral representation:

$$\psi(g,\phi) = \int^{(g,\phi)} \mathcal{D}g\mathcal{D}\phi e^{iS}$$

What is the class of paths?

Hartle-Hawking wave function

Hartle & Hawking (1983)

$$\psi_{HH}(g,\phi) = \int^{(g,\phi)} e^{-S_E}$$

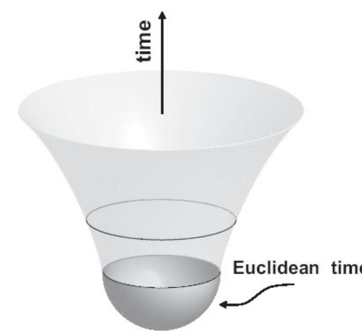
Euclidean metrics

Tunneling wave function

A.V. (1984)

$$\psi_T(g,\phi) = \int_{\varnothing}^{(g,\phi)} e^{iS}$$

Lorentzian metrics



Creation of the universe from "nothing"

I will focus on the tunneling proposal.
Based on work with Masaki Yamada (2018, 2019)

Tunneling wave function

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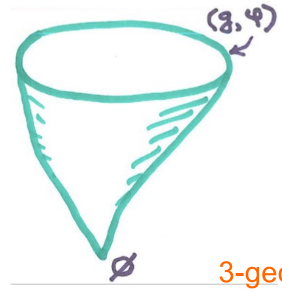
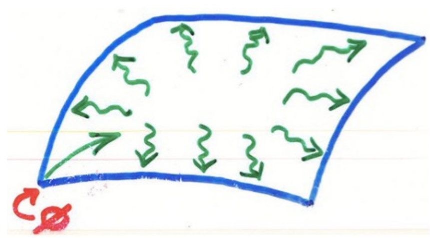


3-geometries of vanishing volume ("Nothing")

Corresponding b.c. in superspace:



- Outgoing wave boundary condition for the WDW equation
- ψ should give a normalizable probability distribution (regularity condition)

A.V. (1986)

Are these formulations equivalent?

Critique: The tunneling wave function predicts runaway instability of matter fields.

Halliwell & Hartle *(1990)*
Bousso & Hawking (1996)
Feldbrugge, Lehners & Turok (2017)

Perturbative minisuperspace model

$$S = \int \sqrt{-g} d^4x \left(\frac{R}{2} - \rho_v \right) + S_m + S_B$$

$$S_m = \int \sqrt{-g} d^4x \left[-\frac{1}{2} (\nabla \phi)^2 - \frac{1}{2} m^2 \phi^2 - \frac{1}{12} R \phi^2 \right]$$

$$ds^2 = a^2(\eta) \left(N^2 d\eta^2 - d\Omega^2 \right), \quad N = \mathrm{const}$$

$$\phi(\mathbf{x}, \eta) = \frac{1}{a(\eta)} \sum_n f_n(\eta) Q_n(\mathbf{x})$$

Treat ϕ as a small perturbation.

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Treat ϕ as a small perturbation.

WDW equation (dropping some numerical factors)

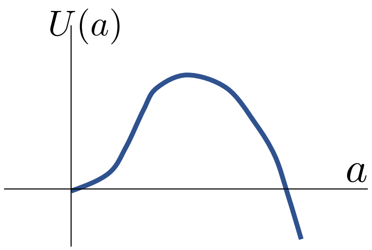
$$\left[\frac{\partial^2}{\partial a^2} - U(a) - \sum_n \mathcal{H}_n \right] \psi(a, f_n) = 0$$

$$\mathcal{H}_n = \frac{1}{2}\frac{\partial^2}{\partial f_n^2} - \frac{1}{2}\omega_n^2(a) f_n^2$$

$$U(a) = a^2(1 - H^2 a^2)$$

$$H^2 = \rho_v/3$$

$$\omega_n^2(a) = n^2 + m^2 a^2$$



TUNNELING BOUNDARY CONDITIONS

WKB ansatz:

$$\psi(a, f_n) = A \exp\left[-S(a) - \frac{1}{2}\sum_n R_n(a) f_n^2\right]$$

WDW eq. $\Rightarrow$

$$(S')^2 - U(a) = 0$$

$$S' R_n' - R_n^2 + \omega_n^2(a) = 0$$

Boundary conditions:

(1) Only outgoing wave in $S(a)$ at $a \to \infty$.

