2008/12/21 RIKEN Symposium
"Towards New Developments in Fields and String Theories"

Ginsparg-Wilson formulation of 2D $\mathcal{N} = (2,2)$ SQCD with exact lattice supersymmetry

Fumihiko Sugino
(Okayama Inst. Quantum Phys.)

Mainly based on

- **F. S., JHEP 0403 (2004) 067** [arXiv:hep-lat/0401027].
- **F. S., Nucl. Phys. B 808 (2009) 292** [arXiv:0807.2683 [hep-lat]].
- **Y. Kikukawa and F. S.,** arXiv:0811.0916 [hep-lat].

1 Introduction

◇ Physics beyond the Standard Model (Experiments: LHC, WMAP, etc)

↑

Supersymmetric gauge theories, Superstring theories,...

◇ In particular, nonperturbative aspects of these theories are important to understand our universe!

⇒ Nonperturbative formulations (e.g. "lattice formulations") are desired.

◇ Difficulty for realization of SUSY on lattice

In general, $(\text{SUSY})^2 \sim$ (infinitesimal translation)

↑

Not a symmetry of lattice

◇ A part of supercharges can be preserved on the lattice: (We focus on it.)

[$\mathcal{N}=1$ SUSY theory] $\Rightarrow$ (dim. red.) $\Rightarrow$ [Nonchiral theory with extended SUSY]

$y_1, \cdots, y_m$

on $(x_0, \cdots, x_n, y_1, \cdots, y_m)$ on $(x_0, \cdots, x_n)$

Q_a: supercharges related to translations along $y_1, \cdots, y_m$

$\{Q_a, Q_b\} \sim$ (internal symmetry transf.)

$\Downarrow$

(Some of) Q_a could be preserved on the lattice.

$\nearrow$

(Not all internal symmetry can be preserved on the lattice.)

e.g.) 2D $\mathcal{N}=(2,2)$ SYM case (we will see):

[Two R-symmetries]	[on lattice]	["nilpotent" supercharges]
$U(1)_A$	O.K.	$Q \equiv -\frac{1}{\sqrt{2}}(Q_L + \bar{Q}_R)$ preserved on lattice
$U(1)_V$	$\times$	$Q' \equiv -\frac{1}{\sqrt{2}}(Q_L - \bar{Q}_R)$ broken on lattice

◇ **Q_a form a "nilpotent" SUSY algebra.**

(⇔ **scalar supercharges from topological twist**)

- **2D Wess-Zumino model** [Sakai-Sakamoto, Kikukawa-Nakayama, Catterall]
- **pure SYM models** [Kaplan et al, Ishii et al] ← **deconstruction (via orbifolding)**,
  [F.S., Catterall] ← **TFT approach**
- **SYM + matter fields** [Endre-Kaplan, Matsuura] ← **deconstruction via orbifolding**,
  This Talk ← **TFT approach**

Here, we construct lattice models for
2D $\mathcal{N} = (2,2)$ SQCD
(SYM + n_+ fundamental and n_- anti-fundamental matter multiplets)
with $G = \mathrm{U}(N)$ (or $\mathrm{SU}(N)$)
2D regular lattice (with the spacing a)
compact gauge fields $U_\mu = e^{iA_\mu}$
general matter superpotentials and general twisted mass terms,
keeping one of the supercharges Q.

◇ **Our models are closest to the conventional lattice gauge model compared to the other approaches.**
(most practical for numerical simulation)

◇ Plan of Talk

§ 1: Introduction

§ 2: Continuum 2D $\mathcal{N} = (2, 2)$ SQCD

§ 3: The SYM part on lattice

§ 4: Lattice Formulation of SQCD (1) ← **"naive construction"**

§ 5: Lattice Formulation of SQCD (2) ← **Ginsparg-Wilson formulation**

§ 6: Lattice Formulation of Gauged Linear Sigma Models

§ 7: Summary and Discussion

Appendix A: Gauged Linear Sigma Models ⇒ Grassmannian

Appendix B: Admissibility Conditions

2 Continuum 2D $\mathcal{N} = (2,2)$ SQCD

The continuum Lagrangian $\Leftarrow$ dimensional reduction from 4D $\mathcal{N} = 1$ SQCD:

$$
\begin{aligned}
\mathcal{L} &= \mathcal{L}_{\rm SYM} + \mathcal{L}_{\rm mat} + \mathcal{L}_{\rm pot} + \mathcal{L}_{{\rm FI},\vartheta}, \\
\mathcal{L}_{\rm SYM} &= \frac{1}{8g^2} {\rm tr}\left(W^\alpha W_\alpha|_{\theta\theta} + \bar{W}_{\dot\alpha}\bar{W}^{\dot\alpha}\Big|_{\bar\theta\bar\theta}\right), \\
\mathcal{L}_{\rm mat} &= \left[\sum_{I=1}^{n_+} \Phi^\dagger_{+I} e^{V - \widetilde{V}_{+I}} \Phi_{+I} + \sum_{I=1}^{n_-} \Phi_{-I} e^{-V + \widetilde{V}_{-I}} \Phi^\dagger_{-I} \right]\Bigg|_{\theta\theta\bar\theta\bar\theta} \\
\mathcal{L}_{\rm pot} &= W(\Phi_+, \Phi_-)|_{\theta\theta} + \overline{W}(\Phi^\dagger_+, \Phi^\dagger_-)\Big|_{\bar\theta\bar\theta} \\
\mathcal{L}_{{\rm FI},\vartheta} &= {\rm tr}\left(-\kappa D + \frac{\vartheta}{2\pi} F_{01} \right),
\end{aligned}
$$

where $\widetilde{V}_{\pm I} \equiv 2\theta_R \bar\theta_L \widetilde{m}_{\pm I} + 2\theta_L \bar\theta_R \widetilde{m}^*_{\pm I}$: twisted masses.

- $V = (A_\mu, \phi, \bar\phi; \lambda; D) \Leftarrow$ 4D $\mathcal{N} = 1$ vector superfield
- $\Phi_{+I} = (\phi_{+I}; \psi_{+IR}, \psi_{+IL}; F_{+I}) \Leftarrow$ 4D $\mathcal{N} = 1$ chiral superfield
  (fundamental repre., flavors: $I = 1, \cdots, n_+$)
- $\Phi_{-I} = (\phi_{-I}; \psi_{-IR}, \psi_{-IL}; F_{-I}) \Leftarrow$ 4D $\mathcal{N} = 1$ chiral superfield
  (anti-fundamental repre., flavors: $I = 1, \cdots, n_-$)

Note

Two kinds of fermion mass terms can be introduced.

- **Complex mass terms** $(\subset W, \bar{W})$:
  $m_I \left(\psi_{-IL}\psi_{+IR} - \psi_{-IR}\psi_{+IL}\right) + m_I^* \left(\bar{\psi}_{+IR}\bar{\psi}_{-IL} - \bar{\psi}_{+IL}\bar{\psi}_{-IR}\right)$
- **Twisted mass terms** $(\not\subset W, \bar{W})$:
  $\widetilde{m}_{+I}\bar{\psi}_{+IL}\psi_{+IR} + \widetilde{m}^*_{+I}\bar{\psi}_{+IR}\psi_{+IL} + \widetilde{m}_{-I}\psi_{-IR}\bar{\psi}_{-IL} + \widetilde{m}^*_{-I}\psi_{-IL}\bar{\psi}_{-IR}$

◇ **Flavor symmetry of $\mathcal{L}_{\rm mat}$:**

$$\mathrm{U}(n_+) \times \mathrm{U}(n_-) \ \text{ for } \ \widetilde{m}_{\pm 1} = \cdots = \widetilde{m}_{\pm n_\pm}, \widetilde{m}^*_{\pm 1} = \cdots = \widetilde{m}^*_{\pm n_\pm}$$

$$\updownarrow$$

$$\mathrm{U}(1)^{n_+} \times \mathrm{U}(1)^{n_-} \ \text{ for } \ \text{general } \widetilde{m}_{\pm I}, \widetilde{m}^*_{\pm I}$$

3 Lattice Formulation of the SYM Part

4D $\mathcal{N}=1$ SYM	$\Rightarrow$ (dim. red.) $\Rightarrow$	2D $\mathcal{N}=(2,2)$ SYM
$A_\mu \quad (\mu=0,1)$	(x_2, x_3)	$A_\mu \Rightarrow U_\mu(x)$ (link variables on the lattice)
A_2, A_3		$\phi(x), \bar{\phi}(x)$ (site variables)
Rotational symmetry on (x_2, x_3)		$\mathrm{U}(1)_A$ R-symmetry

Fermions : 4-component Majorana spinor

$\Psi(x) = \left(\psi_0(x), \psi_1(x), \chi(x), \frac{1}{2}\eta(x)\right)^T$ (site variables)

3 Lattice formulation of the SYM Part

4D $\mathcal{N}=1$ SYM	$\Rightarrow$ (dim. red.) $\Rightarrow$	2D $\mathcal{N}=(2,2)$ SYM
$A_\mu \quad (\mu=0,1)$	(x_2, x_3)	$A_\mu \Rightarrow U_\mu(x)$ (link variables on the lattice)
A_2, A_3		$\phi(x), \bar{\phi}(x)$ (site variables)
Rotational symmetry on (x_2, x_3)		$\mathrm{U}(1)_A$ R-symmetry

Fermions : 4-component Majorana spinor
$\Psi(x) = \left(\psi_0(x), \psi_1(x), \chi(x), \frac{1}{2}\eta(x)\right)^T$ (site variables)

Exact Q-SUSY on the lattice
For admissible gauge fields ($||1-U_{01}(x)||<\epsilon$)

$$QU_\mu(x) = i\psi_\mu(x)U_\mu(x)$$
$$Q\psi_\mu(x) = i\psi_\mu(x)\psi_\mu(x) + ia\nabla_\mu\phi(x)$$
$$Q\phi(x) = 0$$
$$Q\bar{\phi}(x) = \eta(x), \quad Q\eta(x) = [\phi(x), \bar{\phi}(x)]$$
$$Q\chi(x) = iD(x) + \frac{i}{2}\widehat{\Phi}(x), \quad QD(x) = -\frac{1}{2}Q\widehat{\Phi}(x) - i[\phi(x), \chi(x)],$$

where $a\nabla_\mu\phi(x) \equiv U_\mu(x)\phi(x+\hat{\mu})U_\mu(x)^{-1} - \phi(x)$,

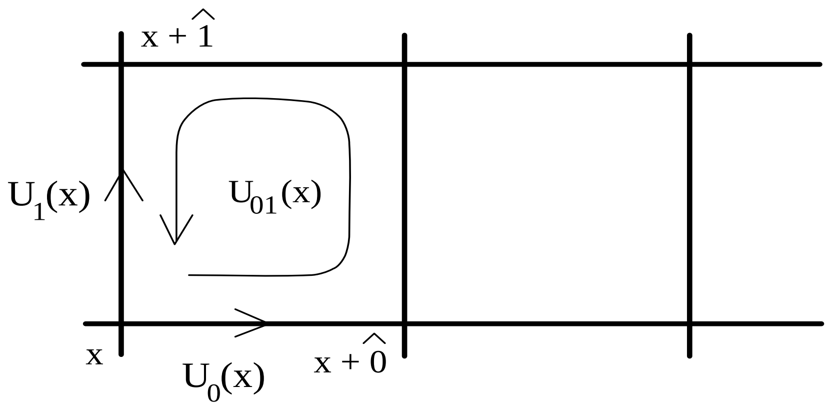

Figure 1: Link variables $U_\mu(x)$ and plaquette field $U_{01}(x)$. $U_{10}(x) = U_{01}(x)^{-1}$.

