Non-Abelian Vortices

— *Five Years Since the Discovery* —

Towards New Developments in Field and String Theories
12/22/2008 @ RIKEN

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Anyone is welcome to join us anytime !

§1. Introduction: What are Vortices?

Vortices are topological solitons

- **of codimension 2: point-like in $d = 2+1$, string in $d = 3+1$,**
- **to exist when symmetry is broken $G \to H$ with**

$$\boxed{\pi_1(G/H) \simeq \pi_0(H) \simeq H/H_0 \neq 0}$$ **for simply connected G,**

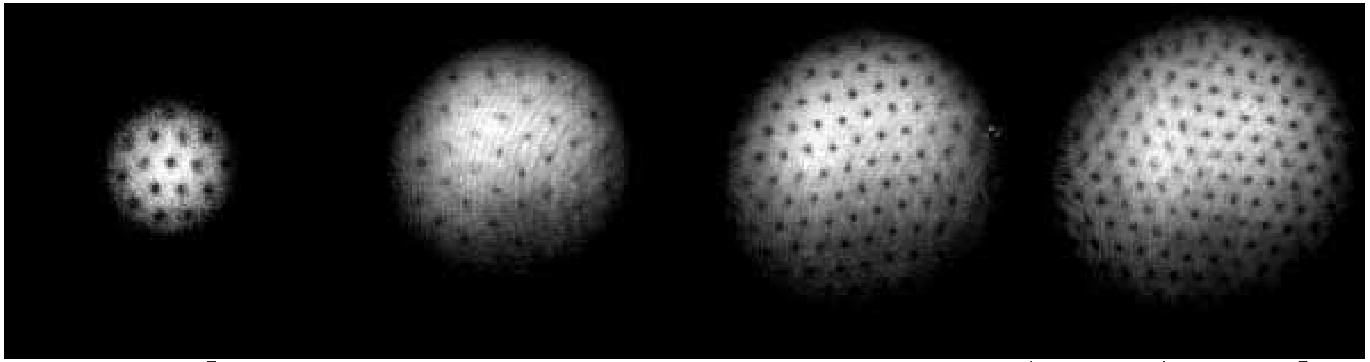
- **formed via the Kibble-Zurek mechanism or rotation of media,**
- **carrying magnetic flux or circulation which is quantized.**

π_n	**Defects** codim $n+1$	**Textures** codim n	**Gauge Structure** codim $n+1$
π_0	domain walls(kinks)		
π_1	vortices	nonlinear kinks(sine-Gordon)	
π_2	monopoles	lumps(2D skyrmions)	
π_3		Skyrmions (textures)	YM instantons

They appear in various area of physics:

1. **condensed matter physics**

- **superconductor** (Abrikosov lattice) **Abrikosov('57)**
- **superfluid ^{4}He Onsager('49), Feynman('55)**
  superfluid ^{3}He
- **(skyrmions in) quantum Hall effects**
- **(Bloch line in) Ferromagnets**
- **atomic gas Bose-Einstein condensation (cold atom) ('01-)**
- **quantum turbulence (Kolmogorov law)**



MIT [Abo-Shaer et.al, Science 292 (2001) 476]

2. cosmology and astrophysics

- a candidate of cosmic strings

  Phase transition occurs in the early Universe.

  ⇒ vortices must form (Kibble mechanism) Kibble ('76)

  (cf: monopoles ⇒ monopole problem Preskill, Guth('79))

  Suggested as a source of structure formation ('80s – early'90)

  ⇒ ruled out by Cosmic Microwave Background ('98 - '01)

- vortex-ring(=vorton): candidate of dark matter, ultra high energy cosmic ray

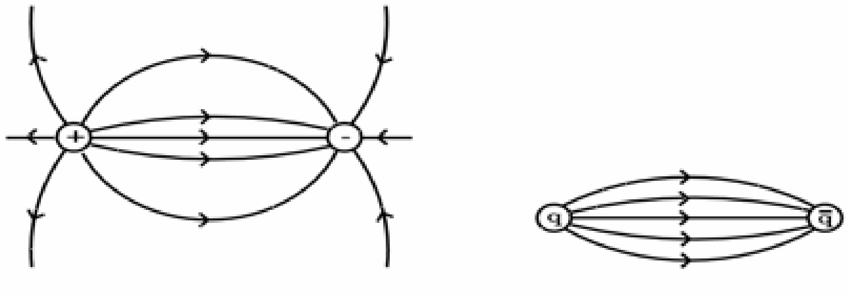
- Recent revivals of cosmic strings ('03 - present):

(a) cosmic superstrings (F/D-strings) in string theory, brane inflation Dvali-Tye, Polchinski etc ('04) (p,q) string network

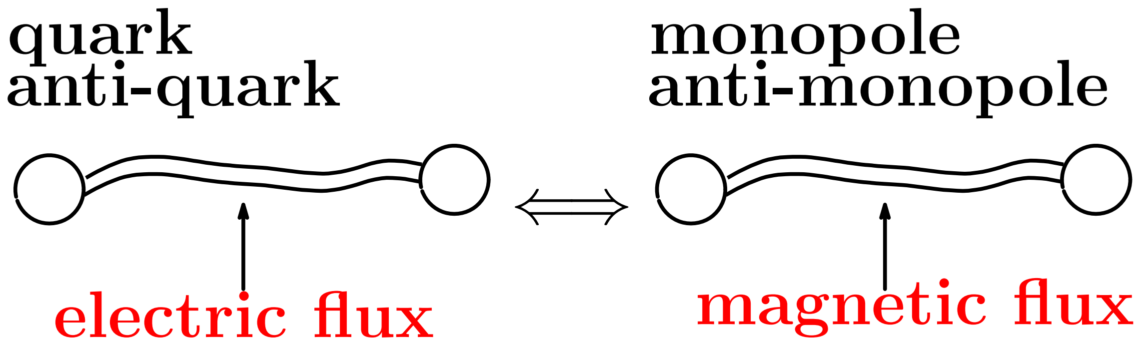
(b) possible detection of cosmic strings by CMB, gravitational lensing, gravitational wave

3. high energy physics

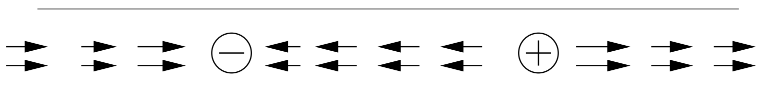
- magnetic flux tube confining monopoles Nielsen-Olesen('73)
  = dual superconductor 'tHooft, Nambu, Mandelstam ('74)



dual Meissner effect



- The center vortex mechanism 'tHooft, Cornwall etc ('79)
  trying to extend it to color(non-Abelian) gauge symmetry

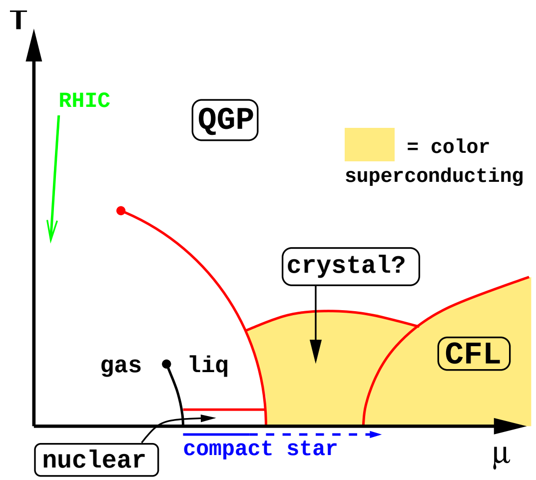


lattice sim. Ambjorn et.al ('00)

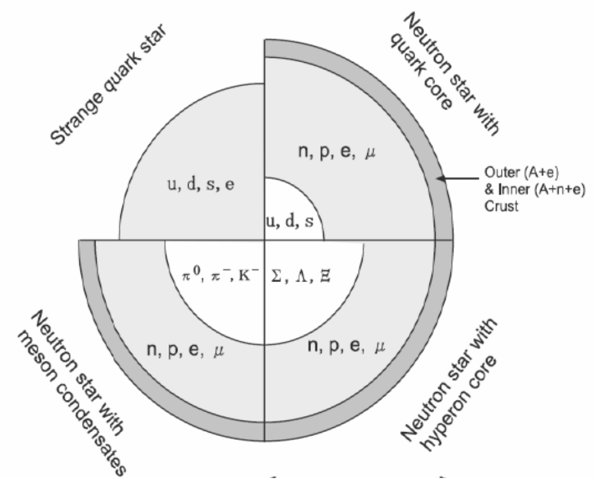
- Supersymmetric QCD Hanany-Tong, Konishi group(Pisa), Shifman-Yung(Minnesota), TITech ('03-)
- Weinberg-Salam, Nambu('77), Vachaspati('92)
- SO(10) GUT Kibble ('82), SUSY GUTs Jeannerot *et al* ('03)

