QED in intense background fields: Foundations

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Outline

• The electromagnetic interaction
• Typical scales of classical electrodynamics (CED) and quantum electrodynamics (QED)
• Sources of strong electromagnetic fields
• Furry picture in strong-field QED
• The special case of a plane-wave background field
  – Volkov states
  – Brief discussion about some processes
• Conclusions
**Additional reading information**

- **Books:**

- **Reviews:**
  2. V. I. Ritus, J. Sov. Laser Res. 6, 497 (1985)
  10. F. Karbstein, Particles 3, 39 (2020)
Electromagnetic interaction

• The electromagnetic interaction is one of the four fundamental interactions in Nature

• By accounting only for the lightest charged particles (electrons and positrons), the Lagrangian of the theory depends on two parameters:
  
  – Electron mass \( m = 9.1 \times 10^{-28} \text{ g} \)
  
  – Electron charge \( e \), with \( |e| = 4.8 \times 10^{-10} \text{ statcoulomb} \)

• The typical scales of classical electrodynamics (CED) are determined by combining \( m \) and \( e \) with another fundamental constant:
  
  – Speed of light \( c = 3.0 \times 10^{10} \text{ cm/s} \)

• In quantum electrodynamics (QED) the dynamics is richer:
  
  – Reduced Planck constant \( \hbar = 1.1 \times 10^{-27} \text{ erg s} \)
## Typical scales of CED and QED

<table>
<thead>
<tr>
<th></th>
<th>CED</th>
<th>QED</th>
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</thead>
<tbody>
<tr>
<td><strong>Energy</strong></td>
<td>Electron’s rest energy: $\varepsilon_0=mc^2=0.5$ MeV</td>
<td></td>
</tr>
<tr>
<td><strong>Momentum</strong></td>
<td>$p_0=\varepsilon_0/c=0.5$ MeV/c</td>
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<tr>
<td><strong>Length</strong></td>
<td>Classical electron’s radius: $r_0=e^2/mc^2=2.8\times10^{-13}$ cm</td>
<td>Compton wavelength: $\lambda_C=\hbar/p_0=3.9\times10^{-11}$ cm</td>
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<td>(from the Thomson cross section)</td>
<td>(from Heisenberg uncertainty principle)</td>
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<tr>
<td><strong>Time</strong></td>
<td>$r_0/c=1.0\times10^{-23}$ s</td>
<td>$\lambda_C/c=1.3\times10^{-21}$ s</td>
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$r_0=\alpha\lambda_C$, where $\alpha=e^2/4\pi\hbar c\approx1/137$ is the fine-structure constant

### Field scales of QED (critical or Schwinger field)

$$E_{cr} = \frac{m^2c^3}{\hbar|e|} = 1.3 \times 10^{16} \text{ V/cm}$$

$$B_{cr} = \frac{m^2c^3}{\hbar|e|} = 4.4 \times 10^{13} \text{ G}$$

$$I_{cr} = cE_{cr}^2 = 4.6 \times 10^{29} \text{ W/cm}^2$$
QED critical fields and vacuum physics

- In classical physics the vacuum is a region of space where neither particles nor fields are present.
- In quantum field theory the vacuum is the lowest-energy state where no real particles (electrons, positrons, photons etc...) are present.
  - Virtual particles are present.
  - They “live” for a very short time and cover a very short distance (for electrons and positrons $\tau_C = \hbar/mc^2 \sim 10^{-21}$ s and $\lambda_C = \hbar/mc \sim 10^{-11}$ cm, respectively).

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Quantum Mechanics ➔ Quantum Field Theory ➔ Special Relativity

Time-Energy Uncertainty Principle $\Delta \varepsilon \Delta t \gtrsim \hbar$

Virtual Particles, i. e., Quantum Vacuum Fluctuations

Mass-Energy Equivalence $\varepsilon = mc^2$
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• Physical meaning of the critical fields:
\[ |e|E_{cr} \times \frac{\hbar}{mc} = mc^2 \]

• Vacuum instability and electromagnetic cascades (more on the second part)

• The interaction energy of a Bohr magneton with a magnetic field of the order of \( B_{cr} \) is of the order of the electron rest energy

• In the presence of background electromagnetic fields of the order of the critical ones a new regime of QED, the strong-field QED regime, opens:

1. where the properties of the vacuum are substantially altered by the fields
2. where a tight interplay unavoidably exists between collective (plasma-like) and quantum effects
3. which is inaccessible to conventional accelerators because it requires coherent fields
Sources of strong electromagnetic fields