$$\widehat{\Phi}(x) = \frac{-i(U_{01}(x) - U_{10}(x))}{1 - \frac{1}{\epsilon^2}||1 - U_{01}(x)||^2} \sim 2F_{01}$$

$\Rightarrow Q^2 =$ (infinitesimal gauge tr. with the parameter $\phi(x)$)

Lattice Action: Q-exact form $\Rightarrow$ Exact Q-SUSY $QS_{\rm SYM}^{(\rm lat)} = 0$

For admissible gauge fields ($||1 - U_{01}(x)|| < \epsilon$ for $\forall x$),

$$S_{\rm SYM}^{(\rm lat)} = Q\frac{1}{g_0^2}\sum_x \mathrm{tr}\left[\chi(x)\left\{-\frac{i}{2}\widehat{\Phi}(x) + iD(x)\right\} + \frac{1}{4}\eta(x)[\phi(x),\bar{\phi}(x)] - i\sum_\mu \psi_\mu(x) a\nabla_\mu\bar{\phi}(x)\right]$$

$$\begin{aligned} = \frac{1}{g_0^2}\sum_x \mathrm{tr}\Bigg[&\frac{1}{4}\widehat{\Phi}(x)^2 + a^2\sum_\mu \nabla_\mu\phi(x)\nabla_\mu\bar{\phi}(x) + i\chi(x)Q\widehat{\Phi}(x) + i\sum_\mu \psi_\mu(x)a\nabla_\mu\eta(x) \\ &+\frac{1}{4}[\phi(x),\bar{\phi}(x)]^2 - \chi(x)[\phi(x),\chi(x)] - \frac{1}{4}\eta(x)[\phi(x),\eta(x)] \\ &-\sum_\mu \psi_\mu(x)\psi_\mu(x)\left(\bar{\phi}(x) + U_\mu(x)\bar{\phi}(x+\hat{\mu})U_\mu(x)^{-1}\right) - D(x)^2\Bigg], \end{aligned}$$

For the other cases, $S_{\rm SYM}^{(\rm lat)} = +\infty$. (i.e. The Boltzmann weight is zero.)

Note

Without the admissibility and the denominator of $\widehat{\Phi}$

$\Rightarrow$ gauge kinetic terms

$$\sim -\mathrm{tr}\,(U_{01}(x) - U_{10}(x))^2 = \mathrm{tr}\,(2 - U_{01}(x)^2 - U_{10}(x)^2)$$

$\Rightarrow$ The configurations

$$U_{01}(x) = \begin{pmatrix} \pm 1 & & \\ & \ddots & \\ & & \pm 1 \end{pmatrix} \qquad \text{(up to gauge tr.)}$$

for $\forall x$ give the classical minima of the action.

Huge degeneracy! ($\sharp$ of minima) $\sim \mathcal{O}$ ($\sharp$ of plaquettes)

$\Downarrow$

We should take into account fluctuations around all the minima.

$\Downarrow$

The connection of the lattice model to the continuum theory becomes unclear.

$\diamondsuit$ To avoid such situation, we employ the admissibilty and $\widehat{\Phi}$ to smoothly single out the vacuum $U_{01}(x) = 1$. Note: Q-SUSY is kept preserved.

c.f.) The Wilson lattice gauge action: $\mathrm{tr}\,(2 - U_{01}(x) - U_{10}(x))$

$\Rightarrow$ The unique minimum $U_{01}(x) = 1$.

$\diamondsuit$ The lattice action clearly becomes the continuum action in the naive continuum limit.

How about in the quantum sense?

Dimensional analysis $\Rightarrow$ $\mathbb{1}, \varphi, \varphi^2$ are relevant or marginal.

Fermion masses ψ^2 are irrelevant. (mass dimension 3)

The Q-SUSY forbids the mass term $\phi\bar{\phi}$ appearing as radiative corrections in the lattice perturbation.
$\mathrm{U}(1)_A$ symmetry forbits $\phi, \bar{\phi}$.

$\Downarrow$

The continuum theory is expected to be constructed without any fine-tuning.
(Computer simulations will give the nonperturbative check [Kanamori-Suzuki].
$\Rightarrow$ Care of the flat directions!)

4 Lattice Formulation of SQCD (1)

◇ Forward (backward) covariant differences $D_\mu (D_\mu^*)$:

$$
\begin{aligned}
aD_\mu \Phi_{+I}(x) &\equiv U_\mu(x)\Phi_{+I}(x+\hat{\mu}) - \Phi_{+I}(x) \\
aD_\mu^* \Phi_{+I}(x) &\equiv \Phi_{+I}(x) - U_\mu(x-\hat{\mu})^{-1}\Phi_{+I}(x-\hat{\mu}) \\
aD_\mu \Phi_{-I}(x) &\equiv \Phi_{-I}(x+\hat{\mu})U_\mu(x)^{-1} - \Phi_{-I}(x) \\
aD_\mu^* \Phi_{-I}(x) &\equiv \Phi_{-I}(x) - \Phi_{-I}(x-\hat{\mu})U_\mu(x-\hat{\mu}) \\
&\vdots
\end{aligned}
$$

and

$$
D_\mu^S \equiv \frac{1}{2}\left(D_\mu + D_\mu^*\right), \qquad D_\mu^A \equiv \frac{1}{2}\left(D_\mu - D_\mu^*\right), \qquad D^A \equiv \sum_\mu D_\mu^A.
$$

Q-SUSY on the lattice [Consider the case $n_+ = n_- \equiv n$]

$$Q\phi_{+I}(x) = -\psi_{+IL}(x), \quad Q\psi_{+IL}(x) = -(\phi(x) - \widetilde{m}_{+I})\phi_{+I}(x),$$

$$Q\psi_{+IR}(x) = a\left(D_0^S + iD_1^S\right)\phi_{+I}(x) + F_{+I}(x) - raD^A\phi_{-I}(x)^\dagger, \quad \longleftarrow \text{Wilson term}$$

$$\begin{aligned} QF_{+I}(x) = (\phi(x) - \widetilde{m}_{+I})\psi_{+IR}(x) + a\left(D_0^S + iD_1^S\right)\psi_{+IL}(x) - raD^A\bar{\psi}_{-IR}(x) \\ -a\left(Q(D_0^S + iD_1^S)\right)\phi_{+I}(x) + ra\left(QD^A\right)\phi_{-I}(x)^\dagger, \end{aligned}$$

$$Q\phi_{-I}(x)^\dagger = -\bar{\psi}_{-IR}(x), \quad Q\bar{\psi}_{-IR}(x) = -(\phi(x) - \widetilde{m}_{-I})\phi_{-I}(x)^\dagger,$$

$$\begin{aligned} Q\bar{\psi}_{-IL}(x) &= a\left(D_0^S - iD_1^S\right)\phi_{-I}(x)^\dagger + F_{-I}(x)^\dagger - raD^A\phi_{+I}(x), \\ QF_{-I}(x)^\dagger &= (\phi(x) - \widetilde{m}_{-I})\bar{\psi}_{-IL}(x) + a\left(D_0^S - iD_1^S\right)\bar{\psi}_{-IR}(x) - raD^A\psi_{+IL}(x) \\ &\quad -a\left(Q(D_0^S - iD_1^S)\right)\phi_{-I}(x)^\dagger + ra\left(QD^A\right)\phi_{+I}(x), \\ &\vdots \end{aligned}$$

$\Rightarrow$ Q is "nilpotent" for variables besides $F_{\pm I}$.

However, we have, for example,

$$Q^2F_{+I}(x) = (\phi(x) - \widetilde{m}_{+I})F_{+I}(x) + (\widetilde{m}_{+I} - \widetilde{m}_{-I})raD^A\phi_{-I}(x)^\dagger.$$

$\Rightarrow$ **When $\widetilde{m}_{+I} = \widetilde{m}_{-I} (\equiv \widetilde{m}_I)$, Q is "nilpotent" for all variables, i.e.**

$$Q^2 = \text{(infinitesimal gauge tr. with the parameter } \phi(x)\text{)} + \text{(infinitesimal U(1)}^n \text{ flavor rotation with the parameter } \widetilde{m}_I\text{)}.$$

$\uparrow$

$$\delta\Phi_{\pm I} = \mp \widetilde{m}_I \Phi_{\pm I}, \quad \delta\Phi^{\dagger}_{\pm I} = \pm \widetilde{m}_I \Phi^{\dagger}_{\pm I}$$

$\diamondsuit$ **Without the Wilson terms (set $r = 0$),**
doubler modes would appear both in bosons and fermions.

($\nwarrow$ consistent to the Q-SUSY)

The Wilson terms suppress the bosonic and fermionic doublers.

Lattice Action: Q-exact form $\qquad S^{(\text{lat})}_{\text{mat}} = S^{(\text{lat})}_{\text{mat},+} + S^{(\text{lat})}_{\text{mat},-}$

$$
\begin{aligned}
S^{(\text{lat})}_{\text{mat},+} = Q \sum_x \sum_{I=1}^{n} \Big[& \frac{1}{2}\bar{\psi}_{+IL}(x)\left\{a\left(D_0^S + iD_1^S\right)\phi_{+I}(x) - F_{+I}(x) - raD^A\phi_{-I}(x)^\dagger\right\} \\
& + \frac{1}{2}\left\{a\left(D_0^S - iD_1^S\right)\phi_{+I}(x)^\dagger - F_{+I}(x)^\dagger - raD^A\phi_{-I}(x)\right\}\psi_{+IR}(x) \\
& + \frac{1}{2}\bar{\psi}_{+IR}(x)(\bar{\phi}(x) - \widetilde{m}^*_{+I})\phi_{+I}(x) \\
& - \frac{1}{2}\phi_{+I}(x)^\dagger(\bar{\phi}(x) - \widetilde{m}^*_{+I})\psi_{+IL}(x) \\
& + i\phi_{+I}(x)^\dagger\chi(x)\phi_{+I}(x)\Big],
\end{aligned}
$$

$$
\begin{aligned}
S^{(\text{lat})}_{\text{mat},-} = Q \sum_x \sum_{I=1}^{n} \Big[& \frac{1}{2}\left\{a\left(D_0^S + iD_1^S\right)\phi_{-I}(x) - F_{-I}(x) - raD^A\phi_{+I}(x)^\dagger\right\}\bar{\psi}_{-IL}(x) \\
& + \frac{1}{2}\psi_{-IR}(x)\left\{a\left(D_0^S - iD_1^S\right)\phi_{-I}(x)^\dagger - F_{-I}(x)^\dagger - raD^A\phi_{+I}(x)\right\} \\
& + \frac{1}{2}\psi_{-IL}(x)(\bar{\phi}(x) - \widetilde{m}^*_{-I})\phi_{-I}(x)^\dagger \\
& - \frac{1}{2}\phi_{-I}(x)(\bar{\phi}(x) - \widetilde{m}^*_{-I})\bar{\psi}_{-IR}(x) \\
& - i\phi_{-I}(x)\chi(x)\phi_{-I}(x)^\dagger\Big],
\end{aligned}
$$

◇ **Superpotential terms are also Q-exact: (i: gauge group index)**

$$S_{\text{pot}}^{(\text{lat})} = Q\sum_x \sum_I \sum_{i=1}^{N} \left[-\frac{\partial W}{\partial \phi_{+Ii}(x)}\psi_{+IRi}(x) - \frac{\partial W}{\partial \phi_{-Ii}(x)}\bar{\psi}_{-IRi}(x) \right.$$
$$\left. -\bar{\psi}_{+ILi}(x)\frac{\partial \bar{W}}{\partial \phi^*_{+Ii}(x)} - \psi_{-ILi}(x)\frac{\partial \bar{W}}{\partial \phi^*_{-Ii}(x)} \right]$$

Note

Due to the Wilson terms,

- **the flavor symmetry of $S_{\text{mat}}^{\text{LAT}}$ is down to $\mathrm{U}(1)^n$ (diagonal subgroup of $\mathrm{U}(1)^n \times \mathrm{U}(1)^n$).**
- **the superpotential terms are not exactly holomorphic or anti-holomorphic on the lattice.**

⇒ The lattice action is Q-SUSY invariant when $\widetilde{m}_{+I} = \widetilde{m}_{-I} (\equiv \widetilde{m}_I)$.
(We can still choose $\widetilde{m}^*_{+I}, \widetilde{m}^*_{-I}$ freely!)