4. hadron physics

- proton vortices and neutron vortices in hadronic phase of neutron stars ⇒ pulsar glitch Anderson-Itoh('75)
- color superconductivity (core of neutron stars) Iida-Baym etc('01), Balachandran-Digal-Matsuura('05), Nakano-MN-Matsuura('07)
- chiral phase transition Brandenberger('97), Balachandran-Digal('01), MN-Shiki,Nakano-MN-Matsuura('07)
- YM plasma Chernodub-Zakharov, Liao-Shuryak('07-)



Alford et.al



Hatsuda et.al

Abelian Vortices

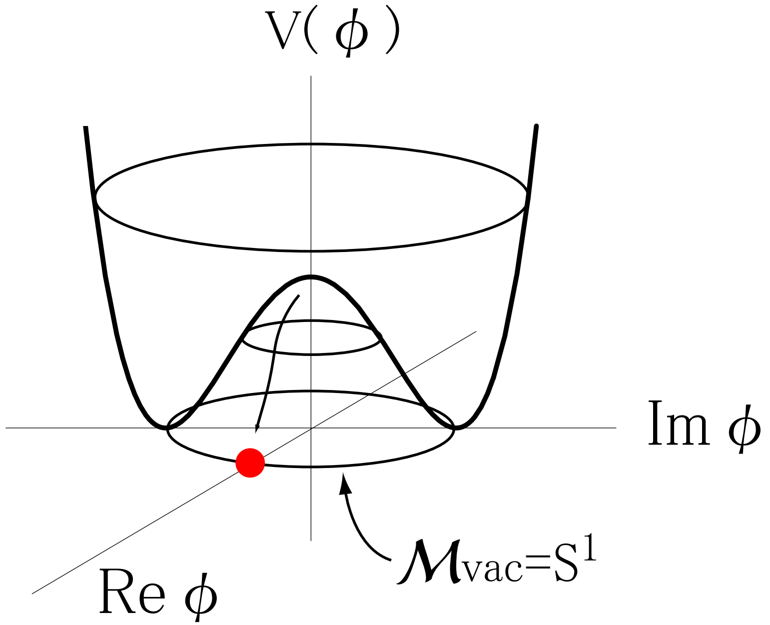
Vortices appear when $U(1)$ local sym. is spontaneously broken.

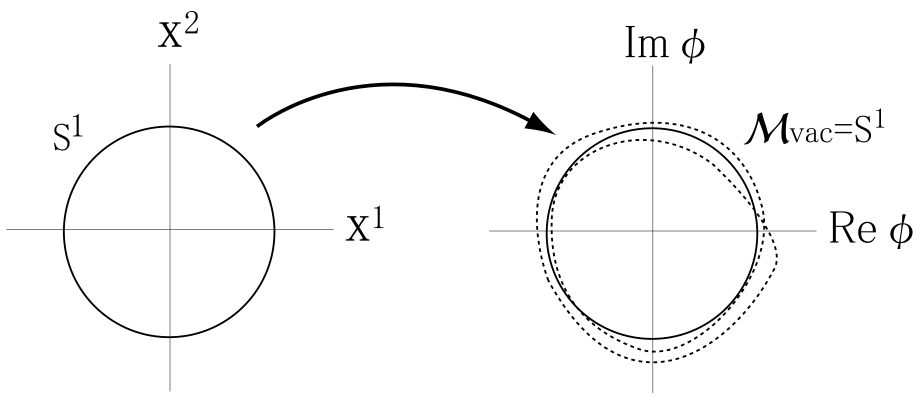
The Abelian Higgs model [(gauged) Laudau-Ginzburg model]

$$H = \int d^2x \left[\frac{1}{2e^2}(\mathbf{E}^2 + \mathbf{B}^2) + |(\nabla - i\mathbf{A})\phi|^2 + \underbrace{\frac{\lambda}{4}(|\phi|^2 - c)^2}_{V(\phi)} \right] \qquad (1)$$

e**: gauge coupling,** λ**: Higgs scalar coupling,** $v = \langle \phi \rangle = \sqrt{c}$

local(=gauge) symmetry: $\phi(x) \to e^{i\alpha(x)}\phi(x)$, $\mathbf{A} \to \mathbf{A} + \nabla\alpha(x)$

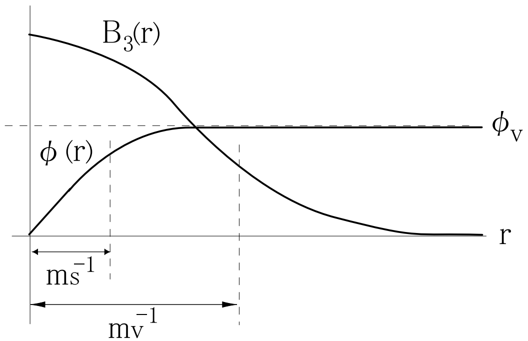


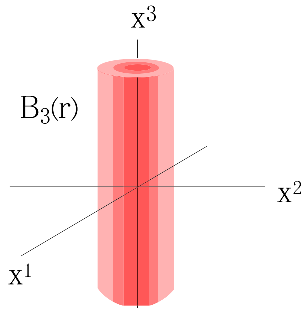


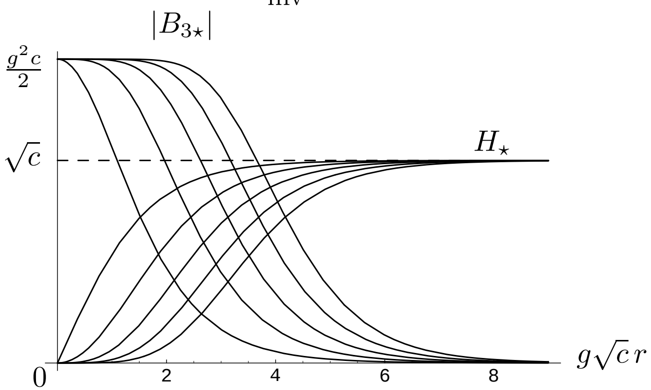
Magnetic flux is quantized to be integer.
Vortex(winding) #(=vorticity) is given by 1st homotopy class:

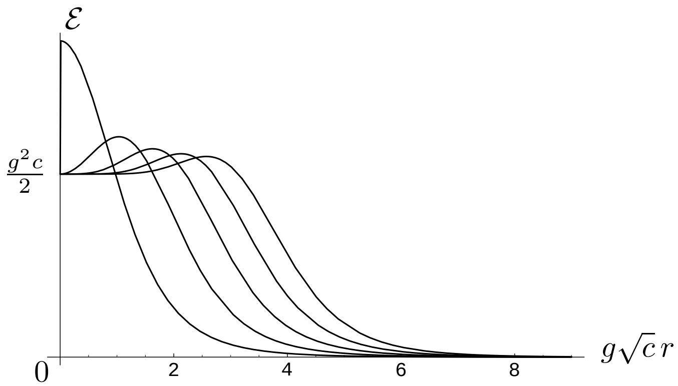
$$\int d^2x B_3 = 2\pi c\, k, \qquad k \in \pi_1[U(1)] = \mathbf{Z}.$$

Abrikosov('57) and Nielsen-Olesen('73) (ANO vortices).









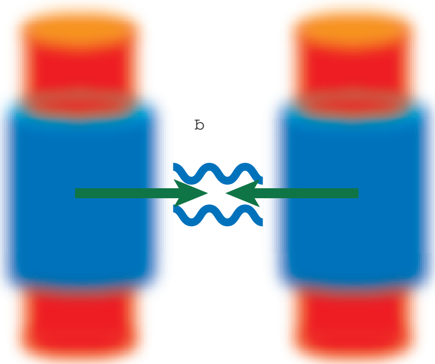
U(1) gauge symmetry is recovered in the core

e: gauge coupling, λ: Higgs scalar coupling, v: VEV of scalar

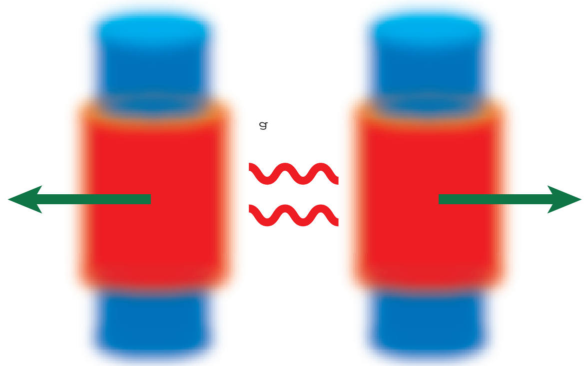
gauge mass: $m_v \simeq \sqrt{2}ev \Rightarrow$ penetration depth: $r_v = m_v^{-1} \simeq (\sqrt{2}ev)^{-1}$

scalar mass: $m_s \simeq \sqrt{\lambda}v \Rightarrow$ coherence length: $r_s = m_s^{-1} \simeq (\lambda v)^{-1}$

type	range	static force	stability under B
type I	$r_v < r_s$ $(2e^2 > \lambda)$	attractive force	unstable
type II	$r_v > r_s$ $(2e^2 < \lambda)$	repulsive force	stable Abrikosov lattice
critical	$r_v = r_s$ $(2e^2 = \lambda)$	non ($\rightarrow$ moduli dynamics)	



type I



type II

Critical coupling (Bogomol'nyi-Prasad-Sommerfield = BPS)

$$H = \int d^2x \left[\frac{1}{2e^2} B_z^2 + |(\nabla - i\mathbf{A})\phi|^2 + \frac{\lambda}{4}\left(|\phi|^2 - c\right)^2 \right] \quad (2)$$

$\lambda = 2e^2$ **(critical)** (← **realized by** *Supersymmetry*)

$$\begin{aligned} H &= \int d^2x \left[|(\partial_x - iA_x)\phi + i(\partial_y - iA_y)\phi|^2 + \frac{1}{2e^2}\{B_z + e^2(|\phi|^2 - c)^2\}^2 \right] \\ &\quad + c\int d^2x B_z \\ &\geq c\int d^2x B_z = 2\pi c\, k, \qquad k \in \mathbf{Z} \end{aligned} \quad (3)$$

"=" ⇔ **Bogomol'nyi bound** **(energy minimum)**

The most *stable* for a fixed vortex number k.