- Highly-charged ions (Bethe and Heitler 1934, Bethe and Maximon 1954)

\[ \varepsilon(\hbar \omega) \rightarrow Z|e| \]

High-order nonlinear QED effects (Coulomb corrections) only depend on the parameter \( Z\alpha \) (\( \alpha \approx 1/137 \))

- Crystals and channeling (Uggerhøj 2005): ultrarelativistic charged particles interact coherently with the atoms aligned in the crystal

- Magnetars (Turolla et al. 2015): rotating neutron stars whose surrounding magnetic fields are estimated to even exceeding the critical one
• Ultrarelativistic electron-positron bunches (Chen 1987): in collisions between $e^+e^-$ beams as those presumably occurring in future linear colliders, strong-field QED effects may limit the performances of such colliders (beamstrahlung)

• Intense lasers (Di Piazza et al. 2012):
  - Astra Gemini
  - ELI Beamlines
  - LUXE at DESY (Abramowicz et al. 2019)
  - E-320 at FACET II (SLAC) (Meuren et al. 2020)
Electromagnetic field as classical field

- Following Bohr, an electromagnetic field can be treated as a classical field if the occupation numbers \( n_{k,\lambda} \) corresponding to the operators \( N_{k,\lambda} = a_{k,\lambda}^\dagger a_{k,\lambda} \) are large.
- If all \( n_{k,\lambda} \) are large, the energy of the field would be infinite.
- If the field varies on a typical time scale \( \Delta t \), its typical Fourier components have angular frequencies less or equal than \( \omega_0 = 1/\Delta t \).
- Require that \( n_{k,\lambda} \gg 1 \) for \( \omega = c|\mathbf{k}| < \omega_0 \).
- Typical occupation number \( n \) in terms of the fields \( (E, B) \):
  \[
  n \sim \frac{\text{Total field energy}}{\text{Typical number of states} \times \text{Typical states' energy}} \\
  \sim \frac{\frac{1}{2}(E^2 + B^2)V}{2 \times \frac{V (2\pi\hbar)^3}{(2\pi)^3} \frac{4}{3} \pi (\frac{\hbar \omega_0}{c})^3 \times (\hbar \omega_0)}
  
  \]
- Constant fields can always be treated classically.
- The condition \( n \gg 1 \) is easily fulfilled for available optical lasers.
- Physical processes should not significantly alter the classical spacetime evolution of the field (Seipt et al. 2017, Ilderton et al. 2017).
Furry picture in strong-field QED

• The Lagrangian density of QED in the presence of a background field $F_{\mu\nu}^B = \partial_\mu A_\nu^B - \partial_\nu A_\mu^B$ produced by a four-current $J_\mu^B$ is given by

$$L_{\text{QED}} = \bar{\psi} \left[ \gamma^\mu (i\partial_\mu - eA_\mu - eA_{B,\mu}) - m \right] \psi + L_B - J_\mu^B(A_\mu + A_{B,\mu}) - \frac{1}{4} (F_{\mu\nu} + F_{B,\mu\nu})(F^{\mu\nu} + F_B^{\mu\nu})$$

• The background field and four-current are given functions fulfilling Maxwell’s equations $\partial_\mu F_{\mu\nu}^B = J_\nu^B$, and we can drop the “constant” terms

$$L_{\text{QED}} \sim \bar{\psi} \left[ \gamma^\mu (i\partial_\mu - eA_\mu - eA_{B,\mu}) - m \right] \psi - J_\mu^B A_\mu - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} F_{\mu\nu} F_B^{\mu\nu}$$

• By integrating by parts the last term

$$F^{\mu\nu} F_{B,\mu\nu} = (\partial_\mu A_\nu - \partial_\nu A_\mu) F_{B,\mu\nu} = 2(\partial_\mu A_\nu) F_{B,\mu\nu} = 2\partial_\mu (A_\nu F_{B,\mu\nu}) - 2A_\nu (\partial_\mu F_{B,\mu\nu})$$

$$= 2\partial_\mu (A_\nu F_{B,\mu\nu}) - 2A_\nu J_{B,\nu}$$

we obtain (Furry 1951)

$$L_{\text{QED}} \sim \bar{\psi} \left[ \gamma^\mu (i\partial_\mu - eA_\mu - eA_{B,\mu}) - m \right] \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