4.1 $\mathrm{U}(1)_A$ Anomaly

◇ $\mathrm{U}(1)_A$-symmetry with the charges:

$$
\begin{array}{rcl}
 & & \text{[SYM]} \qquad \text{[Matter]} \\
+2 & : & \phi \\
+1 & : & \psi_\mu, \quad \psi_{\pm IL}, \quad \bar{\psi}_{\pm IR} \\
-1 & : & \chi, \quad \eta, \quad \psi_{\pm IR}, \quad \bar{\psi}_{\pm IL} \\
-2 & : & \bar{\phi}, \\
0 & : & \text{the others}
\end{array}
$$

is realized in the lattice action, when all the twisted masses are zero.

In particular, the Wilson terms are consistent with the $\mathrm{U}(1)_A$-symmetry.

$\mathrm{U}(1)_A$ can be anomalous at the quantum level.

Note

- The gaugino fields (ψ_μ, χ, η) in the adjoint representation
  $\Rightarrow$ No contribution to the anomaly
- The present lattice theory has no source for the anomaly.
  In fact, $\mathrm{U}(1)_A$ is not anomalous when $n_+ = n_-$. O.K.

◇ **U(1)$_A$-WT identity:**

$$\partial^*_\mu \left\langle j_\mu^{U(1)_A}(x) \right\rangle = \left\langle \sum_{I=1}^{n} \left(\mathcal{M}_{+I}(x) + \mathcal{M}_{-I}(x)\right) \right\rangle,$$

with ∂^*_μ: **backward difference operators,**

$$\begin{aligned}
\mathcal{M}_{+I}(x) \;&=\; 2\widetilde{m}_I \left(\phi_{+I}(x)^\dagger \bar{\phi}(x)\phi_{+I}(x) + \bar{\psi}_{+IL}(x)\psi_{+IR}(x)\right) \\
&\quad -2\widetilde{m}^*_{+I} \left(\phi_{+I}(x)^\dagger \phi(x)\phi_{+I}(x) + \bar{\psi}_{+IR}(x)\psi_{+IL}(x)\right) \\
\mathcal{M}_{-I}(x) \;&=\; 2\widetilde{m}_I \left(\phi_{-I}(x)\bar{\phi}(x)\phi_{-I}(x)^\dagger + \psi_{-IR}(x)\bar{\psi}_{-IL}(x)\right) \\
&\quad -2\widetilde{m}^*_{-I} \left(\phi_{-I}(x)\phi(x)\phi_{-I}(x)^\dagger + \psi_{-IL}(x)\bar{\psi}_{-IR}(x)\right).
\end{aligned}$$

◇ **U(1)$_A$-WT identity:**

$$\partial_\mu^* \left\langle j_\mu^{U(1)_A}(x) \right\rangle = \left\langle \sum_{I=1}^{n} \left(\mathcal{M}_{+I}(x) + \mathcal{M}_{-I}(x)\right) \right\rangle ,$$

with ∂_μ^*: backward difference operators,

$$\begin{aligned}
\mathcal{M}_{+I}(x) &= 2\widetilde{m}_I \left(\phi_{+I}(x)^\dagger \bar{\phi}(x)\phi_{+I}(x) + \bar{\psi}_{+IL}(x)\psi_{+IR}(x)\right) \\
&\quad -2\widetilde{m}^*_{+I} \left(\phi_{+I}(x)^\dagger \phi(x)\phi_{+I}(x) + \bar{\psi}_{+IR}(x)\psi_{+IL}(x)\right) \\
\mathcal{M}_{-I}(x) &= 2\widetilde{m}_I \left(\phi_{-I}(x)\bar{\phi}(x)\phi_{-I}(x)^\dagger + \psi_{-IR}(x)\bar{\psi}_{-IL}(x)\right) \\
&\quad -2\widetilde{m}^*_{-I} \left(\phi_{-I}(x)\phi(x)\phi_{-I}(x)^\dagger + \psi_{-IL}(x)\bar{\psi}_{-IR}(x)\right).
\end{aligned}$$

Let us investigate the general case of $n_+ \neq n_-$, by sending
$\widetilde{m}^*_{+I} \to \infty$ $(I = n_+ + 1, \cdots, n)$, $\widetilde{m}^*_{-I'} \to \infty$ $(I' = n_- + 1, \cdots, n)$
before the continuum limit $(a \to 0)$.

$$\Uparrow$$

We expect
corresponding fields decoupled and $\Phi_{+1}, \cdots, \Phi_{+n_+}, \Phi_{-1}, \cdots, \Phi_{-n_-}$ remain.

$\diamondsuit$ Regarding $\mathrm{U}(1)_A$-anomaly, we can check that such decoupling is achieved in the lattice perturbation.

Anomaly term comes from matter-fermion one-loop diagrams of $\mathcal{M}_{+I}$ $(I > n_+)$ and $\mathcal{M}_{-I'}$ $(I' > n_-)$.

$\Downarrow$

The anomalous WT-identity for n_+ fundamentals and n_- anti-fundamentals

$$\partial_\mu^* \left\langle j_\mu^{U(1)_A}(x) \right\rangle = -\frac{1}{\pi}(n_+ - n_-)\mathrm{tr}\, F_{01}(x) + \left\langle \sum_{I=1}^{n_+} \mathcal{M}_{+I}(x) + \sum_{I=1}^{n_-} \mathcal{M}_{-I}(x) \right\rangle.$$

(The SYM fields are assumed to be smooth.)

Note

- The decoupling is not completely trivial, because the holomorphic parts $\widetilde{m}_I$ are kept finite.
- The Q-supersymmetry plays an important role to achieve the decoupling. (tr ϕ terms, seeming to be left finite, cancel between the bosonic and fermionic sectors.)

5 Lattice Formulation of SQCD (2)

◇ Here, we introduce the overlap Dirac operator to construct the lattice action for general $n_\pm$ and general twisted masses.

5.1 Doublet Notation

Prepare $n_0 (\equiv \max\{n_+, n_-\})$ fundamentals and anti-fundamentals.
Combine them as doublets:

$$\Phi_I \equiv \begin{pmatrix} \phi_{+I} \\ \phi^\dagger_{-I} \end{pmatrix}, \qquad \Phi^\dagger_I \equiv \left(\phi^\dagger_{+I}, \phi_{-I}\right),$$
$$\Psi_{uI} \equiv \begin{pmatrix} \psi_{+IL} \\ \bar{\psi}_{-IR} \end{pmatrix}, \qquad \Psi_{dI} \equiv \begin{pmatrix} \bar{\psi}_{-IL} \\ \psi_{+IR} \end{pmatrix},$$
$$\Psi^\dagger_{uI} \equiv \left(\bar{\psi}_{+IL}, \psi_{-IR}\right), \qquad \Psi^\dagger_{dI} \equiv \left(\psi_{-IL}, \bar{\psi}_{+IR}\right),$$
$$F_I \equiv \begin{pmatrix} F_{+I} \\ F^\dagger_{-I} \end{pmatrix}, \qquad F^\dagger_I \equiv \left(F^\dagger_{+I}, F_{-I}\right) \qquad (I = 1, \cdots, n_0).$$

The upper and down components of each doublet have the same gauge transformation property.

◇ **Notations:**

$$\gamma_0 \equiv \sigma_1, \qquad \gamma_1 \equiv \sigma_2, \qquad \gamma_3 \equiv -i\gamma_0\gamma_1 = \sigma_3,$$
$$\bar{\Psi}_{uI} \equiv \Psi^{\dagger}_{uI}\gamma_0, \qquad \bar{\Psi}_{dI} \equiv \Psi^{\dagger}_{dI}\gamma_0.$$

The fundamental or anti-fundamental degrees of freedom are extracted by acting the chiral projectors $P_{\pm} = \frac{1}{2}(1 \pm \gamma_3)$ to the doublets.

⇒ **Q-SUSY in the continuum**

$$\begin{aligned}
&Q\Phi_I = -\Psi_{uI}, \qquad Q\Psi_{uI} = -\left(\phi - \widetilde{m}_{+I}P_+ - \widetilde{m}_{-I}P_-\right)\Phi_I, \\
&Q\Psi_{dI} = \not{D}\Phi_I + \gamma_0 F_I, \\
&Q(\gamma_0 F_I) = \left(\phi - \widetilde{m}_{+I}P_- - \widetilde{m}_{-I}P_+\right)\Psi_{dI} + \not{D}\Psi_{uI} - i\gamma_\mu\psi_\mu\Phi_I, \\
&Q\Phi^{\dagger}_I = -\bar{\Psi}_{dI}, \qquad Q\bar{\Psi}_{dI} = \Phi^{\dagger}_I\left(\phi - \widetilde{m}_{+I}P_+ - \widetilde{m}_{-I}P_-\right), \\
&Q\bar{\Psi}_{uI} = \Phi^{\dagger}_I\not{D}^{\dagger} + F^{\dagger}_I\gamma_0, \\
&Q(F^{\dagger}_I\gamma_0) = -\bar{\Psi}_{uI}\left(\phi - \widetilde{m}_{+I}P_- - \widetilde{m}_{-I}P_+\right) + \bar{\Psi}_{dI}\not{D}^{\dagger} + i\Phi^{\dagger}_I\gamma_\mu\psi_\mu.
\end{aligned} \tag{5.1}$$

Note

For each I, (5.1) splits into four irreducible parts consisting of

$$\{P_+\Phi_I, P_+\Psi_{uI}, P_-\Psi_{dI}, P_+F_I\}, \quad \{\Phi_I^\dagger P_+, \bar{\Psi}_{dI}P_+, \bar{\Psi}_{uI}P_-, F_I^\dagger P_+\},$$
$$\{P_-\Phi_I, P_-\Psi_{uI}, P_+\Psi_{dI}, P_-F_I\}, \quad \{\Phi_I^\dagger P_-, \bar{\Psi}_{dI}P_-, \bar{\Psi}_{uI}P_+, F_I^\dagger P_-\}.$$

⇒ Chiral decomposition OK.