(2) Regularity: $Re[R_n(a)] > 0$

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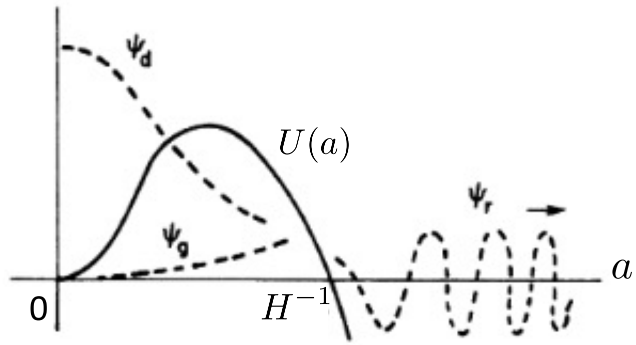
Leading order:

$$S(a < H^{-1}) = \pm \int_a^{H^{-1}} \sqrt{U(a')} da'$$
$$S(a > H^{-1}) = \pm i \int_{H^{-1}}^{a} \sqrt{-U(a')} da'$$

Select "-"
Corresponds to expanding universe.



$\psi_d \sim \psi_g \sim \psi_r$

at $a \sim H^{-1}$

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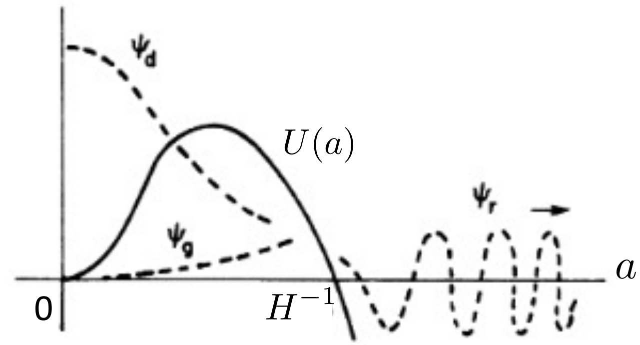

$\psi_d \sim \psi_g \sim \psi_r$
at $a \sim H^{-1}$

Including perturbations

The regularity condition is satisfied in the entire classically allowed region $(U(a) < 0)$ if it is satisfied at any point in that region.

Same for ψ_d in the classically forbidden region.

For ψ_g the regularity condition is satisfied in the entire classically forbidden region if it is satisfied at $a = 0$.

Vachaspati & A.V (1988)
A.V. & Yamada (2018)
Damour & A.V. (2019)

Boundary condition at $a = 0$

$$S' R'_n - R_n^2 + \omega_n^2(a) = 0 \qquad\qquad \omega_n^2(a) = n^2 + m^2 a^2$$

For $a < H^{-1}$ introduce (conformal) Euclidean time τ:

$$\frac{da}{d\tau} = S'(a) = \pm\sqrt{U(a)} \qquad\qquad a(\tau) = (H\cosh\tau)^{-1}$$

S^4

$$\frac{dR_n}{d\tau} - R_n^2 + \omega_n^2(a) = 0$$

$a \to 0$ corresponds to $\tau \to \pm\infty$.

(+ for ψ_g , - for ψ_d.)

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(+ for ψ_g , - for ψ_d.)

This is a Riccati equation.

$$R_n(\tau) = -\frac{1}{\nu_n}\frac{d\nu_n}{d\tau} \Rightarrow \frac{d^2\nu_n}{d\tau^2} - \omega_n^2 \nu_n = 0$$

Same as eq. for f_n with $N\eta \to i\tau$.

This formalism is equivalent to QFT in curved spacetime.

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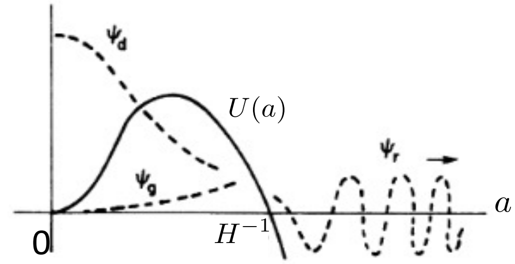
This formalism is equivalent to QFT in curved spacetime.

For $a \to 0$: $\quad \omega_n^2 \approx n^2, \quad \nu_n \approx A_n e^{-n\tau} + B_n e^{n\tau}$

$$R_n(\tau) \approx n\frac{A_n - B_n e^{2n\tau}}{A_n + B_n e^{2n\tau}}$$

$R_n(\tau \to \infty) = -n$, unless $B_n = 0$. $\implies$ Require $B_n = 0$.



$B_n = 0 \quad \Rightarrow \quad \nu_n(\tau \to \infty) \propto e^{-n\tau}$

$$\frac{d\nu_n}{d\tau} = -n\nu_n \quad (\tau \to \infty)$$

Robin boundary condition

This determines $\nu_n(\tau)$ for ψ_g. Selects the Bunch-Davies vacuum state.

ν_n for ψ_d are determined from the matching conditions at $a \sim H^{-1}$. $\Rightarrow$ $\nu_n(\tau \to -\infty) \propto e^{-n\tau}$

The regularity condition is equivalent to Robin b.c. at $a \to 0$.

This is now really a b.c.