The BPS equation (vortex equation)

$$(\mathcal{D}_x + i\mathcal{D}_y)\phi = 0, \qquad B_z + e^2(|\phi|^2 - c) = 0 \quad (4)$$

BPS solitons allow the moduli space $\mathcal{M}_k$.

1. **All possible configurations.**
2. **Dynamics/scattering = geodesic motion on the moduli space (geodesic/Manton approx.).**
3. **Collective coordinate quantization.**
4. **Integration over the instanton moduli space (Nekrasov).**
5. **Topological invariants (mathematics)**

The moduli space of ANO(Abelian) vortices

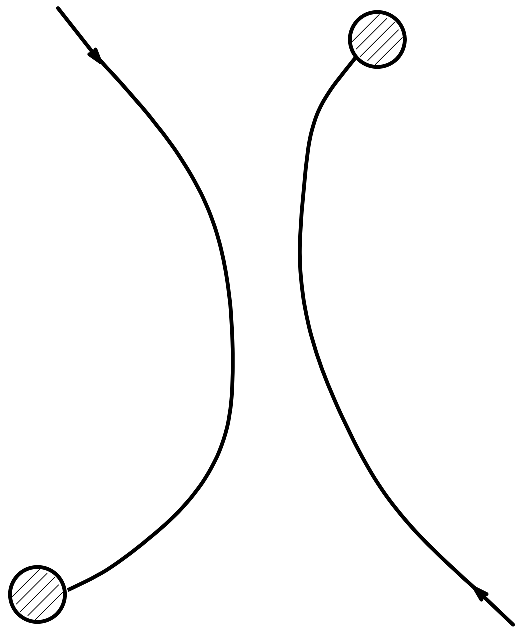
E.Weinberg ('79)
The index theorem counting zero modes: $\dim \mathcal{M}_k = 2k$.

Taubes ('80) Rigorous proof of the existence and uniqueness of multiple vortex solutions.
The moduli space is symmetric product: $\mathcal{M}_k = \mathbf{C}^k/\mathfrak{S}_k$.

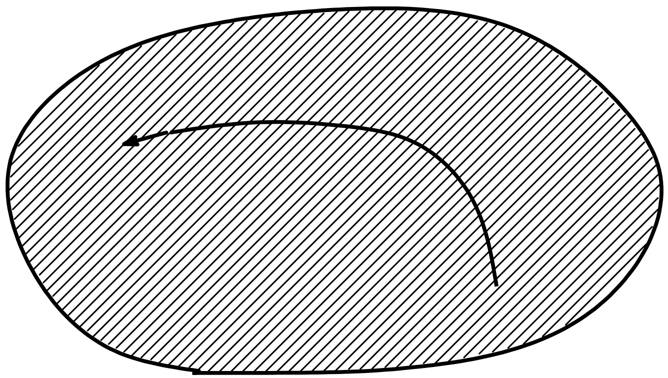
Samols ('92) The moduli space metric. The right-angle (90 **degree) scattering in head-on collisions.**

The moduli space ⇒ Dynamics

If solitons move slowly there appear force between them.
The moduli space describes classical dynamics of solitons, the scattering of solitons. The moduli (geodesic, Manton's) approx.



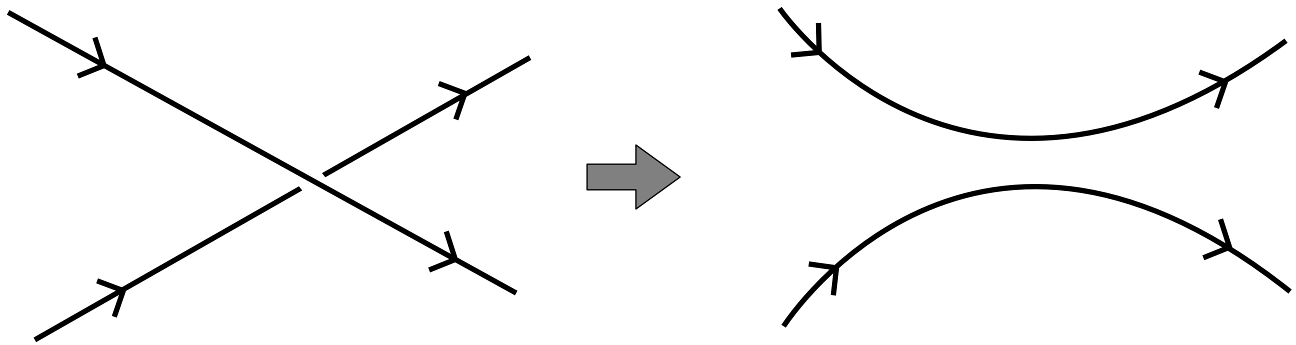
Soliton Scattering ⇔



Geodesics in Moduli Space

ex.) For instance, a scattering of two BPS monopoles is described by a geodesic on the Atiyah-Hitchin metric.

Reconnection (intercommutation, recombination) **of vortex-strings (in $d = 3 + 1$) is very important.**



1. **Essential process for (quantum) turbulence (Kolmogorov law)**
2. **superconductor, superfluid ^{4}He.**
3. **Cosmic Strings**

When two cosmic strings collide with angle they may reconnect.

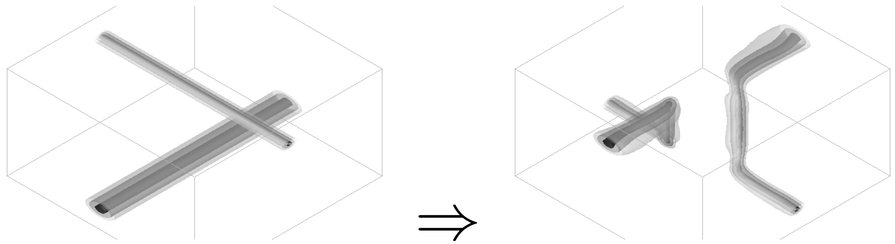
Reconnection probability P is very important.

$P \sim 1 \Longrightarrow$ **# density of strings is low.**

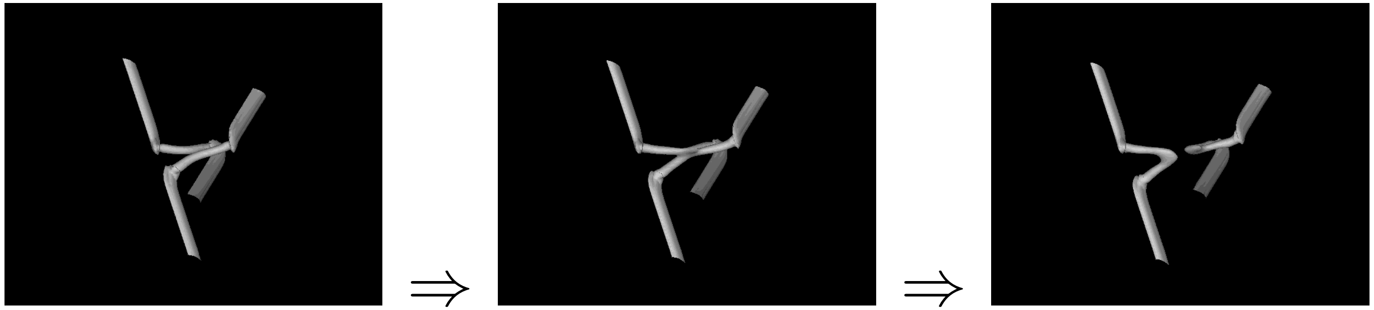
$P \sim 0 \Longrightarrow$ **# density is high (contradict to observation).**

Many computer simulations have been performed:

1. local strings in the Abelian-Higgs model $P \sim 1$ ('80s)
2. semi-local strings $P \sim 1$
   Laguna, Natchu, Matzner and Vachaspati, PRL[hep-th/0604177]
   Two different sizes vary to concide with each other.