• The effect of the external field is to give rise to an additional vertex corresponding to the interaction term $L_{i,B} = -e \bar{\psi} \gamma^\mu \psi A_{B,\mu}$
• The Lagrangian density one employs in strong-field QED is

\[ \mathcal{L}_{SFQED} = \bar{\psi} [\gamma^\mu (i \partial_\mu - e A_\mu - e A_{B,\mu}) - m] \psi - \frac{1}{4} F_{\mu \nu} F^{\mu \nu} \]

• The effect of the external field on a QED process

\[ + \cdots = e^- \]

• The contribution of the external field has to be taken into account exactly in the calculations for

The expression of the quantity \( \rho = |e| A_B / m \) depends on the external field:
1. HCI: \( \rho = Z \alpha / \lambda_c m = Z \alpha \)
2. PW: \( \rho = |e| E_L / \omega_L m = \xi \)
• Analytical solutions are available only for highly symmetric fields: plane wave, constant fields, Coulomb field (Bagrov et al. 2014).

\[ \mathcal{L}_{SFQED} = \mathcal{L}_e + \mathcal{L}_\gamma + \mathcal{L}_i \]
\[ \mathcal{L}_e = \bar{\psi}[\gamma^\mu (i\partial_\mu - eA_{B,\mu}) - m]\psi \]
\[ \mathcal{L}_\gamma = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \]
\[ \mathcal{L}_i = -e\bar{\psi} \gamma^\mu \psi A_\mu \]

• The quantization of the spinor field in the presence of the background field implies the ability of solving analytically the "dressed" Dirac equation
\[ [\gamma^\mu (i\partial_\mu - eA_{B,\mu}) - m]\psi = 0 \]

• Does the presence of the background field allow for a particle description of the Dirac field?

• Does the eigenvalue equation
\[ \{ \alpha \cdot [-i\nabla - eA_B(x)] + \beta m + e\Phi_B(x) \} \psi(x) = \varepsilon(t) \psi(x) \]
for the one-particle Hamiltonian admit positive- and negative-energy eigenvalues separated by a finite gap?

• Explicit time dependence
• Furry considered external fields where the gap between positive and negative energies was finite for all times (Furry 1951)

• The quantization procedure is conceptually straightforward:
  1. Find the base of the dressed positive-energy solutions $U_n(x)$ and the negative-energy solutions $V_n(x)$ of the Dirac equation $[\gamma^\mu(i\partial_\mu - eA_{B,\mu}) - m]\psi=0$ characterized by quantum numbers collected here in the index $n$ and expand the field operator in the Heisenberg representation (interaction representation once you include the interaction with the photons)
    $$\psi(x) = \sum_n [c_n U_n(x) + d_n^\dagger V_n(x)]$$
    in terms of creation and annihilation operators
  2. By applying Wick’s theorem, one also needs the causal dressed propagator $G(x, y)$ by solving the equation
    $$[\gamma^\mu(i\partial_\mu - eA_{B,\mu}) - m]G(x, y) = \delta^4(x - y)$$
    with Feynman prescription $m^2 \rightarrow m^2 - i\varepsilon$
  2. Identify the Feynman diagrams contributing to the process at hand
  3. Calculate the total amplitude and then the cross section (or the rate) using “dressed” states and propagators

• Furry’s approach covers relevant examples like the constant magnetic field and the plane-wave field (even if it is time-dependent, details in a few slides)
• The fact that positive- and negative-energy states are always separated by a gap implies that the vacuum is stable in the presence of those background fields
• Spontaneous pair production from vacuum (Schwinger mechanism) is forbidden
• Furry’s approach does not cover the case of a (time-dependent) uniform electric field
• In a more general approach (Fradkin et al. 1991) one assumes that the gap exists only asymptotically for \( t \to \pm \infty \):

1. Find the two bases of the in-states \( U_n^{(\text{in})}(x) \) and \( V_n^{(\text{in})}(x) \) and the out-states \( U_n^{(\text{out})}(x) \) and \( V_n^{(\text{out})}(x) \) with positive and negative energies for \( t \to -\infty \) and \( t \to +\infty \), respectively and expand the field operator

\[
\psi(x) = \sum_n \left[ \bar{c}_n^{(\text{in})} U_n^{(\text{in})}(x) + \bar{d}_n^{(\text{in})} V_n^{(\text{in})}(x) \right] = \sum_n \left[ \bar{c}_n^{(\text{out})} U_n^{(\text{out})}(x) + \bar{d}_n^{(\text{out})} V_n^{(\text{out})}(x) \right]
\]

in terms of in- and out-creation and annihilation operators

2. Initial (final) states are built out of the in-vacuum (out-vacuum) states by means of the corresponding creation and annihilation operators