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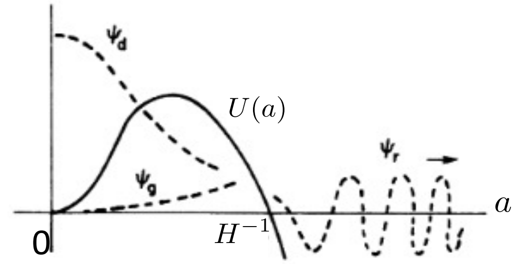

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This is now really a b.c.

Note: $\nu_n(\tau \to -\infty) \to \infty$. But $R_n = -\dfrac{1}{\nu_n}\dfrac{d\nu_n}{d\tau} \to n$ are well behaved.

Conclusion: The tunneling b.c. uniquely determine the wave function. It describes a dS universe nucleating with the field ϕ in the Bunch-Davies state.

Vachaspati & A.V (1988)
A.V. & Yamada (2018)

PATH INTEGRAL APPROACH

$$\psi(a, f_n) = \int_0^\infty dN \int \mathcal{D}a \int \mathcal{D}f_n \ e^{iS}$$

Leading order

$$\psi_0(a_1) = \int_0^\infty dN \int \mathcal{D}a \ e^{iS(a,N)}$$

$$S(a, N) = 6\pi^2 \int_{-\infty}^{\eta_1} \left[-\frac{\dot{a}^2}{N} + Na^2 \left(1 - H^2 a^2\right) \right]$$

$$a(\eta_1) = a_1, \quad a(\eta \to -\infty) = 0$$

The path integral over a can be done exactly:

Halliwell & Louko *(1990)*

$$\psi_0(a_1) = \int_0^\infty \frac{dN}{N^{1/2}} e^{iS_0(a_1,N)}$$

$$S_0(a_1, N) = 6\pi^2 \left[N^3(H^4/12) + N\left(1 - H^2 a_1^2/2\right) - a_1^4/4N \right]$$

Use saddle point approximation.

$$\psi_0(a_1) = \int_0^\infty \frac{dN}{N^{1/2}} e^{iS_0(a_1,N)}$$

Saddle points: $\partial S/\partial N = 0$

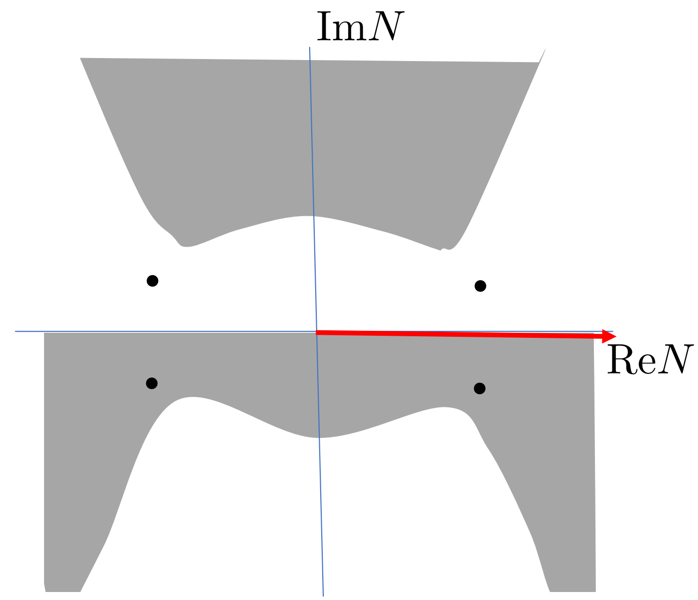
$$N = \pm H^{-2}\left(i \pm \sqrt{H^2 a_1^2 - 1}\right)$$

Pickard-Lefshetz prescription:

Deform the contour so that it passes through a saddle point following steepest ascent/descent lines.

Only one saddle point is relevant: $N = H^{-1}\left(i + \sqrt{H^2 a_1^2}\right)$

$(a_1 > H^{-1})$



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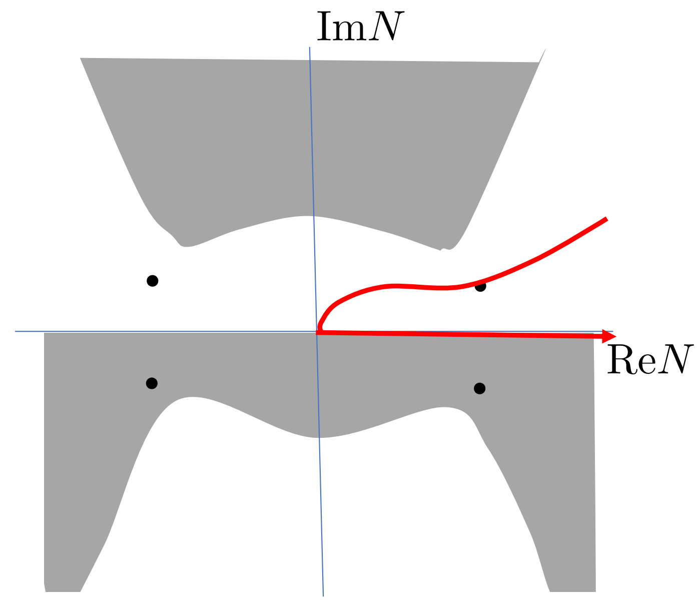
$$\psi(a_1) \sim e^{iS_0(a_1,N)} = \exp\left[-\frac{4\pi^2}{H^2} - i\frac{4\pi^2}{H^2}\left(H^2 a_1^2 - 1\right)^{3/2}\right]$$