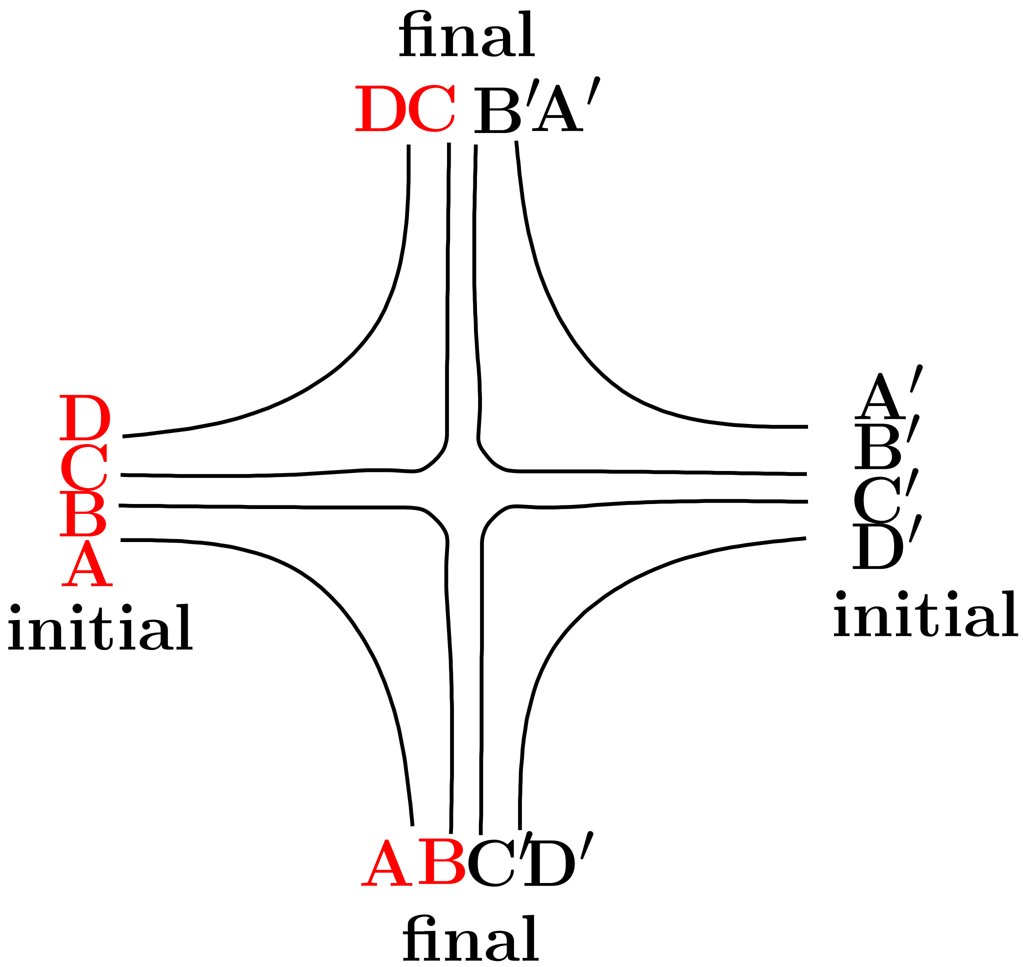


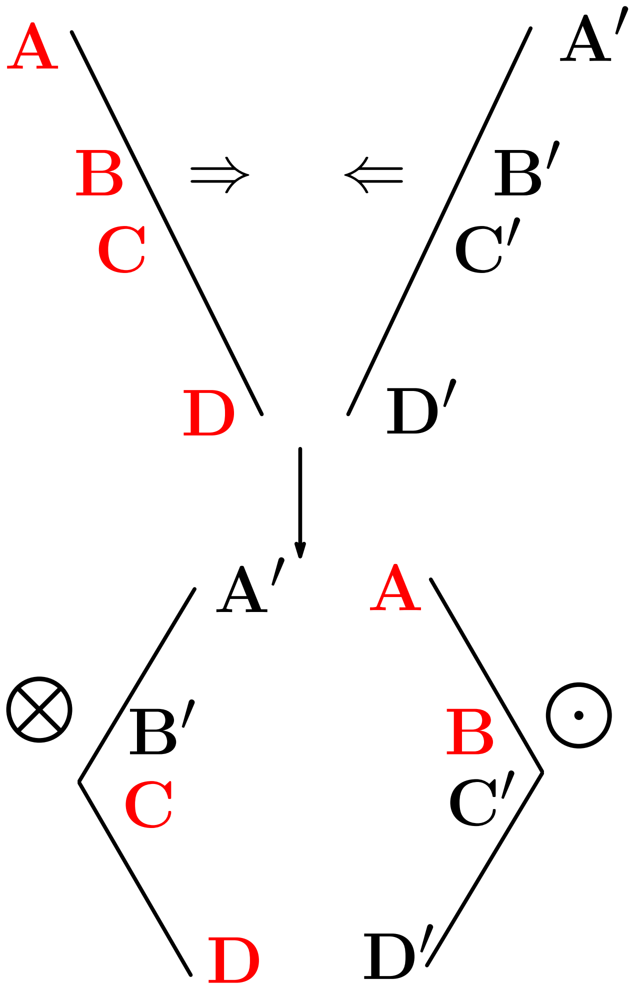
3. non-intercommutation in high speed collision, $P \neq 1$
   Achucarro and de Putter, PRD[hep-th/0605084]



analytical argument

Right angle scattering of vortex-particles in head-on collisions
⇕ Copeland-Turok, Shellard ('88)
Reconnection of vortex-strings





interlude: **How "non-Abelian" are non-Abelian vortices??**

$$\pi_1(G/H) \simeq \pi_0(H) \tag{5}$$

Different definitions of "non-Abelian" vortices: (3 $\Rightarrow$ 2 $\Rightarrow$ 1)

1. **G is non-Abelian**

   ex) $G = SU(N)$ with N adjoint Higgs

   $H \simeq \mathbf{Z}_N$: **Abelian,** $\quad \pi_1(G/H) \simeq \mathbf{Z}_N$: **Abelian**

2. **H is non-Abelian** $\leftarrow$ **Our definition**

3. **$\pi_1(G/H)$ is non-Abelian**

   ex1) biaxial nematics: $SO(3)$ with 5 (sym.tensor) real Higgs

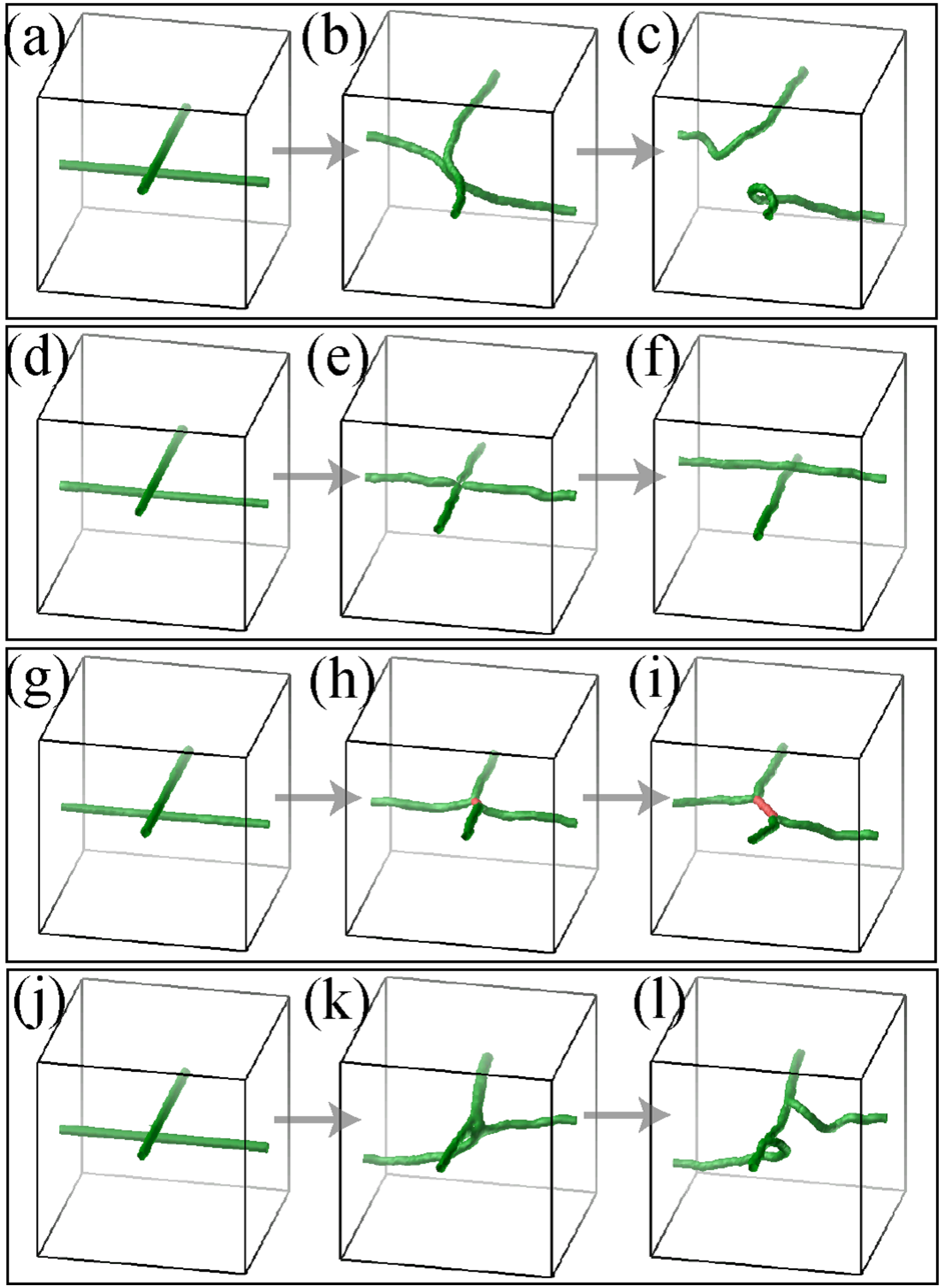
   $SO(3)/\mathbf{K} \simeq SU(2)/\mathbf{Q}_8$ (**$\mathbf{Q}_8$: quaternion**), $\quad \pi_1 \simeq \mathbf{Q}_8$

   ex2) spinor BEC ($F = 2$), cyclic phase:

   $SO(3) \times U(1)$ with 5 (sym.tensor) complex Higgs

   $[SO(3) \times U(1)]/T$ (**T: tetrahedral**)

   Kobayashi, Kawaguchi, MN and Ueda [arXiv:0810.5441]



a model for

(p, q) **web of cosmic strings**

Kobayashi, Kawaguchi, MN and Ueda [arXiv:0810.5441]

Knot soliton: $\pi_3(S^2) \simeq \mathbf{Z}$

Kawaguchi, MN and Ueda

PRL [arXiv:0802.1968]

cover

Plan of My Talk

§1. Introduction: What are Vortices? (14+3 pages)

§2. Non-Abelian Vortices: Review (13+5 pages)

§3. Moduli Matrix Formalism (16+1 pages)

§4. Conclusion / Discussion (2 pages)

§2. Non-Abelian Vortices: Review

The non-Abelian extension has been discovered recently.
Hanany-Tong ('03), Konishi et.al ('03)

- Vortices in the color-flavor locking vacuum.
- Each carries a non-Abelian magnetic flux.
- It is characterized by non-Abelian orientational moduli $\mathbf{C}P^{N-1}$ ($U(2)$ gauge $\Rightarrow$ $\mathbf{C}P^1 \simeq S^2$: sphere).
- Half properties of Yang-Mills instantons (on a NC $\mathbf{R}^4$).

We call these **non-Abelian vortices**.

The non-Abelian Higgs model (bosonic part of $N = 2$ **SUSY**)

$U(N)$ gauge theory with N Higgs in the fund. rep. H ($N \times N$):

$$\mathcal{L} = \mathrm{Tr}_{N_\mathrm{C}} \left[-\frac{1}{2g^2} F_{\mu\nu} F^{\mu\nu} - \mathcal{D}_\mu H \mathcal{D}^\mu H^\dagger - \frac{g^2}{4} \left(c \mathbf{1}_{N_\mathrm{C}} - H H^\dagger \right)^2 \right] \tag{6}$$

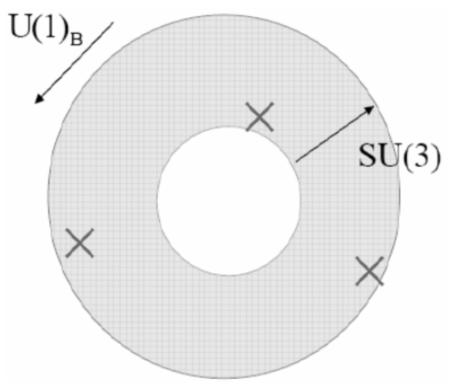
$U(N)$ color(local) $\times$ $SU(N)$ flavor(global) symmetry.

$$H \to g_C(x) H g_F, \quad F_{\mu\nu} \to g_C(x) F_{\mu\nu} g_C(x)^{-1} \tag{7}$$

$$g_C(x) \in U(N), \quad g_F \in SU(N) \tag{8}$$

The system is in the color-flavor locking vacuum: $\boxed{H = \sqrt{c}\mathbf{1}_N}$.

$$\boxed{U(N)_\mathrm{C} \times SU(N)_\mathrm{F} \to SU(N)_\mathrm{C+F}}$$



$$\mathrm{OPS}: \quad \frac{U(N)_\mathrm{C} \times SU(N)_\mathrm{F}}{SU(N)_\mathrm{C+F}} \simeq \frac{U(1) \times SU(N)}{\mathbf{Z}_N}$$

Vortex Equations

The Bogomol'nyi bound for vortices:

$$\mathcal{E} = \int dx^1 dx^2 (\textbf{r.h.s of BPS eqs.})^2 + T_{\text{vortices}} \quad (9)$$

$$\geq T_{\text{vortices}} = -c \int dz d\bar{z}\, \mathrm{Tr}\, F_{12} = 2\pi c\, k, \quad (10)$$

$$k \in \mathbf{N}_+ = \pi_1[U(N)]. \quad (11)$$

The BPS equations (vortex equations):

$$0 = (\mathcal{D}_1 + i\mathcal{D}_2)H, \quad (12)$$

$$0 = F_{12} + \frac{g^2}{2}(c\mathbf{1}_N - HH^\dagger). \quad (13)$$

cf. The $U(1)$ case ($N = 1$) $\rightarrow$ the ANO vortex eqs.

Moduli space for single vortex Hanany-Tong, Konishi et.al ('03)

We can embed the ANO solution $(F_{12}^{\rm ANO}, H^{\rm ANO})$ **(**$z = x^1 + ix^2$**):**

$$F_{12} = \begin{pmatrix} F_{12}^{\rm ANO}(z-z_0) & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}, \quad H = \begin{pmatrix} H^{\rm ANO}(z-z_0) & & & \\ & \sqrt{c} & & \\ & & \ddots & \\ & & & \sqrt{c} \end{pmatrix} \tag{14}$$

This solution breaks $\boxed{SU(N)_{\rm C+F} \to SU(N-1)\times U(1)}$**.**

The moduli space of Nambu-Goldstone modes:

$$\mathcal{M}_{N,k=1} = \mathbf{C} \times \frac{SU(N)_{\rm C+F}}{SU(N-1)\times U(1)} \simeq \mathbf{C} \times \mathbf{C}P^{N-1}\,. \tag{15}$$

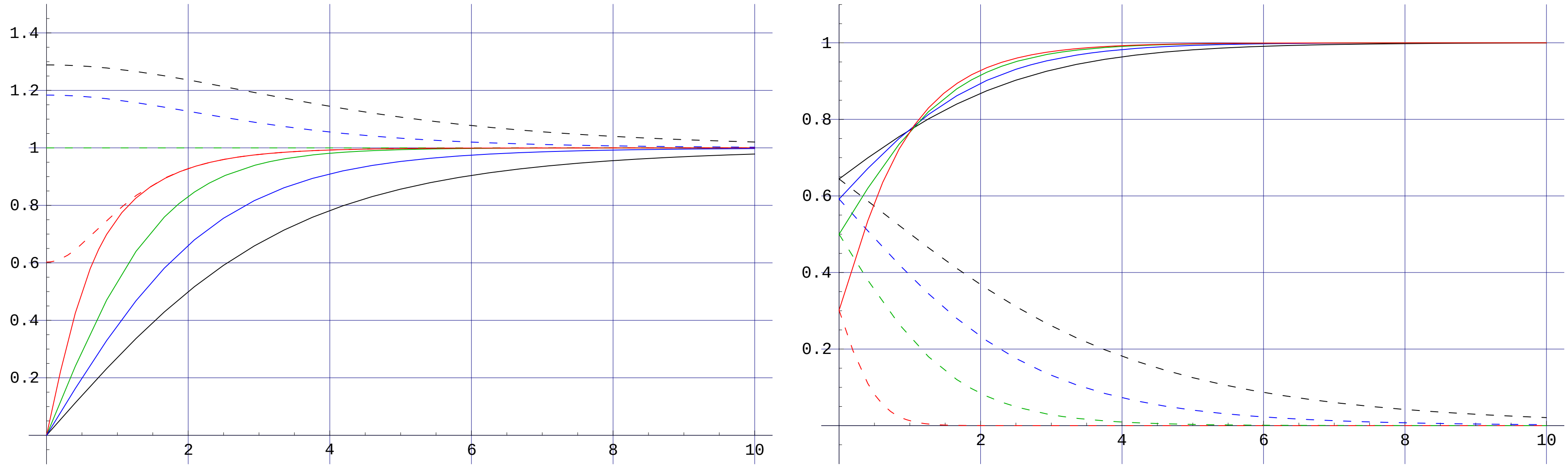
$\uparrow$ **translational** $\quad$ $\uparrow$ **internal symmetry** $\quad$ $(\mathbf{C}P^1 \simeq S^2)$

These are normalizable modes (= localized around the vortex).