◇ The latticization in the previous section corresponds to

$$\not{D} \to D_W \equiv \textstyle\sum_{\mu=0}^{1} \gamma_\mu D_\mu^S - rD^A.$$

⇒ Due to the Wilson terms, the chiral decomposition is not possible on the lattice.

◇ **The previous lattice action is rewritten in the doublet notation as**

$$
\begin{aligned}
S_{\rm mat}^{\rm (lat)} \; = \; Q\sum_x \sum_{I=1}^{n} \frac{1}{2}\Big[& \bar{\Psi}_{uI}(x)\left(aD_W\Phi_I(x)-\gamma_0 F_I(x)\right) \\
& +\left(\Phi_I(x)^\dagger aD_W^\dagger - F_I(x)^\dagger\gamma_0\right)\Psi_{dI}(x) \\
& -\Phi_I(x)^\dagger\left(\bar{\phi}(x)-\widetilde{m}^*_{+I}P_+ - \widetilde{m}^*_{-I}P_-\right)\Psi_{uI}(x) \\
& +\bar{\Psi}_{dI}(x)\left(\bar{\phi}(x)-\widetilde{m}^*_{+I}P_+ - \widetilde{m}^*_{-I}P_-\right)\Phi_I(x) \\
& +2i\Phi_I(x)^\dagger\gamma_3\chi(x)\Phi_I(x)\Big]. \qquad (5.2)
\end{aligned}
$$

In order to resolve the difficulty, we introduce the overlap Dirac operator.

5.2 The Overlap Dirac Operator

The overlap Dirac operator $\widehat{D}$ satisfies the Ginsparg-Wilson relation

$$\gamma_3 \widehat{D} + \widehat{D} \gamma_3 = a \widehat{D} \gamma_3 \widehat{D}.$$

$\widehat{D}$ has been explicitly given by [Neuberger]

$$\widehat{D} \equiv \frac{1}{a}\left(1 - X \frac{1}{\sqrt{X^\dagger X}}\right), \qquad X = 1 - aD_W.$$

In order for $\widehat{D}$ to express the propagation of physical modes with doublers decoupled, we have to take $r > \frac{1}{2}$. [Kikukawa-Yamada, Suzuki].
In what follows, r is fixed to $r = 1$.

Note

The requirement $||X^\dagger X|| > 0$

$\Downarrow$

the admissibility condition with $0 < \epsilon < \frac{1}{5}$ ← 2D case of [Hernandez-Jansen-Lüscher]

◇ **In the kinetic part of the action (5.2), $D_W \to \widehat{D}$:**

$$\bar{\Psi}_{uI}(x) a\widehat{D}\Phi_I(x) + \Phi_I(x)^\dagger a\widehat{D}^\dagger \Psi_{dI}(x),$$

there are two possibilities of the chiral decomposition:

$$\bar{\Psi}_{uI}(x) P_\pm a\widehat{D}\Phi_I(x) + \Phi_I(x)^\dagger a\widehat{D}^\dagger P_\pm \Psi_{dI}(x) \;\Rightarrow\; \text{Formulation I},$$
$$\bar{\Psi}_{uI}(x) a\widehat{D} P_\pm \Phi_I(x) + \Phi_I(x)^\dagger P_\pm a\widehat{D}^\dagger \Psi_{dI}(x) \;\Rightarrow\; \text{Formulation II}.$$

Formulation I

$$\widehat{P}_\pm \equiv \frac{1 \pm \widehat{\gamma}_3}{2}, \qquad \widehat{\gamma}_3 \equiv \gamma_3(1 - a\widehat{D})$$

are projection operators ($\widehat{P}_\pm^2 = \widehat{P}_\pm$).

$$P_\pm \widehat{D} = \widehat{D}\widehat{P}_\mp, \qquad \widehat{D}^\dagger P_\pm = \widehat{P}_\mp \widehat{D}^\dagger, \qquad \widehat{P}_\pm^\dagger = \widehat{P}_\pm.$$

Formulation II

$$\bar{P}_\pm \equiv \frac{1 \pm \bar{\gamma}_3}{2}, \qquad \bar{\gamma}_3 \equiv (1 - a\widehat{D})\gamma_3$$

are projection operators ($\bar{P}_\pm^2 = \bar{P}_\pm$).

$$\bar{P}_\pm \widehat{D} = \widehat{D} P_\mp, \qquad \widehat{D}^\dagger \bar{P}_\pm = P_\mp \widehat{D}^\dagger, \qquad \bar{P}_\pm^\dagger = \bar{P}_\pm.$$

Fundamental matters ($I = 1, \cdots, n_+$):

$$\widehat{P}_+\Phi_I, \qquad \widehat{P}_+\Psi_{uI}, \qquad P_-\Psi_{dI}, \qquad P_+F_I \quad \text{as chiral fields,} \tag{5.3}$$

$$\Phi_I^\dagger\widehat{P}_+, \qquad \bar{\Psi}_{dI}\widehat{P}_+, \qquad \bar{\Psi}_{uI}P_-, \qquad F_I^\dagger P_+ \quad \text{as anti-chiral fields} \tag{5.4}$$

Anti-fundamental matters ($I' = 1, \cdots, n_-$):

$$\Phi_{I'}^\dagger\widehat{P}_-, \qquad \bar{\Psi}_{dI'}\widehat{P}_-, \qquad \bar{\Psi}_{uI'}P_+, \qquad F_{I'}^\dagger P_- \quad \text{as chiral fields,} \tag{5.5}$$

$$\widehat{P}_-\Phi_{I'}, \qquad \widehat{P}_-\Psi_{uI'}, \qquad P_+\Psi_{dI'}, \qquad P_-F_{I'} \quad \text{as anti-chiral fields} \tag{5.6}$$

Remark

If we use a naive transformation in the previous section,

$$\begin{aligned} Q(\widehat{P}_+\Phi_I(x)) &= \widehat{P}_+(Q\Phi_I(x)) + (Q\widehat{P}_+)\Phi_I(x) \\ &= -\widehat{P}_+\Psi_{uI}(x) + (Q\widehat{P}_+)\widehat{P}_+\Phi_I(x) + (Q\widehat{P}_+)\widehat{P}_-\Phi_I(x). \end{aligned}$$

$Q\widehat{P}_\pm$ generally do not vanish, since $\widehat{P}_\pm$ contain the link variables!
Due to the last term in the r.h.s., the transformation does not close among the chiral fields (5.3).

Instead,

we regard (5.3), (5.4), (5.5), (5.6) as fundamental contents of the theory, and define their Q-transformation by starting with

$$\begin{aligned} Q(\widehat{P}_+\Phi_I(x)) &= -\widehat{P}_+\Psi_{uI}(x) + (Q\widehat{P}_+)\widehat{P}_+\Phi_I(x), \\ Q(\Phi_I^\dagger\widehat{P}_+(x)) &= -\bar{\Psi}_{dI}\widehat{P}_+(x) + \Phi_I^\dagger\widehat{P}_+(Q\widehat{P}_+)(x), \\ Q(\widehat{P}_-\Phi_{I'}(x)) &= -\widehat{P}_-\Psi_{uI'}(x) + (Q\widehat{P}_-)\widehat{P}_-\Phi_{I'}(x), \\ Q(\Phi_{I'}^\dagger\widehat{P}_-(x)) &= -\bar{\Psi}_{dI'}\widehat{P}_-(x) + \Phi_{I'}^\dagger\widehat{P}_-(Q\widehat{P}_-)(x). \end{aligned}$$

$$\Downarrow$$

Q-SUSY transformation can be consistently determined as a closed form among the (anti-)chiral fields.

$$\begin{aligned} Q(\widehat{P}_+\Phi_I(x)) &= -\widehat{P}_+\Psi_{uI}(x) + (Q\widehat{P}_+)\widehat{P}_+\Phi_I(x), \\ Q(\widehat{P}_+\Psi_{uI}(x)) &= -(\widehat{P}_+\phi - \widetilde{m}_{+I})\widehat{P}_+\Phi_I(x) + (Q\widehat{P}_+)\widehat{P}_+\Psi_{uI}(x) - (Q\widehat{P}_+)^2\widehat{P}_+\Phi_I(x), \\ Q(P_-\Psi_{dI}(x)) &= a\widehat{D}\widehat{P}_+\Phi_I(x) + \gamma_0 P_+F_I(x), \\ Q(\gamma_0 P_+F_I(x)) &= (\phi(x) - \widetilde{m}_{+I})P_-\Psi_{dI}(x) + a\widehat{D}\widehat{P}_+\Psi_{uI}(x) - P_-Q(a\widehat{D})\widehat{P}_+\Phi_I(x), \\ &\vdots \end{aligned}$$

$\diamondsuit$ **Q is nilpotent in the sense of**

$$
\begin{aligned}
Q^2 = & \text{ (infinitesimal gauge transformation with the parameter } \phi(x)) \\
& +(\text{infinitesimal } \mathrm{U}(1)^{n_+} \times \mathrm{U}(1)^{n_-} \text{ flavor rotations (5.7) and (5.8))}
\end{aligned}
$$

with

$$
\begin{aligned}
\delta(\widehat{P}_+\Phi_I) &= -\widetilde{m}_{+I}\widehat{P}_+\Phi_I, & \delta(\Phi_I^\dagger\widehat{P}_+) &= \widetilde{m}_{+I}\Phi_I^\dagger\widehat{P}_+, \\
\delta(\widehat{P}_+\Psi_{uI}) &= -\widetilde{m}_{+I}\widehat{P}_+\Psi_{uI}, & \delta(\bar{\Psi}_{uI}P_-) &= \widetilde{m}_{+I}\bar{\Psi}_{uI}P_-, \\
\delta(P_-\Psi_{dI}) &= -\widetilde{m}_{+I}P_-\Psi_{dI}, & \delta(\bar{\Psi}_{dI}\widehat{P}_+) &= \widetilde{m}_{+I}\bar{\Psi}_{dI}\widehat{P}_+, \\
\delta(P_+F_I) &= -\widetilde{m}_{+I}P_+F_I, & \delta(F_I^\dagger P_+) &= \widetilde{m}_{+I}F_I^\dagger P_+,
\end{aligned}
\tag{5.7}
$$

$$
\begin{aligned}
\delta(\Phi_{I'}^\dagger\widehat{P}_-) &= \widetilde{m}_{-I'}\Phi_{I'}^\dagger\widehat{P}_-, & \delta(\widehat{P}_-\Phi_{I'}) &= -\widetilde{m}_{-I'}\widehat{P}_-\Phi_{I'}, \\
\delta(\bar{\Psi}_{uI'}P_+) &= \widetilde{m}_{-I'}\bar{\Psi}_{uI'}P_+, & \delta(\widehat{P}_-\Psi_{uI'}) &= -\widetilde{m}_{-I'}\widehat{P}_-\Psi_{uI'}, \\
\delta(\bar{\Psi}_{dI'}\widehat{P}_-) &= \widetilde{m}_{-I'}\bar{\Psi}_{dI'}\widehat{P}_-, & \delta(P_+\Psi_{dI'}) &= -\widetilde{m}_{-I'}P_+\Psi_{dI'}, \\
\delta(F_{I'}^\dagger P_-) &= \widetilde{m}_{-I'}F_{I'}^\dagger P_-, & \delta(P_-F_{I'}) &= -\widetilde{m}_{-I'}P_-F_{I'}.
\end{aligned}
\tag{5.8}
$$

OK for general $n_\pm$ and general twisted masses.