• A non-vanishing overlap between the negative-energy out-states with the positive-energy in states signals vacuum instability
• The singular case of the Coulomb-field case for \( Z \geq 137 \)
New effects

- Depending on the structure of the external field total energy and/or total momentum of the Dirac and the electromagnetic field may not be conserved.
- The instability of the vacuum in a background field is related to a change in its dielectric properties too: the virtual electron-positron dipoles in the vacuum give rise to nonlinear a self-interaction of the background field (Euler-Heisenberg Lagrangian density)
  \[ \mathcal{L}_{EH} = \frac{1}{2}(E^2 - B^2) + \frac{2\alpha^2}{45m^4} [(E^2 - B^2)^2 + 7(E \cdot B)^2] \]
- The vacuum becomes a birefringent medium in the presence of an electromagnetic background field, i.e., the refractive index depends on the polarization direction of the propagating wave:
  \[ n_\parallel = 1 + \frac{4\alpha}{90\pi} \frac{E^2}{E_{cr}^2} \]
  \[ n_\perp = 1 + \frac{7\alpha}{90\pi} \frac{E^2}{E_{cr}^2} \]
- The vacuum four-current induced by the presence of the background field may also induce observable effects, which were supposed to vanish until recently (Karbstein 2016 and 2017, Edwards et al. 2017, Ahmadiniaz et al. 2017 and 2019).
Furry picture in a strong plane-wave field

- Plane-wave field: \( A_B^\mu(x) = A^\mu(\phi) \), where \( \phi = (nx) = t - n \cdot x \)

- Three components of the four-momentum are conserved:
  \[
p_- = \varepsilon - p_\parallel \quad \text{and} \quad p_\perp = \mathbf{p} - p_\parallel \mathbf{n}
\]
with \( p_\parallel = \mathbf{n} \cdot \mathbf{p} \) and can be used as quantum numbers to classify the electron states

- By means of these conservation rules and the on-shell condition \( p^2 = m^2 \), the classical energy of an electron at an arbitrary phase is
  \[
  \varepsilon(\phi) = \frac{p_-}{2} + \frac{m^2 + [p_{0,\perp} - eA_\perp(\phi) + eA_\perp(\phi_0)]^2}{2p_-}
  \]

- Positive- and negative-energy states are characterized by \( p_- \geq 0 \) and physically they are disjoint [in the figure \( m^* = (m^2 + p_\perp^2)^{1/2} \)]

- One sees classically that \( \varepsilon(\phi) \geq m \) \( \varepsilon(\phi) \leq - m \) for \( p_\geq 0 \) \( p_- < 0 \) (no bound states)

- This is the physical reason why Furry’s approach can be applied to the plane-wave case
**Volkov states**

- **Assumptions about the four-vector potential:**
  1. \( \lim_{\phi \to \pm \infty} A^\mu(\phi) = 0 \)
  2. Lorenz gauge: \( \partial_\mu A^\mu(\phi) = 0 \) plus \( A^0(\phi) = 0 \), which imply \( A^\mu(\phi) = (0, A_\perp(\phi), 0) \)

- **Positive-energy Volkov in-states** (Volkov 1935):
  \[
  U^{(\text{in})}_{p,\sigma}(x) = \left[ 1 + \frac{e}{2p_-} \hat{n} \hat{A} \right] \frac{u_{p,\sigma}}{\sqrt{2\varepsilon}} e^{i S^{(\text{in})}_p(x)}
  \]
  \[
  S^{(\text{in})}_p(x) = -(px) - \int_{-\infty}^{\phi} d\phi' \left[ \frac{e(pA)}{p_-} - \frac{e^2 A^2}{2p_-} \right]
  \]

- **Quantum numbers:** four-momentum components \( p_- \) and \( p_\perp \) [the remaining component \( p_+ = (\varepsilon + p_\parallel)/2 \) is derived from the on-shell condition \( p^2 = m^2 \): \( p_+ = (m^2 + p_\perp^2)/2p_- \)] and spin quantum number

- **Spin dynamics** (\( \hat{\nu} = \gamma_\mu v^\mu \)): The average spin pseudo-four-vector fulfills the classical \( \text{Bargmann-Michel-Telegdi (BMT)} \) equation