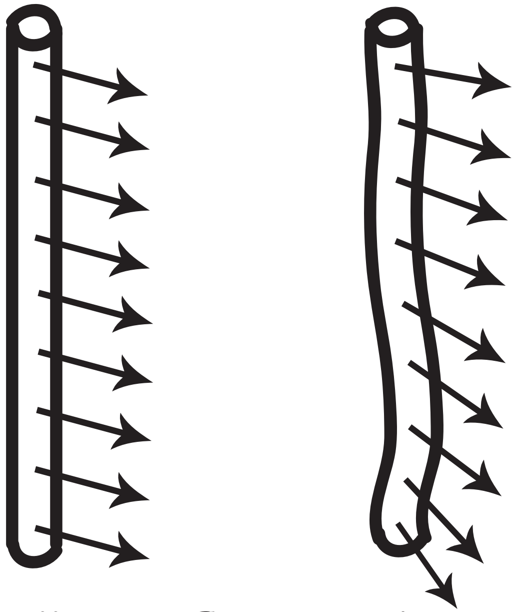
$(F_{12}^{\rm ANO}, H^{\rm ANO}) \to (0, \sqrt{c})$ **as** $z \to \infty$

No more moduli: $\dim_{\mathbf{C}} \mathcal{M}_{N,k=1} = N$ **from the index theorem.**

interlude: When gauge couplings for $U(1)$ and $SU(N)$ are different, it's not just an embedding of the ANO solution.



The effective theory is the $\mathbf{C}P^{N-1}$ model.

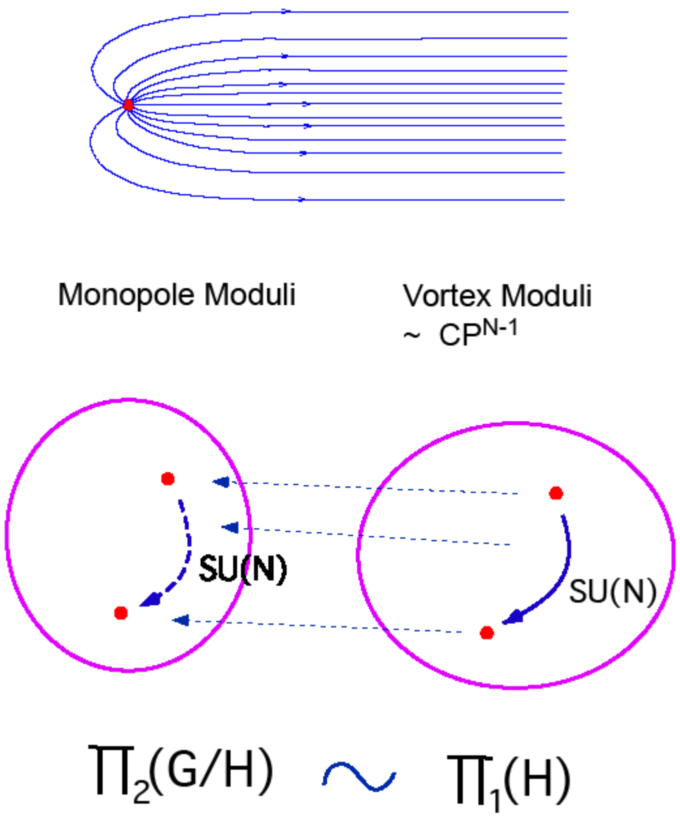


1. **It carries a flux of a linear combination of $U(1)$ and one generator T of $SU(N)_{\rm C}$, which is recovered inside the vortex core.** $SU(N-1)_{\rm C}$ **is still locked with** $SU(N-1)_{\rm F}[\subset SU(N)_{\rm F}]$.
2. **Choice of recovering $U(1)$** $\Longleftrightarrow_{1:1}$ **a point at $\mathbf{C}P^{N-1}$.**
3. **The tension of $k=1$ vortex is $1/N$ of ANO.**

Motivation of the Konishi group

extension of Seiberg-Witten to non-Abelian duality
Goddard-Nuyts-Olive-Weinberg (GNOW, Langrands) duality

But, NA monopoles have a problem of non-normalizable moduli.
$\Rightarrow$ NA monopole confined by NA vortices



GNOW dual $\tilde{G}$

G	$SO(2M)$	$USp(2M)$	$SO(2M+1)$
$\tilde{G}$	$SO(2M)$	$SO(2M+1)$	$USp(2M)$

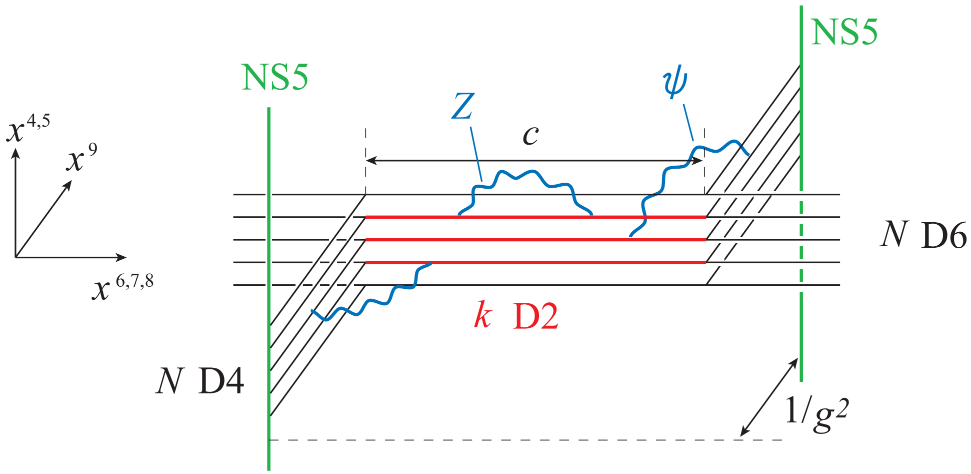
1. Multiple-vortex moduli space $\mathcal{M}_{N,k}$??
2. Multi-vortex solution??

$\Downarrow$

- **String Theory (D-brane construction)**
  $\rightarrow$ Kähler quotient ("half ADHM") Hanany-Tong ('03)
  only moduli space topology, *nothing about solutions*

- **The Moduli Matrix Approach TITech ('05, '06-)**
  Solutions. Moduli space with *the metric*.
  Dynamics(Scattering of vortices/reconnection of strings) .

D-brane construction of vortices Hanany-Tong ('03)



$d = 4$ **theory**

2 NS5	:	012345				
N D6	:	0123		678		
N D4	:	0123				9

vortices

k D2	:	0	3		8

$\mathcal{M}_{N,k}$ = **Higgs branch of** $U(k)$ **gauge theory on** k **D2's (Kähler quotient):**

$$\mathcal{M}^{\mathrm{ST}}_{N,k} = \left\{ Z, \Psi \middle| \pi c [Z^\dagger, Z] + \Psi^\dagger \Psi = \frac{4\pi}{g^2} 1_k \right\} \Big/ U(k)$$
$$\simeq \left\{ Z, \Psi \right\} /\!\!/ GL(k, \mathbb{C})$$

with Z **adjoint** $(k \times k)$ **and** Ψ **fundamental** $(N \times k)$**.**
"Half ADHM"

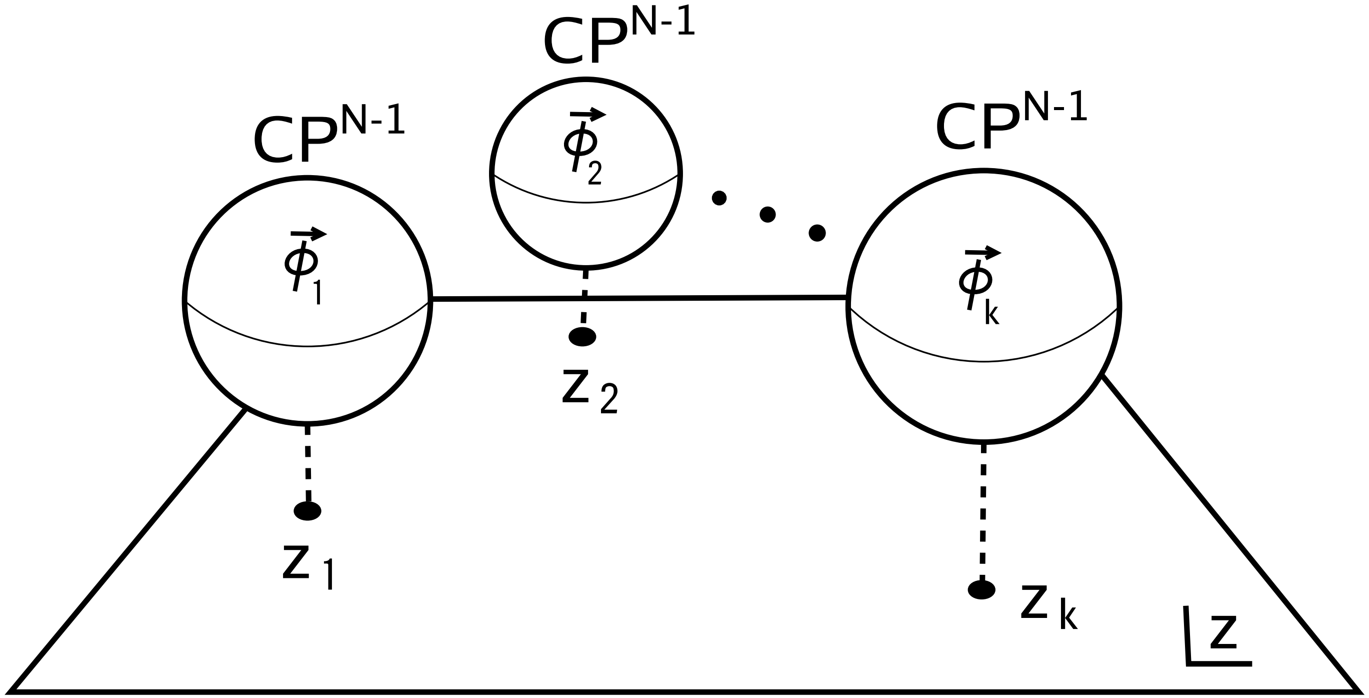
Full k-vortex moduli space in $U(N)$ gauge theory:

TiTech group (moduli matrix formalism): PRL [hep-th/0511088]

$$\mathcal{M}_{N,k} \leftarrow \left(\mathbf{C} \times \mathbf{C}P^{N-1}\right)^k / \mathfrak{S}_k \tag{16}$$

full space   **separated = symmetric product**

smooth   **very singular** (“$\leftarrow$” = resolution of sing.)



