Computing moments of inertia

The moment of inertia of a rigid continuous object is given by

$$I = \int r^2 dm.$$

The formulas for various homogeneous rigid objects are listed in **Table 10.2** of the textbook. These are,

1. Hoop (or thin cylindrical shell) of radius R

$$I_{CM} = MR^2 \tag{1}$$

2. Hollow cylinder of inner radius R_1 and outer radius R_2

$$I_{CM} = \frac{1}{2}M(R_1^2 + R_2^2) \tag{2}$$

3. Solid cylinder (or disk) of radius R

$$I_{CM} = \frac{1}{2}MR^2 \tag{3}$$

4. Rectangular plate

$$I_{CM} = \frac{1}{12}M(a^2 + b^2) \tag{4}$$

5. Rod of length L, around its centre

$$I_{CM} = \frac{1}{12}ML^2 \tag{5}$$

6. Rod of length L, around one of its ends

$$I_{end} = \frac{1}{3}ML^2 \tag{6}$$

7. Solid sphere of radius R

$$I_{CM} = \frac{2}{5}MR^2 \tag{7}$$

8. Thin spherical shell of radius R

$$I_{CM} = \frac{2}{3}MR^2 \tag{8}$$

In all cases, M is the mass of the object. In the textbook, 1, 2, 3 are derived in **Example 10.5**, and 5, 6 are derived in **Example 10.4** and **10.6**. Let us derive the formulae for the remaining cases below.

1 Rectangular plate

The moment of inertia for the rectangular plate of sides a and b can be found by using the formula (5) and the parallel axis theorem. The moment of inertia of a rod of mass M and length L, with axis separated by distance x from the original one (through the centre of mass), is

$$I_x = I_{CM} + Mx^2 = \frac{1}{12}ML^2 + Mx^2. \tag{9}$$

Now replacing $L \to a$, $M \to dm = \sigma a dx$, and integrating over x from $-b/2$ to $b/2$, one obtains

$$\begin{aligned} I &= \int_{-b/2}^{b/2} (\frac{1}{12}a^3\sigma + ax^2\sigma)dx \\ &= \frac{1}{12}\sigma(a^3b + ab^3) \\ &= \frac{1}{12}M(a^2 + b^2), \end{aligned} \tag{10}$$

where $M = \sigma ab$ has been used.

2 Thin spherical shell

Consider a thin spherical shell of radius R and mass M. We take spherical coordinates with azimuthal angle θ and zenith angle ϕ (see for example `http://mathworld.wolfram.com/SphericalCoordinates.html`). On the spherical shell the mass element is

$$dm = \sigma R \sin\theta d\phi R d\phi, \tag{11}$$

where $\sigma = M/4\pi R^2$ is the surface mass density, and the distance from the rotational axis is $r = R\sin\phi$. Hence the moment of inertia to be calculated is

$$I = \int r^2 dm = 2\pi\sigma R^4 \int_0^\pi \sin^3\phi d\phi. \tag{12}$$

Noting that

$$\begin{aligned} \int_0^\pi \sin^3\phi d\phi &= \int_0^\pi \sin\phi(1-\cos^2\phi)d\phi \\ &= \int_0^\pi \sin\phi d\phi - \int_{-1}^1 u^2 du \\ &= \Big[-\cos\phi\Big]_{\phi=0}^{\pi} - \Big[\frac{1}{3}u^3\Big]_{-1}^{1} = \frac{4}{3} \end{aligned} \tag{13}$$

(the variable has been changed as $u = \cos\phi$ and $du = d\cos\phi = -\sin\phi d\phi$), we now find

$$I = \frac{2}{3}MR^2. \tag{14}$$

3 Solid sphere

The moment of inertia for a solid sphere of radius R and mass M can be obtained by integrating the result for the disk (3) over changing distance from the axis. Choosing the z-axis as the axis of rotation and letting the distance from it to the mass element on the shell as r, we have

$$r^2 = R^2 - z^2. \tag{15}$$

Now $M \to dm = \pi r^2 \rho dz$ and $R^2 \to r^2$ in (3), we have

$$\begin{aligned} I &= \int_{z=-R}^{R} \frac{1}{2}\pi r^2 \rho \cdot r^2 dz \\ &= \frac{1}{2}\pi\rho \int_{z=-R}^{R} (R^4 - 2R^2 z^2 + z^4) dz \\ &= \frac{2}{5} M R^2, \end{aligned} \tag{16}$$

where the mass of the sphere is

$$M = \frac{4}{3}\pi R^3 \rho. \tag{17}$$

Alternatively, we may integrate the result for the spherical shell above, over the shell radius r. Rewriting $M \to dm = 4\pi r^2 \rho dr$, $R \to r$ and integrating from $r = 0$ to $r = R$, we find

$$\begin{aligned} I &= \frac{2}{3} \cdot 4\pi\rho \int_0^R r^4 dr \\ &= \frac{8\pi}{15} \rho R^5 \\ &= \frac{2}{5} M R^2. \end{aligned} \tag{18}$$