◇ **The matter-part action:**

$$
\begin{aligned}
S^{\rm LAT}_{{\rm mat},+} \;=\; & Q\sum_x \sum_{I=1}^{n_+} \frac{1}{2}\Big[\bar{\Psi}_{uI}(x)P_-\left(a\widehat{D}\widehat{P}_+\Phi_I(x)-\gamma_0 P_+F_I(x)\right)\\
& +\left(\Phi_I^\dagger\widehat{P}_+(x)\,a\widehat{D}^\dagger - F_I(x)^\dagger P_+\gamma_0\right)P_-\Psi_{dI}(x)\\
& -\Phi_I^\dagger\widehat{P}_+(x)\,\left(\bar{\phi}(x)-\widetilde{m}^*_{+I}\right)\widehat{P}_+\Psi_{uI}(x)\\
& +\bar{\Psi}_{dI}\widehat{P}_+(x)\,\left(\bar{\phi}(x)-\widetilde{m}^*_{+I}\right)\widehat{P}_+\Phi_I(x)\\
& +2i\Phi_I^\dagger\widehat{P}_+(x)\,\chi(x)\widehat{P}_+\Phi_I(x)\Big]\,,
\end{aligned}
\tag{5.9}
$$

$$
\begin{aligned}
S^{\rm LAT}_{{\rm mat},-} \;=\; & Q\sum_x \sum_{I'=1}^{n_-} \frac{1}{2}\Big[\bar{\Psi}_{uI'}(x)P_+\left(a\widehat{D}\widehat{P}_-\Phi_{I'}(x)-\gamma_0 P_-F_{I'}(x)\right)\\
& +\left(\Phi_{I'}^\dagger\widehat{P}_-(x)\,a\widehat{D}^\dagger - F_{I'}(x)^\dagger P_-\gamma_0\right)P_+\Psi_{dI'}(x)\\
& -\Phi_{I'}^\dagger\widehat{P}_-(x)\,\left(\bar{\phi}(x)-\widetilde{m}_{-I'}\right)\widehat{P}_-\Psi_{uI'}(x)\\
& +\bar{\Psi}_{dI'}\widehat{P}_-(x)\,\left(\bar{\phi}(x)-\widetilde{m}^*_{-I'}\right)\widehat{P}_-\Phi_{I'}(x)\\
& -2i\Phi_{I'}^\dagger\widehat{P}_-(x)\,\chi(x)\widehat{P}_-\Phi_{I'}(x)\Big]\,.
\end{aligned}
\tag{5.10}
$$

After the Q operation,

$$
\begin{aligned}
S^{\rm LAT}_{\rm mat,+} = \sum_x \sum_{I=1}^{n_+} \Big[& a^2 \Phi_I^\dagger \widehat{P}_+(x)\, \widehat{D}^\dagger \widehat{D} \widehat{P}_+ \Phi_I(x) - \left(F_I(x)^\dagger P_+\right) \left(P_+ F_I(x)\right) \\
& + \bar{\Psi}_{uI}(x) P_-\, a\widehat{D} \widehat{P}_+ \Psi_{uI}(x) - \bar{\Psi}_{dI} \widehat{P}_+(x)\, a\widehat{D}^\dagger P_- \Psi_{dI}(x) \\
& + \frac{1}{2} \Phi_I^\dagger \widehat{P}_+(x) \left\{ \phi \widehat{P}_+ - \widetilde{m}_{+I}, \bar{\phi} \widehat{P}_+ - \widetilde{m}^*_{+I} \right\} \widehat{P}_+ \Phi_I(x) \\
& - \Phi_I^\dagger \widehat{P}_+(x) \left(D(x) + \frac{1}{2} \widehat{\Phi}(x) \right) \widehat{P}_+ \Phi_I(x) \\
& + \bar{\Psi}_{uI}(x) P_- \left(\phi(x) - \widetilde{m}_{+I}\right) P_- \Psi_{dI}(x) + \bar{\Psi}_{dI} \widehat{P}_+(x) \left(\bar{\phi}(x) - \widetilde{m}^*_{+I} \right) \widehat{P}_+ \Psi_{uI}(x) \\
& - \bar{\Psi}_{uI}(x) P_-\, Q(a\widehat{D}) \widehat{P}_+ \Phi_I(x) + \Phi_I^\dagger \widehat{P}_+(x)\, Q(a\widehat{D}^\dagger) P_- \Psi_{dI}(x) \\
& - \bar{\Psi}_{dI} \widehat{P}_+(x) \left(\frac{1}{2} \eta(x) + i\chi(x) \right) \widehat{P}_+ \Phi_I(x) \\
& - \Phi_I^\dagger \widehat{P}_+(x) \left(\frac{1}{2} \eta(x) - i\chi(x) \right) \widehat{P}_+ \Psi_{uI}(x) \\
& - \frac{1}{2} \Phi_I^\dagger \widehat{P}_+(x) \left\{ (Q\widehat{P}_+), \bar{\phi} \right\} \widehat{P}_+ \Psi_{uI}(x) - \frac{1}{2} \bar{\Psi}_{dI} \widehat{P}_+(x) \left\{ (Q\widehat{P}_+), \bar{\phi} \right\} \widehat{P}_+ \Phi_I(x) \\
& + \frac{1}{2} \Phi_I^\dagger \widehat{P}_+(x) \left\{ (Q\widehat{P}_+)^2, \bar{\phi} \right\} \widehat{P}_+ \Phi_I(x) + i \Phi_I^\dagger \widehat{P}_+(x) \left[(Q\widehat{P}_+), \chi \right] \widehat{P}_+ \Phi_I(x) \Big],
\end{aligned}
$$

↑

lattice artifacts

$$
\begin{aligned}
S^{\mathrm{LAT}}_{\mathrm{mat},-} \;=\; \sum_x \sum_{I'=1}^{n_-} \Big[& a^2 \Phi^\dagger_{I'} \widehat{P}_-(x)\, \widehat{D}^\dagger \widehat{D} \widehat{P}_- \Phi_{I'}(x) - \left(F_{I'}(x)^\dagger P_-\right) \left(P_- F_{I'}(x)\right) \\
& + \bar{\Psi}_{uI'}(x) P_+\, a\widehat{D} \widehat{P}_- \Psi_{uI'}(x) - \bar{\Psi}_{dI'} \widehat{P}_-(x)\, a\widehat{D}^\dagger P_+ \Psi_{dI'}(x) \\
& + \frac{1}{2} \Phi^\dagger_{I'} \widehat{P}_-(x) \left\{ \phi \widehat{P}_- - \widetilde{m}_{-I'}, \bar{\phi} \widehat{P}_- - \widetilde{m}^*_{-I'} \right\} \widehat{P}_- \Phi_{I'}(x) \\
& + \Phi^\dagger_{I'} \widehat{P}_-(x) \left(D(x) + \frac{1}{2} \widehat{\Phi}(x) \right) \widehat{P}_- \Phi_{I'}(x) \\
& + \bar{\Psi}_{uI'}(x) P_+ \left(\phi(x) - \widetilde{m}_{-I'}\right) P_+ \Psi_{dI'}(x) + \bar{\Psi}_{dI'} \widehat{P}_-(x) \left(\bar{\phi}(x) - \widetilde{m}^*_{-I'} \right) \widehat{P}_- \Psi_{uI'}(x) \\
& - \bar{\Psi}_{uI'}(x) P_+\, Q(a\widehat{D}) \widehat{P}_- \Phi_{I'}(x) + \Phi^\dagger_{I'} \widehat{P}_-(x)\, Q(a\widehat{D}^\dagger) P_+ \Psi_{dI'}(x) \\
& - \bar{\Psi}_{dI'} \widehat{P}_-(x) \left(\frac{1}{2} \eta(x) - i\chi(x) \right) \widehat{P}_- \Phi_{I'}(x) \\
& - \Phi^\dagger_{I'} \widehat{P}_-(x) \left(\frac{1}{2} \eta(x) + i\chi(x) \right) \widehat{P}_- \Psi_{uI'}(x) \\
& - \frac{1}{2} \Phi^\dagger_{I'} \widehat{P}_-(x) \left\{ (Q\widehat{P}_-), \bar{\phi} \right\} \widehat{P}_- \Psi_{uI'}(x) - \frac{1}{2} \bar{\Psi}_{dI'} \widehat{P}_-(x) \left\{ (Q\widehat{P}_-), \bar{\phi} \right\} \widehat{P}_- \Phi_{I'}(x) \\
& + \frac{1}{2} \Phi^\dagger_{I'} \widehat{P}_-(x) \left\{ (Q\widehat{P}_-)^2, \bar{\phi} \right\} \widehat{P}_- \Phi_{I'}(x) - i \Phi^\dagger_{I'} \widehat{P}_-(x) \left[(Q\widehat{P}_-), \chi \right] \widehat{P}_- \Phi_{I'}(x) \Big].
\end{aligned}
$$

$\uparrow$

lattice artifacts

Fundamental matters ($I = 1, \cdots, n_+$):

$$P_+\Phi_I, \qquad P_+\Psi_{uI}, \qquad \bar{P}_-\Psi_{dI}, \qquad \bar{P}_-\gamma_0 F_I \quad \text{as chiral fields,} \tag{5.11}$$

$$\Phi_I^\dagger P_+, \qquad \bar{\Psi}_{dI}P_+, \qquad \bar{\Psi}_{uI}\bar{P}_-, \qquad F_I^\dagger\gamma_0\bar{P}_- \quad \text{as anti-chiral fields} \tag{5.12}$$

Anti-fundamental matters ($I' = 1, \cdots, n_-$):

$$\Phi_{I'}^\dagger P_-, \qquad \bar{\Psi}_{dI'}P_-, \qquad \bar{\Psi}_{uI'}\bar{P}_+, \qquad F_{I'}^\dagger\gamma_0\bar{P}_+ \quad \text{as chiral fields,} \tag{5.13}$$

$$P_-\Phi_{I'}, \qquad P_-\Psi_{uI'}, \qquad \bar{P}_+\Psi_{dI'}, \qquad \bar{P}_+\gamma_0 F_{I'} \quad \text{as anti-chiral fields} \tag{5.14}$$

Q-SUSY transformation:

$$\begin{aligned}
Q(P_+\Phi_I(x)) &= -P_+\Psi_{uI}(x), \\
Q(P_+\Psi_{uI}(x)) &= -(\phi(x) - \widetilde{m}_{+I})P_+\Phi_I(x), \\
Q(\bar{P}_-\Psi_{dI}(x)) &= a\widehat{D}P_+\Phi_I(x) + \bar{P}_-\gamma_0 F_I(x) + (Q\bar{P}_-)\bar{P}_-\Psi_{dI}(x), \\
Q(\bar{P}_-\gamma_0 F_I(x)) &= (\bar{P}_-\phi - \widetilde{m}_{+I})\bar{P}_-\Psi_{dI}(x) + a\widehat{D}P_+\Psi_{uI}(x) - \bar{P}_-Q(a\widehat{D})P_+\Phi_I(x) \\
&\quad +(Q\bar{P}_-)\bar{P}_-\gamma_0 F_I(x) + (Q\bar{P}_-)^2\bar{P}_-\Psi_{dI}(x) \\
&\quad \vdots
\end{aligned}$$

is nilpotent in the similar sense.