For Abelian (ANO) $N = 1$, $\mathcal{M}_{N=1,k} \simeq \mathbf{C}^k/\mathfrak{S}_k$.

1. How are the orbifold singularities resolved in $\mathcal{M}_{N,k}$??

2. How do NA vortices collide?

$\Downarrow$

The moduli matrix provides all necessary tools.

interlude

Separated k-instantons in $U(N)$ gauge theory on NC $\mathbf{R}^4$:

$$\mathcal{I}_{N,k} \leftarrow \left(\mathbf{C}^2 \times T^*\mathbf{C}P^{N-1}\right)^k / \mathfrak{S}_k \tag{17}$$

full space **separated = symmetric product**

smooth **very singular**

NC instantons: "Hilbert scheme" (H.Nakajima)

Confined Monopoles Tong('03), Shifman-Yung('04)

The Bogomol'nyi bound (Higgs H masses, and adj. Higgs Σ introduced)

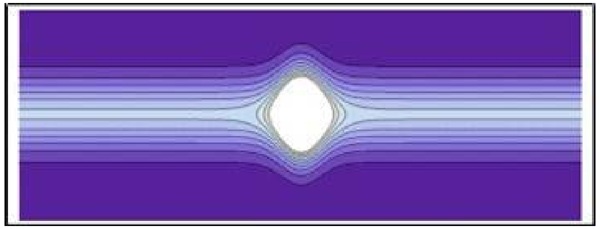
$$\mathcal{H} \geq \underbrace{\mathrm{tr}[\partial_3(c\Sigma)]}_{\text{walls}} - \underbrace{c\,\mathrm{tr}[B_3]}_{\text{vortices}} + \underbrace{\frac{1}{g^2}\mathrm{tr}[\partial_a(\Sigma B_a)]}_{\text{monopoles}}, \qquad B_a \equiv \frac{1}{2}\epsilon_{abc}F_{bc}$$

1/4 BPS equations

$$0 = (\mathcal{D}_3 + \Sigma) H + HM, \qquad 0 = (\mathcal{D}_1 + i\mathcal{D}_2) H \quad (18)$$

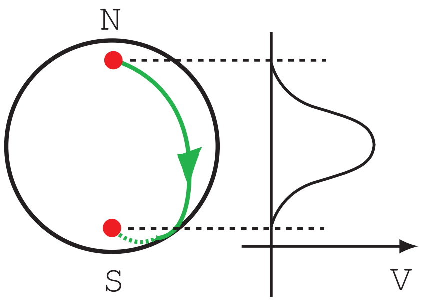
$$0 = B_3 - \mathcal{D}_3\Sigma + \frac{g^2}{2}(c - HH^\dagger) \quad (19)$$

$$0 = F_{23} - \mathcal{D}_1\Sigma = F_{31} - \mathcal{D}_2\Sigma \quad (20)$$

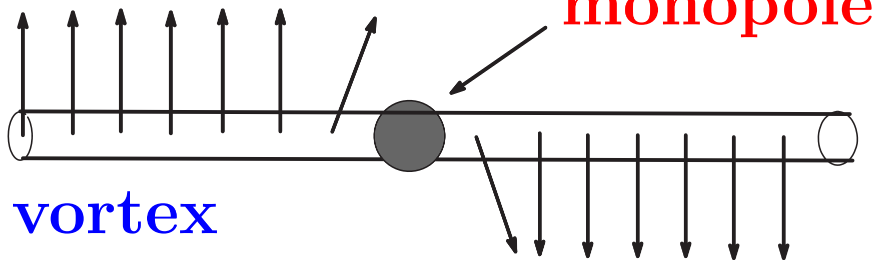


a numerical solution

kink in $\mathbf{C}P^1$

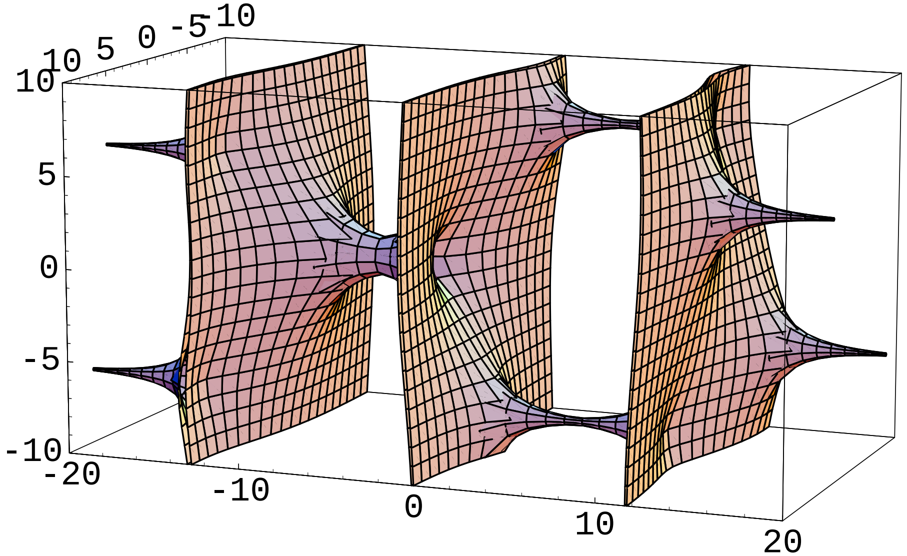


$\Longrightarrow$



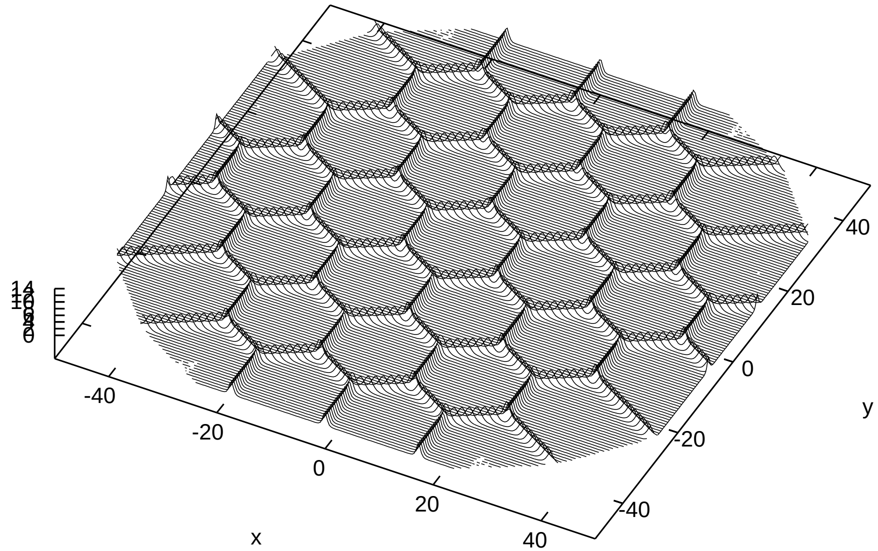
Composite Solitons

TITech PRD[hep-th/0405129]
domain wall+vortex
"*D-brane soliton*"
exact(analytic) solution



resembling with D-brane in superstring theory.