Large *H* are favored

Expanding universe
→ Satisfies the outgoing wave condition.



Halliwell & Louko *(1990)*
Feldbrugge, Lehners & Turok (2017)

Including perturbations

$$\psi(a_1, f_{n1}) = e^{iS_0(a_1,N)} \prod_n \int \mathcal{D} f_n e^{iS_n(f_n,N)}$$

$$N = H^{-1}\left(i + \sqrt{H^2 a_1^2}\right)$$

$$S_n(f_n, N) = \frac{1}{2}\int_{-\infty}^{\eta_1} d\eta \left(\frac{1}{N}\dot{f}_n^2 - N\omega_n^2 f_n^2\right) + S_{Bn}$$

$$f_n(\eta_1) = f_{n1}, \qquad \frac{df_n}{d\eta} = inNf_n \quad (\eta_0 \to -\infty)$$

To ensure regularity

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This path integral can be done exactly.

Disregard S_{Bn} for now.

$$\psi(a_1, f_{n1}) = e^{iS_0(a_1,N)} \prod_n e^{iS_{n0}}$$

$$S_{n0} = \frac{1}{2N}\left(f_{n1}\dot{f}_{n1} - f_{n0}\dot{f}_{n0}\right)$$

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But $\quad f_{n0} \propto e^{inN\eta_0} \to \infty \qquad (\eta_0 \to -\infty)$

(since $\mathrm{Im}N > 0$)

⇨ $iS_{n0} \to \infty$

If we drop the Robin b.c., then we lose regularity

⇨ uncontrolled fluctuations.

The boundary term

The choice of the boundary term depends on the boundary conditions.
Dirichlet b.c. (fixing f_n at η_0 and η_1) do not require any boundary term.

We have Dirichlet b.c. at η_1 but Robin b.c. $\dfrac{df_n}{d\eta} = inNf_n$ at η_0.

$\Rightarrow$ Add a boundary term $S_{Bn} = \dfrac{in}{2} f_n^2(\eta_0)$. Then variation of the action gives

This comes from
the boundary term

$$\delta S_n = \delta f_n \frac{1}{N}\frac{df_n}{d\eta}(\eta_1) - \delta f_n \left(\frac{1}{N}\frac{df_n}{d\eta} - inf_n\right)(\eta_0)$$

= 0   = 0

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This comes from the boundary term

The action with the boundary term:

$$S_{n0} = \frac{1}{2N}\left(f_{n1}\dot{f}_{n1} - f_{n0}\dot{f}_{n0}\right) + S_{Bn} = \frac{1}{2N} f_{n1}\dot{f}_{n1}$$

The infinity canceled out!
Note: this boundary term is required for consistency.

We can rewrite this as

$$S_{n0} = \frac{i}{2} R_n f_{n1}^2 \qquad R_n = -\frac{i}{N}\frac{\dot{f}_{n1}}{f_{n1}}$$

– same as in the WDW approach.

A.V. & Yamada (2018)

Conclusions

We discussed two approaches to defining the tunneling wave function $\psi(a, f_n)$
in a minisuperspace model:

1) Tunneling boundary conditions in superspace.

2) Lorentzian path integral over histories starting at $a = 0$ with scalar field modes f_n
satisfying Robin b.c.

Both approaches give identical wave functions with well behaved scalar field fluctuations.

Extension beyond perturbative minisuperspace?

What replaces the Robin b.c.?

Some comments on the HH wave function

Original proposal: $$\psi_{HH}(g,\phi) = \int^{(g,\phi)} e^{-S_E}$$

The Euclidean action is unbounded from below, so the integral is divergent.

Solution: Integrate over complex metrics, in particular over complex lapse contours.

How do we choose the contour?

Most recent version: $$\psi_{HH}(g,\phi) = \sum_j d_j \; e^{-S_j(g,\phi)}$$

Contribution of j-th saddle point

Which saddle points should be included?

What are d_j?

Both ψ_{HH} and ψ_T are now "work in progress".

The question is: What is the general law of boundary conditions?