The matter-part action :

$$\begin{aligned}
S^{\rm LAT}_{\rm mat,+} \;=\; & Q\sum_x \sum_{I=1}^{n_+} \frac{1}{2}\Big[\bar{\Psi}_{uI}\bar{P}_-(x)\left(a\widehat{D}P_+\Phi_I(x) - \bar{P}_-\gamma_0 F_I(x)\right) \\
& + \left(\Phi_I(x)^\dagger P_+\, a\widehat{D}^\dagger - F_I^\dagger\gamma_0\bar{P}_-(x)\right)\bar{P}_-\Psi_{dI}(x) \\
& -\Phi_I(x)^\dagger P_+\,\left(\bar{\phi}(x) - \widetilde{m}^*_{+I}\right)P_+\Psi_{uI}(x) \\
& +\bar{\Psi}_{dI}(x)P_+\,\left(\bar{\phi}(x) - \widetilde{m}^*_{+I}\right)P_+\Phi_I(x) \\
& +2i\Phi_I(x)^\dagger P_+\,\chi(x)P_+\Phi_I(x)\Big],
\end{aligned}$$

$$\begin{aligned}
S^{\rm LAT}_{\rm mat,-} \;=\; & Q\sum_x \sum_{I'=1}^{n_-} \frac{1}{2}\Big[\bar{\Psi}_{uI'}\bar{P}_+(x)\left(a\widehat{D}P_-\Phi_{I'}(x) - \bar{P}_+\gamma_0 F_{I'}(x)\right) \\
& + \left(\Phi_{I'}(x)^\dagger P_-\, a\widehat{D}^\dagger - F_{I'}^\dagger\gamma_0\bar{P}_+(x)\right)\bar{P}_+\Psi_{dI'}(x) \\
& -\Phi_{I'}(x)^\dagger P_-\,\left(\bar{\phi}(x) - \widetilde{m}^*_{-I'}\right)P_-\Psi_{uI'}(x) \\
& +\bar{\Psi}_{dI'}(x)P_-\,\left(\bar{\phi}(x) - \widetilde{m}^*_{-I'}\right)P_-\Phi_{I'}(x) \\
& -2i\Phi_{I'}(x)^\dagger P_-\,\chi(x)P_-\Phi_{I'}(x)\Big].
\end{aligned}$$

$\downarrow$

Interaction terms without $\widehat{D}$-dependent projectors

$\Rightarrow$ **Simpler expressions than Formulation I**

After the Q operation,

$$
\begin{aligned}
S^{\mathrm{LAT}}_{\mathrm{mat},+} \;=\; \sum_x \sum_{I=1}^{n_+} \Big[& a^2 \Phi_I(x)^\dagger P_+ \widehat{D}^\dagger \widehat{D} P_+ \Phi_I(x) - \left(F_I^\dagger \gamma_0 \bar{P}_-(x)\right)\left(\bar{P}_- \gamma_0 F_I(x)\right) \\
& + \bar{\Psi}_{uI} \bar{P}_-(x)\, a\widehat{D} P_+ \Psi_{uI}(x) - \bar{\Psi}_{dI}(x) P_+\, a\widehat{D}^\dagger \bar{P}_- \Psi_{dI}(x) \\
& + \frac{1}{2} \Phi_I(x)^\dagger P_+ \left\{ \phi(x) - \widetilde{m}_{+I}, \bar{\phi}(x) - \widetilde{m}^*_{+I} \right\} P_+ \Phi_I(x) \\
& - \Phi_I(x)^\dagger P_+ \left(D(x) + \frac{1}{2} \widehat{\Phi}(x) \right) P_+ \Phi_I(x) \\
& + \bar{\Psi}_{uI} \bar{P}_-(x) \left(\phi(x) - \widetilde{m}_{+I}\right) \bar{P}_- \Psi_{dI}(x) + \bar{\Psi}_{dI}(x) P_+ \left(\bar{\phi}(x) - \widetilde{m}^*_{+I}\right) P_+ \Psi_{uI}(x) \\
& - \bar{\Psi}_{uI} \bar{P}_-(x)\, Q(a\widehat{D}) P_+ \Phi_I(x) + \Phi_I(x)^\dagger P_+\, Q(a\widehat{D}^\dagger) \bar{P}_- \Psi_{dI}(x) \\
& - \bar{\Psi}_{dI}(x) P_+ \left(\frac{1}{2} \eta(x) + i\chi(x) \right) P_+ \Phi_I(x) \\
& - \Phi_I(x)^\dagger P_+ \left(\frac{1}{2} \eta(x) - i\chi(x) \right) P_+ \Psi_{uI}(x) \\
& + \bar{\Psi}_{uI} \bar{P}_-(x)\, (Q\bar{P}_-)^2 \bar{P}_- \Psi_{dI}(x) \Big],
\end{aligned}
$$

↑

lattice artifact

$$
\begin{aligned}
S^{\mathrm{LAT}}_{\mathrm{mat},-} = \sum_x \sum_{I'=1}^{n_-} \Big[& a^2 \Phi_{I'}(x)^\dagger P_- \widehat{D}^\dagger \widehat{D} P_- \Phi_{I'}(x) - \left(F^\dagger_{I'} \gamma_0 \bar{P}_+(x)\right)\left(\bar{P}_+ \gamma_0 F_{I'}(x)\right) \\
& + \bar{\Psi}_{uI'} \bar{P}_+(x)\, a\widehat{D} P_- \Psi_{uI'}(x) - \bar{\Psi}_{dI'}(x) P_-\, a\widehat{D}^\dagger \bar{P}_+ \Psi_{dI'}(x) \\
& + \frac{1}{2} \Phi_{I'}(x)^\dagger P_- \left\{ \phi(x) - \widetilde{m}_{-I'}, \bar{\phi}(x) - \widetilde{m}^*_{-I'} \right\} P_- \Phi_{I'}(x) \\
& + \Phi_{I'}(x)^\dagger P_- \left(D(x) + \frac{1}{2} \widehat{\Phi}(x) \right) P_- \Phi_{I'}(x) \\
& + \bar{\Psi}_{uI'} \bar{P}_+(x) \left(\phi(x) - \widetilde{m}_{-I'}\right) \bar{P}_+ \Psi_{dI'}(x) + \bar{\Psi}_{dI'}(x) P_- \left(\bar{\phi}(x) - \widetilde{m}^*_{-I'}\right) P_- \Psi_{uI'}(x) \\
& - \bar{\Psi}_{uI'} \bar{P}_+(x)\, Q(a\widehat{D}) P_- \Phi_{I'}(x) + \Phi_{I'}(x)^\dagger P_-\, Q(a\widehat{D}^\dagger) \bar{P}_+ \Psi_{dI'}(x) \\
& - \bar{\Psi}_{dI'}(x) P_- \left(\frac{1}{2} \eta(x) - i\chi(x) \right) P_- \Phi_{I'}(x) \\
& - \Phi_{I'}(x)^\dagger P_- \left(\frac{1}{2} \eta(x) + i\chi(x) \right) P_- \Psi_{uI'}(x) \\
& + \bar{\Psi}_{uI'} \bar{P}_+(x) \left(Q \bar{P}_+\right)^2 \bar{P}_+ \Psi_{dI'}(x) \Big],
\end{aligned}
$$

↑

lattice artifact

Since Formulation II seems to give simpler expressions than Formulation I, we will mainly develop Formulation II.

Superpotentials

$$
\begin{aligned}
S_{\mathrm{pot}}^{\mathrm{LAT}} = {} & Q \sum_x \sum_{i=1}^{N} \sum_{I=1}^{n_+} \left[-\frac{\partial W}{\partial (P_+ \Phi_I(x))_i} \left(\gamma_0 \bar{P}_- \Psi_{dI}(x)\right)_i - \left(\bar{\Psi}_{uI} \bar{P}_-(x) \gamma_0\right)_i \frac{\partial \bar{W}}{\partial (\Phi_I(x)^\dagger P_+)_i} \right] \\
& + Q \sum_x \sum_{i=1}^{N} \sum_{I'=1}^{n_-} \left[-\frac{\partial \bar{W}}{\partial (P_- \Phi_{I'}(x))_i} \left(\gamma_0 \bar{P}_+ \Psi_{dI'}(x)\right)_i - \left(\bar{\Psi}_{uI'} \bar{P}_+(x) \gamma_0\right)_i \frac{\partial W}{\partial (\Phi_{I'}(x)^\dagger P_-)_i} \right]
\end{aligned}
$$

with

$$
W = W(P_+ \Phi_I, \Phi_{I'}^\dagger P_-), \qquad \bar{W} = \bar{W}(\Phi_I^\dagger P_+, P_- \Phi_{I'}).
$$

Note

$S_{\mathrm{pot}}^{\mathrm{LAT}}$ exactly realizes holomorphic or anti-holomorphic structure on the lattice, i.e.

- terms containing W depend only on the chiral fields (5.11) and (5.13),
- terms containing $\bar{W}$ depend only on the anti-chiral fields (5.12) and (5.14),

besides the SYM fields which come in via $\bar{P}_\pm$ or $Q\bar{P}_\pm$.

◇ Path-integral measure for the SYM part

$$(\mathrm{d}\mu_{2\mathrm{DSYM}}) \equiv \prod_x \left[\prod_{\mu=0}^{1} \mathrm{d}U_\mu(x) \right] \leftarrow \text{Haar measures of } G$$
$$\times \prod_A \mathrm{d}\psi_0^A(x)\,\mathrm{d}\psi_1^A(x)\,\mathrm{d}\chi^A(x)\,\mathrm{d}\eta^A(x)\,\mathrm{d}\phi^A(x)\,\mathrm{d}\bar{\phi}^A(x)\,\mathrm{d}D^A(x)$$

↑

A labels the generators of G.

◇ Path-integral measure for the matter part

$$(\mathrm{d}\mu_{\mathrm{mat}}) = \left(\prod_{I=1}^{n_+} \mathrm{d}\mu_{\mathrm{mat},+I} \right) \left(\prod_{I'=1}^{n_-} \mathrm{d}\mu_{\mathrm{mat},-I'} \right)$$

$$\mathrm{d}\mu_{\mathrm{mat},+I} \equiv \prod_x \prod_{i=1}^{N} \mathrm{d}(P_+\Phi_I(x))_i\,\mathrm{d}(\Phi_I(x)^\dagger P_+)_i\,\mathrm{d}(\bar{P}_-\gamma_0 F_I(x))_i\,\mathrm{d}(F_I^\dagger\gamma_0\bar{P}_-(x))_i$$
$$\times \mathrm{d}(P_+\Psi_{uI}(x))_i\,\mathrm{d}(\bar{\Psi}_{uI}\bar{P}_-(x))_i\,\mathrm{d}(\bar{P}_-\Psi_{dI}(x))_i\,\mathrm{d}(\bar{\Psi}_{dI}(x)P_+)_i,$$

$$\mathrm{d}\mu_{\mathrm{mat},-I'} \equiv \prod_x \prod_{i=1}^{N} \mathrm{d}\left(P_-\Phi_{I'}(x)\right)_i\,\mathrm{d}(\Phi_{I'}(x)^\dagger P_-)_i\,\mathrm{d}(\bar{P}_+\gamma_0 F_{I'}(x))_i\,\mathrm{d}(F_{I'}^\dagger\gamma_0\bar{P}_+(x))_i$$
$$\times \mathrm{d}(P_-\Psi_{uI'}(x))_i\,\mathrm{d}(\bar{\Psi}_{uI'}\bar{P}_+(x))_i\,\mathrm{d}(\bar{P}_+\Psi_{dI'}(x))_i\,\mathrm{d}(\bar{\Psi}_{dI'}(x)P_-)_i.$$

Let us see transformation properties of the matter-part measure.