- **Constant positive-energy free spinor** \( u_{p,\sigma} \)

- **Classical action** \( S_p^{(\text{in})}(x) \) reducing to the free action for \( \phi \to -\infty \)

- **Volkov states** have a **quasiclassical structure** although they are an exact solution of Dirac equation
• Comparison between in- and out-states:

\[ U_{p,\sigma}^{(\text{in})}(x) = \left[ 1 + \frac{e}{2p_-} \hat{n} \hat{A} \right] \frac{u_{p,\sigma}}{\sqrt{2\varepsilon}} e^{i S_p^{(\text{in})}(x)} \quad U_{p,\sigma}^{(\text{out})}(x) = \left[ 1 + \frac{e}{2p_-} \hat{n} \hat{A} \right] \frac{u_{p,\sigma}}{\sqrt{2\varepsilon}} e^{i S_p^{(\text{out})}(x)} \]

\[ S_p^{(\text{in})}(x) = -(px) - \int_{-\infty}^{\phi} d\phi' \left[ \frac{e(pA)}{p_-} - \frac{e^2 A^2}{2p_-} \right] \quad S_p^{(\text{out})}(x) = -(px) - \int_{-\infty}^{\phi} d\phi' \left[ \frac{e(pA)}{p_-} - \frac{e^2 A^2}{2p_-} \right] \]

• In- and out-states are equivalent as they only differ by a constant phase factor: the vacuum is stable in a plane wave (Schwinger 1951)

\[ U_{p,\sigma}^{(\text{out})}(x) = U_{p,\sigma}^{(\text{in})}(x) e^{i \Phi_p(-\infty, \infty)} \]

\[ \Phi_p(-\infty, \infty) = \int_{-\infty}^{\infty} d\phi \left[ \frac{e(pA)}{p_-} - \frac{e^2 A^2}{2p_-} \right] \]

• Physical reason: all photons of a plane wave propagate along the same direction and do not interact with each other

• Comparison between positive- and negative-energy in-states:

\[ U_{p,\sigma}^{(\text{in})}(x) = \left[ 1 + \frac{e}{2p_-} \hat{n} \hat{A} \right] \frac{u_{p,\sigma}}{\sqrt{2\varepsilon}} e^{i S_p^{(\text{in})}(x)} \quad V_{p,\sigma}^{(\text{in})}(x) = \left[ 1 - \frac{e}{2p_-} \hat{n} \hat{A} \right] \frac{v_{p,\sigma}}{\sqrt{2\varepsilon}} e^{i S_p^{(\text{in})}(x)} \]

\[ S_p^{(\text{in})}(x) = \Phi_p(-\infty, 0) - p_- T + p_\perp \cdot x_\perp - \int_{-\infty}^{\phi} d\phi' \frac{m^2 + [p_\perp - e A_\perp(\phi')]^2}{2p_-} \]

• Remaining light-cone coordinates: \( T = (t + x_\parallel)/2 \) and \( x_\perp = x - x_\parallel n \)

• Positive- and negative-energy Volkov in-states (as well as out-states) form a complete set of states (Gitman et al. 1975, Boca et al. 2010, Di Piazza 2018)
Some Examples (plane wave)

- One-vertex strong-field QED processes:
  1. Nonlinear Single Compton Scattering
  2. Nonlinear Breit-Wheeler Pair Production

- Second-order processes
  3. Electron-Positron Annihilation into Two Photons (Hartin 2017, Bragin et al. 2020)

- Radiative corrections
  1. Mass Operator (Baier et al. 1975)
  2. Polarization Operator (Baier et al. 1976, Becker et al. 1976)
  3. Vertex Correction (Di Piazza et al. 2020)
Conclusions

- There is an increasing interest in studying QED processes in the presence of strong background electromagnetic fields
  - Natural physical systems (highly charged ions, magnetars)
  - Experiments designed to test QED in strong fields (lasers, crystals)
- The theoretical investigation of such processes predominantly relies on a semi-perturbative approach (Furry picture) where
  - The interaction between charged particles (electrons and positrons) and the background field is taken into account exactly in the calculations
  - The interaction between charged particles and the radiation field is treated perturbatively
- New effects occur in the presence of strong background electromagnetic fields (e.g., new processes, vacuum birefringence)
- Analytical insight in the plane-wave case: Volkov states