Gauge Invariance

$g(x) = \mathrm{e}^{i\omega(x)} \in G$ ($\omega(x)$: infinitesimal) transformation for fundamentals:

$$
\begin{aligned}
P_+\Phi_I(x) &\longrightarrow g(x)P_+\Phi_I(x) = (1 + i\omega(x)P_+)P_+\Phi_I(x), \\
\Phi_I(x)^\dagger P_+ &\longrightarrow \Phi_I(x)^\dagger P_+ g(x)^{-1} = \Phi_I(x)^\dagger P_+(1 - iP_+\omega(x)), \\
\bar{P}_-\gamma_0 F_I(x) &\longrightarrow g(x)\bar{P}_-\gamma_0 F_I(x) = (1 + i\omega(x)\bar{P}_-)\bar{P}_-\gamma_0 F_I(x), \\
F_I^\dagger\gamma_0\bar{P}_-(x) &\longrightarrow F_I^\dagger\gamma_0\bar{P}_-(x)g(x)^{-1} = F_I^\dagger\gamma_0\bar{P}_-(1 - i\bar{P}_-\omega)(x),
\end{aligned}
$$

$$
\begin{aligned}
P_+\Psi_{uI}(x) &\longrightarrow g(x)P_+\Psi_{uI}(x) = (1 + i\omega(x)P_+)P_+\Psi_{uI}(x), \\
\bar{\Psi}_{uI}\bar{P}_-(x) &\longrightarrow \bar{\Psi}_{uI}\bar{P}_-(x)g(x)^{-1} = \bar{\Psi}_{uI}\bar{P}_-(1 - i\bar{P}_-\omega)(x), \\
\bar{P}_-\Psi_{dI}(x) &\longrightarrow g(x)\bar{P}_-\Psi_{dI}(x) = (1 + i\omega(x)\bar{P}_-)\bar{P}_-\Psi_{dI}(x), \\
\bar{\Psi}_{dI}(x)P_+ &\longrightarrow \bar{\Psi}_{dI}(x)P_+g(x)^{-1} = \bar{\Psi}_{dI}(x)P_+(1 - iP_+\omega(x)).
\end{aligned}
$$

For bosons, $\mathcal{O}(\omega)$ parts of the jacobian cancel with their conjugates.
For fermions, they cancel between $P_+\Psi_{uI}$ and $\bar{\Psi}_{dI}P_+$,
and between $\bar{\Psi}_{uI}\bar{P}_-$ and $\bar{P}_-\Psi_{dI}$.

$\Rightarrow$ Gauge invariance of the measure

Q-SUSY Invariance

Q-SUSY transformation with the Grassmann number ε:

$$
\begin{aligned}
P_+\Phi_I(x) &\longrightarrow (1+i\varepsilon Q)P_+\Phi_I(x) = P_+\Phi_I(x) + \cdots, \\
\Phi_I(x)^\dagger P_+ &\longrightarrow (1+i\varepsilon Q)\Phi_I(x)^\dagger P_+ = \Phi_I(x)^\dagger P_+ + \cdots, \\
\bar{P}_-\gamma_0 F_I(x) &\longrightarrow (1+i\varepsilon Q)\bar{P}_-\gamma_0 F_I(x) = \left[1 + i\varepsilon(Q\bar{P}_-)\bar{P}_-\right]\bar{P}_-\gamma_0 F_I(x) + \cdots, \\
F_I^\dagger\gamma_0\bar{P}_-(x) &\longrightarrow (1+i\varepsilon Q)F_I^\dagger\gamma_0\bar{P}_-(x) = F_I^\dagger\gamma_0\bar{P}_-\left[1 + i\varepsilon\bar{P}_-(Q\bar{P}_-)\right](x) + \cdots,
\end{aligned}
$$

$$
\begin{aligned}
P_+\Psi_{uI}(x) &\longrightarrow (1+i\varepsilon Q)P_+\Psi_{uI}(x) = P_+\Psi_{uI}(x) + \cdots, \\
\bar{\Psi}_{uI}\bar{P}_-(x) &\longrightarrow (1+i\varepsilon Q)\bar{\Psi}_{uI}\bar{P}_-(x) = \bar{\Psi}_{uI}\bar{P}_-\left[1 + i\varepsilon\bar{P}_-(Q\bar{P}_-)\right](x) + \cdots, \\
\bar{P}_-\Psi_{dI}(x) &\longrightarrow (1+i\varepsilon Q)\bar{P}_-\Psi_{dI}(x) = \left[1 + i\varepsilon(Q\bar{P}_-)\bar{P}_-\right]\bar{P}_-\Psi_{dI}(x) + \cdots, \\
\bar{\Psi}_{dI}(x)P_+ &\longrightarrow (1+i\varepsilon Q)\bar{\Psi}_{dI}(x)P_+ = \bar{\Psi}_{dI}(x)P_+ + \cdots,
\end{aligned}
$$

↑

"$\cdots$" correspond to off-diagonal elements of Jacobi matrices and irrelevant

Note

$$
\mathrm{Det}\left[1 + i\varepsilon(Q\bar{P}_-)\bar{P}_-\right] = 1 + i\varepsilon\,\mathrm{Tr}\left[(Q\bar{P}_-)\bar{P}_-\right] = 1 + i\varepsilon\,\mathrm{Tr}\left[\bar{P}_-(Q\bar{P}_-)\bar{P}_-\right] = 1.
$$

($\bar{P}_- = \bar{P}_-^2$ and $\bar{P}_-(Q\bar{P}_-)\bar{P}_- = 0$ was used.)

$\Rightarrow$ **Q-invariance of the measure**

U(1)$_A$ Transformation **(the parameter α infinitesimal)**

$$
\begin{aligned}
P_+\Psi_{uI}(x) &\longrightarrow \mathrm{e}^{i\alpha}P_+\Psi_{uI}(x) = (1+i\alpha P_+)P_+\Psi_{uI}(x),\\
\bar{\Psi}_{uI}\bar{P}_-(x) &\longrightarrow \bar{\Psi}_{uI}\bar{P}_-(x)\mathrm{e}^{-i\alpha} = \bar{\Psi}_{uI}\bar{P}_-(1-i\alpha\bar{P}_-)(x),\\
\bar{P}_-\Psi_{dI}(x) &\longrightarrow \mathrm{e}^{-i\alpha}\bar{P}_-\Psi_{dI}(x) = (1-i\alpha\bar{P}_-)\bar{P}_-\Psi_{dI}(x),\\
\bar{\Psi}_{dI}(x)P_+ &\longrightarrow \bar{\Psi}_{dI}(x)P_+\mathrm{e}^{i\alpha} = \bar{\Psi}_{dI}(x)P_+(1+i\alpha P_+),
\end{aligned}
$$

$$
\begin{aligned}
P_-\Psi_{uI'}(x) &\longrightarrow \mathrm{e}^{i\alpha}P_-\Psi_{uI'}(x) = (1+i\alpha P_-)P_-\Psi_{uI'}(x),\\
\bar{\Psi}_{uI'}\bar{P}_+(x) &\longrightarrow \bar{\Psi}_{uI'}\bar{P}_+(x)\mathrm{e}^{-i\alpha} = \bar{\Psi}_{uI'}\bar{P}_+(1-i\alpha\bar{P}_+)(x),\\
\bar{P}_+\Psi_{dI'}(x) &\longrightarrow \mathrm{e}^{-i\alpha}\bar{P}_+\Psi_{dI'}(x) = (1-i\alpha\bar{P}_+)\bar{P}_+\Psi_{dI'}(x),\\
\bar{\Psi}_{dI'}(x)P_- &\longrightarrow \bar{\Psi}_{dI'}(x)P_-\mathrm{e}^{i\alpha} = \bar{\Psi}_{dI'}(x)P_-(1+i\alpha P_-).
\end{aligned}
$$

$\Rightarrow$ **The measures change as**

$$
\begin{aligned}
\mathrm{d}\mu_{\mathrm{mat},+I} &\longrightarrow \left[1-2i\alpha\mathrm{Tr}(P_+-\bar{P}_-)\right]\mathrm{d}\mu_{\mathrm{mat},+I} = \left[1+i\alpha\mathrm{Tr}(\gamma_3 a\widehat{D})\right]\mathrm{d}\mu_{\mathrm{mat},+I},\\
\mathrm{d}\mu_{\mathrm{mat},-I'} &\longrightarrow \left[1+2i\alpha\mathrm{Tr}(\bar{P}_+-P_-)\right]\mathrm{d}\mu_{\mathrm{mat},-I'} = \left[1-i\alpha\mathrm{Tr}(\gamma_3 a\widehat{D})\right]\mathrm{d}\mu_{\mathrm{mat},-I'}.
\end{aligned}
$$

Thus,

$$\begin{aligned}(\mathrm{d}\mu_{\mathrm{mat}}) &\rightarrow \left[1 + i\alpha\,(n_+ - n_-)\,\mathrm{Tr}(\gamma_3 a\widehat{D})\right](\mathrm{d}\mu_{\mathrm{mat}}) \\ &\simeq \left[1 + i\alpha\,\frac{n_+ - n_-}{\pi}\int \mathrm{d}^2x\,\mathrm{tr}\,F_{01}\right](\mathrm{d}\mu_{\mathrm{mat}}) \qquad (a \rightarrow 0)\end{aligned}$$

for the gauge fields assumed to be smooth [Kikukawa-Yamada]**.**

$$\Downarrow$$

$U(1)_A$ anomaly in the previous section is reproduced.

6 Lattice Formulation of Gauged Linear Sigma Models

◇ Gauged linear sigma models (which we consider here)

2D $\mathcal{N} = (2,2)$ SQCD ($G = \mathrm{U}(N)$) with n_+ fundamental matters
and ℓ_- matters in the $\det^{-q_{\mathtt{A}'}}$-repre.
($\mathtt{A}' = 1, \cdots, \ell_-$, $q_{\mathtt{A}'} \in \mathbf{Z}_{>0}$)
↑
Different kinds of repre. in the $+$ and $-$ sectors

◇ The $\det^{-q_{\mathtt{A}'}}$-matters: charged only under the overall $\mathrm{U}(1)$ of $G = \mathrm{U}(N)$
Gauge-transformation by $g(x) = 1 + i\omega(x) \in G$ ($\omega(x)$ infinitesimal)

$$\Xi_{-\mathtt{A}'}(x) \longrightarrow (\det g(x))^{-q_{\mathtt{A}'}}\,\Xi_{-\mathtt{A}'}(x),$$

or

$$\delta\Xi_{-\mathtt{A}'}(x) = -iq_{\mathtt{A}'}\,(\operatorname{tr}\omega(x))\,\Xi_{-\mathtt{A}'}(x)$$

↓

Covariant derivatives: $\mathcal{D}_\mu\Xi_{-\mathtt{A}'} = (\partial_\mu - iq_{\mathtt{A}'}(\operatorname{tr} A_\mu))\,\Xi_{-\mathtt{A}'}$

↓

Forward (Backward) covariant differences D_μ (D_μ^*):

$$
\begin{aligned}
aD_\mu \, \Xi_{-\mathtt{A}}(x) &= (\det U_\mu(x))^{q_\mathtt{A}} \, \Xi_\mathtt{A}(x+\hat{\mu}) - \Xi_{-\mathtt{A}}(x), \\
aD_\mu^* \, \Xi_{-\mathtt{A}}(x) &= \Xi_{-\mathtt{A}}(x) - (\det U_\mu(x-\hat{\mu}))^{-q_\mathtt{A}} \, \Xi_{-\mathtt{A}}(x-\hat{\mu}).
\end{aligned}
$$

Ginsparg-Wilson formulation preserves the chiral flavor symmetry.

$\Downarrow$

We can latticize the gauged linear sigma models.

$\diamondsuit$ **When $n_+ \geq N$,**

Baryonic chiral superfields: $B_{I_1 \cdots I_N} \equiv \epsilon_{i_1 \cdots i_N} \Phi_{+I_1 i_1} \cdots \Phi_{+I_N i_N}$
gauge-transform as

$$B_{I_1 \cdots I_N}(x) \longrightarrow (\det g(x)) B_{I_1 \cdots I_N}(x).$$

Let $\mathcal{G}_{\mathtt{A}'}(B)$ be a homogeneous polynomial of degree $q_{\mathtt{A}'}$ w.r.t. $B_{I_1 \cdots I_N}$.

$$\Downarrow$$

The superpotential

$$\mathcal{W} = \sum_{\mathtt{A}'=1}^{\ell_-} \Xi_{-\mathtt{A}'} \mathcal{G}_{\mathtt{A}'}(B)$$

is gauge invariant.

Duality:

Gaugued linear sigma models $\Rightarrow$ (Infra-red) $\Rightarrow$ Nonlinear sigma models with target spaces

$\uparrow$

D-and F-term conditions [Witten]

$\diamondsuit$ Target spaces are in Grassmann manifolds ($\supset$ Calabi-Yau manifolds):

$$G(N, n_+) = \frac{\mathrm{U}(n_+)}{\mathrm{U}(N) \times \mathrm{U}(n_+ - N)}$$

Duality $G(N, n_+) \cong G(n_+ - N, n_+)$

$\Downarrow$

An analog of the Seiberg duality between the following gauged linear sigma models ($\ell_- = 1$) [Hori-Tong]:

- $G = \mathrm{U}(N)$, n_+ fundamental matters Φ_{+I}, one $\det^{-q}$-matter Ξ_- with $\mathcal{W} = \Xi_- \mathcal{G}(B)$ ($\mathcal{G}$: degree q)
- $G = \mathrm{U}(n_+ - N)$, n_+ fundamental matters Φ'_{+I}, one $\det^{-q}$-matter Ξ'_- with $\mathcal{W} = \Xi'_- \mathcal{G}'(B')$ ($\mathcal{G}'$: degree q)

where $\mathcal{G}(B) = \mathcal{G}'(B')$ with the replacement $B_{I_1 \cdots I_N} = \epsilon_{I_1 \cdots I_{n_+}} B'_{I_{N+1} \cdots I_{n_+}}$.

7 Summary and Discussion

◇ We have presented a lattice formulation of 2D $\mathcal{N} = (2, 2)$ SQCD (including gauged linear sigma models) with exactly preserving Q-SUSY.

- Gauge Group $G = \mathrm{U}(N)$ (or $\mathrm{SU}(N)$), Compact link variables $U_\mu(x)$
- In order to resolve the matter doublers,
  - Use of D_W ⇒ the lattice action is constructed in the case $n_+ = n_-$
  - Use of $\widehat{D}$ ⇒ the lattice action is constructed for general $n_\pm$
    Exact chiral flavor symmetry on the lattice
    First example of the Ginsparg-Wilson formulation of lattice gauge models with exact SUSY

(c.f. [Kikukawa-Nakayama] for 2D WZ models)

- The Ginsparg-Wilson formulation ⇒ exactly (anti-)holomorphic superpotentials on the lattice
  ⇒ Nonrenormalization theorem on the lattice expected
- Application to the gauged linear sigma models
  Check the duality from the lattice!
  SCFTs from the lattice

FI and ϑ terms ($G = \mathrm{U}(N)$):

◇ The FI and topological ϑ-terms

$$S^{\mathrm{LAT}}_{\mathrm{FI},\,\vartheta} = Q\kappa \sum_x \mathrm{tr}\,(-i\chi(x)) - \frac{\vartheta - 2\pi i\kappa}{2\pi} \sum_x \mathrm{tr}\,\ln U_{01}(x)$$

↑

Q-invariant by its topological nature

($\delta \sum_x \mathrm{tr}\,\ln U_{01}(x) = 0$)

Well-definedness of $\mathrm{tr}\,\ln U_{01}(x)$

$\Rightarrow 0 < \epsilon < \frac{1}{\sqrt{N}}$ for $G = \mathrm{U}(N)$ with ϑ-term

◇ Use of $\widehat{D}$ yields another FI and ϑ-term:

$$S^{\mathrm{LAT}}_{\mathrm{FI},\vartheta\,(\widehat{D})} \equiv Q\kappa \sum_x \mathrm{tr}\,(-i\chi(x)) - \frac{\vartheta - 2\pi i\kappa}{2\pi} ia^2 \sum_x \mathrm{tr}\,\widehat{F}_{01}(x)$$

with $\widehat{F}_{01}(x) \equiv \frac{\pi}{a}\,\mathrm{tr}_{\mathrm{spin}}\left(\gamma_3 \widehat{D}\right)(x,x)$ ($\mathrm{tr}_{\mathrm{spin}}$: trace over the Dirac indices).

$\sum_x \mathrm{tr}\widehat{F}_{01}(x)$ is topological because $\delta\,\mathrm{Tr}\,(\gamma_3 \widehat{D}) = 0$.

◇ Consider the case of all twisted masses zero and $\ell_- = 1$.

Superpotential: $\mathcal{W} = \Xi_- \mathcal{G}(B)$. ($\Xi_-$: $\det^{-q}$-repre., $\mathcal{G}$: degree q)

Bosonic potential is

$$U = |\mathcal{G}(b)|^2 + |\xi_-|^2 \sum_{I=1}^{n_+} \sum_{i=1}^{N} \left| \sum_{I_1 < \cdots < I_N} \frac{\partial \mathcal{G}(b)}{\partial b_{I_1 \cdots I_N}} \frac{\partial b_{I_1 \cdots I_N}}{\partial \phi_{+Ii}} \right|^2 \qquad \leftarrow \text{F-term}$$

$$+ \frac{g^2}{4} \operatorname{tr} \left\{ \left[\sum_{I=1}^{n_+} \phi_{+I} \phi_{+I}^\dagger - \left(q \xi_-^* \xi_- + \kappa \right) \mathbb{1}_N \right]^2 \right\} \qquad \leftarrow \text{D-term}$$

$$+ \frac{1}{4g^2} \operatorname{tr} \left([\phi, \bar{\phi}]^2 \right) + \sum_{I=1}^{n_+} \frac{1}{2} \phi_{+I}^\dagger \left\{ \phi, \bar{\phi} \right\} \phi_{+I} + |q \operatorname{tr} \phi|^2 |\xi_-|^2 ,$$

where $b_{I_1 \cdots I_N}$, ξ_-: the lowest components of the chiral superfields $B_{I_1 \cdots I_N}$, Ξ_-.

For the potential minimum $U = 0$,

The second term ⇒ $\xi_- = 0$ (for generic $\mathcal{G}$),

The third term ⇒ $\sum_{I=1}^{n_+} \phi_{+I} \phi_{+I}^\dagger = \kappa \mathbb{1}_N$

⇒ N vectors $v_1, \cdots, v_N \in \mathbb{C}^{n_+}$ ($(v_i)_I = \phi_{+Ii}$):

Orthogonal and $(\text{length})^2 = \kappa$ ($\kappa > 0$ assumed.)

⇒ $\{v_1, \cdots, v_N\}$ span the space of N-dim. planes in $\mathbb{C}^{n_+}$,

i.e. Grassmann manifold $G(N, n_+) = \frac{\mathrm{U}(n_+)}{\mathrm{U}(N) \times \mathrm{U}(n_+ - N)}$.

$\Rightarrow$ **The F-term and D-term conditions give**

a hypersurface defined by $\mathcal{G}(b) = 0$ in $G(N, n_+)$.

A.1 Gauged Linear Sigma Models $\Rightarrow$ Calabi-Yau

$\diamondsuit$ **For the case $G = \mathrm{U}(1)$** $\quad b_I = \phi_{+I} \quad (I = 1, \cdots, n_+)$

$$U = |\mathcal{G}(\phi_+)|^2 + |\xi_-|^2 \sum_{I=1}^{n_+} \left| \frac{\partial \mathcal{G}(\phi_+)}{\partial \phi_{+I}} \right|^2 + \frac{g^2}{4} \left(\sum_{I=1}^{n_+} \phi_{+I}\phi^*_{+I} - q\xi^*_- \xi_- - \kappa \right)^2 + \sum_{I=1}^{n_+} |\phi|^2 |\phi_{+I}|^2 + |q\phi|^2 |\xi_-|^2,$$

For $U = 0$,

the second and third terms $\Rightarrow \xi_- = 0, \quad \sum_{I=1}^{n_+} \phi_{+I}\phi^*_{+I} = \kappa$

$\downarrow$

represents CP^{n_+-1} under the action of $G = \mathrm{U}(1)$

$\Rightarrow$ **The F-term and D-term conditions give**

a hypersurface defined by $\mathcal{G}(\phi_+) = 0$ (degree q) in CP^{n_+-1}.

$\Downarrow$

When $q = n_+$, this becomes a Calabi-Yau manifold.

$\mathrm{U}(1)_A$ anomaly cancels, κ does not run.

B Admissibility Conditions

Combining the addmissibility conditions from the SYM part and from the matter part, we find

$G = \mathrm{U}(N)$ **without** ϑ**-term :**

$$0 < \epsilon < \frac{1}{5} \quad \text{for } N = 1, 2, \cdots, 100$$
$$0 < \epsilon < \frac{2}{\sqrt{N}} \quad \text{for } N \geq 101,$$

$G = \mathrm{U}(N)$ **with** ϑ**-term :**

$$0 < \epsilon < \frac{1}{5} \quad \text{for } N = 1, 2, \cdots, 25$$
$$0 < \epsilon < \frac{1}{\sqrt{N}} \quad \text{for } N \geq 26,$$

$G = \mathrm{SU}(N)$ **:**

$$0 < \epsilon < \frac{1}{5} \quad \text{for } N = 2, 3, \cdots, 31$$
$$0 < \epsilon < 2 \sin\left(\frac{\pi}{N}\right) \quad \text{for } N \geq 32.$$

$G = \mathrm{U}(N)$ gauged linear sigma model :

$$0 < \epsilon < \frac{1}{8Nq} \qquad \text{with} \qquad q \equiv \max_{\mathtt{A}'=1,\cdots,\ell_-} (q_{\mathtt{A}'}).$$