TITech PRD[hep-th/0506135]
Domain wall network
exact(analytic) solution



interlude: **Vortex Eqs. in Higher Dim.** **PRD [hep-th/0412048]**

$d = 4 + 1$ $U(N_{\rm C})$ **with** $N_{\rm F}$ **fund Higgs**

The Bogomol'nyi bound

$$\mathcal{E} \geq \mathrm{tr}\left[\underbrace{-c(F_{13} + F_{24})}_{\textbf{vortices}} + \underbrace{\frac{1}{2g^2} F_{mn}\tilde{F}_{mn}}_{\textbf{instantons}}\right], \tag{21}$$

1/4 BPS equations **(**W_M**: gauge fields)**

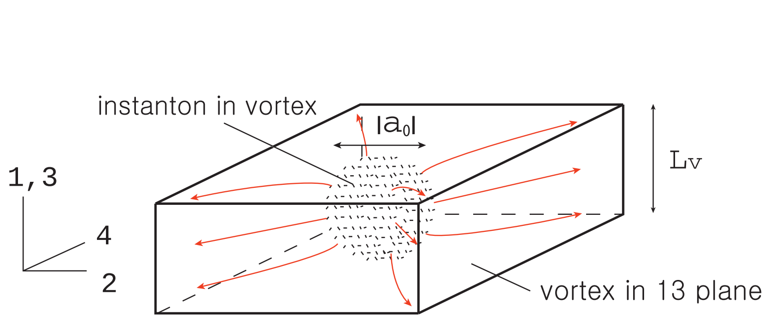
$$F_{12} = F_{34}, \quad F_{23} = F_{14}, \quad F_{13} + F_{24} = -\frac{g^2}{2}\left[c\mathbf{1}_{N_{\rm C}} - HH^\dagger\right]$$
$$\bar{\mathcal{D}}_z H = 0, \quad \bar{\mathcal{D}}_w H = 0. \tag{22}$$

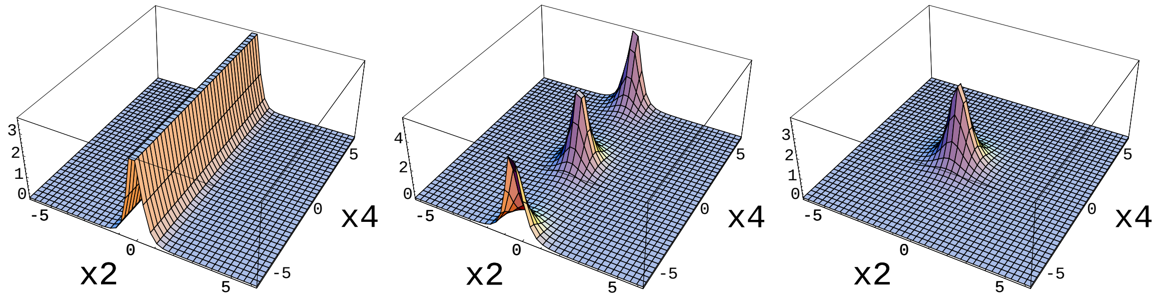
- **Set** $c = 0$, $H = 0 \Rightarrow$ **The SDYM eq. for instantons**
- **Ignore** x^2, x^4 **dep. and** W_2 **and** $W_4 \Rightarrow$ **vortices in** $z = x^1 + ix^3$**.**
- **Ignore** x^1, x^3 **dep. and** W_1 **and** $W_3 \Rightarrow$ **vortices in** $w = x^2 + ix^4$**.**

- **Related to** $d = 6$ **Donaldson-Uhlenbeck-Yau Eqs. at least in the case of** $U(1)$ **gauge th. by** S^2 **equivariant dim. red. (Comm. with A.D.Popov.)**

Instantons + (Intersecting) Vortices PRD [hep-th/0412048]

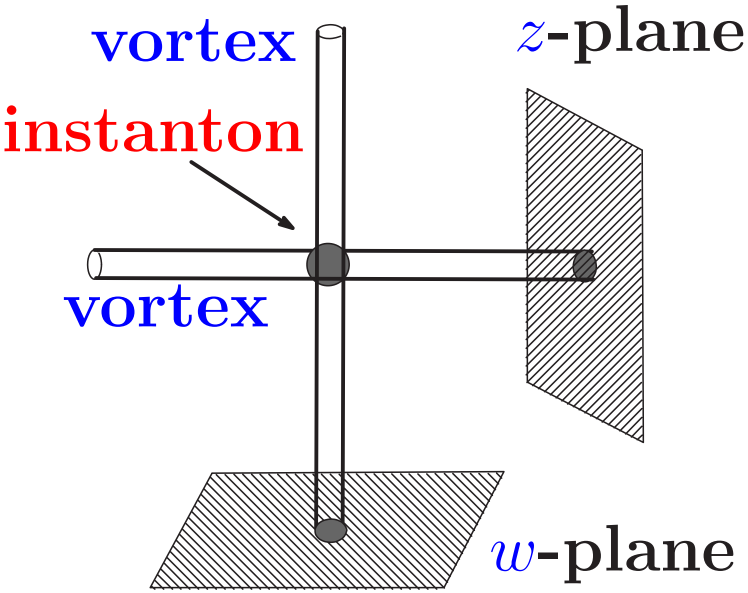
trapped instantons = lumps (CP^1 instantons) in vortex th.

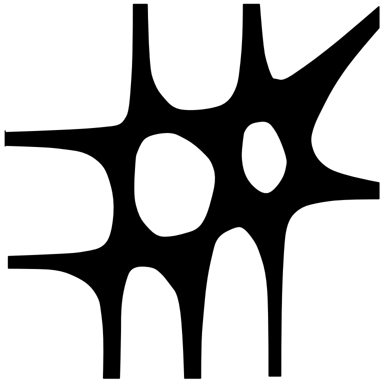


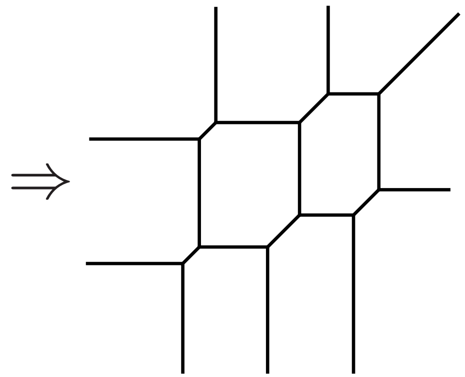


mono-string caloron instanton

Intersecting vortex-membranes with negative instanton charge







Amoeba ⇒ tropical geometry
K.Ohta-Yamazaki + TiTech,
PRD [arXiv:0805.1194]

interlude: **Classification of All BPS eqs** **NPB [hep-th/0506257]**

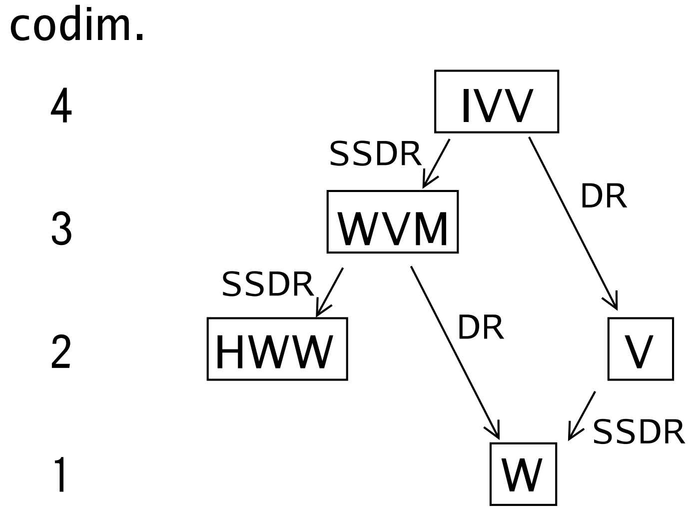
$d = 5+1$: **only vortices and instantons are allowed.**

1/4 **BPS** IVV	0	1	2	3	4	5
Instanton	○	×	×	×	×	○
Vortex	○	×	×	○	○	○
Vortex	○	○	○	×	×	○

1/4 **BPS** VVV	0	1	2	3	4	5
Vortex	○	○	×	×	○	○
Vortex	○	×	○	×	○	○
Vortex	○	×	×	○	○	○

1/8 **BPS** IV^6	0	1	2	3	4	5
Instanton	○	×	×	×	×	○
Vortex	○	○	×	×	○	○
Vortex	○	×	○	×	○	○
Vortex	○	×	×	○	○	○
Vortex	○	×	○	○	×	○
Vortex	○	○	×	○	×	○
Vortex	○	○	○	×	×	○

Dimensional Reduction



The left 1/4 BPS eqs. give previously known BPS eqs. in $d \leq 5$ by dim. reductions. Others are all new!

interlude: Similar non-Abelian vortices in hadron physics

high baryon density QCD (color superconductor)

$$\Phi_{\alpha i} \sim \epsilon_{\alpha\beta\gamma}\epsilon_{ijk}\langle q_j^{T\beta} C\gamma_5 q_k^{\gamma}\rangle \sim v\mathbf{1}_3$$

$U(1)_{\rm B} \times SU(3)_{\rm C} \times SU(3)_{\rm F} \to SU(3)_{\rm C+F}$ **Alford-Rajagopal-Wilczek ('99)**

1. **NA vortices Balachandran, Digal and Matsuura ('05)**
   (a) $U(1)_{\rm B}$ **is** *global*: **superfluid vortex (log div etc)**
   (b) **non-Abelian magnetic flux**
2. **C**P^2 **orientation, long range repulsive force, lattice**
   Nakano, MN and Matsuura, PRD [arXiv:0708.4096 [hep-ph]]
3. **The core of neutron (or quark) stars**
   Sedrakian, Blaschke *et al* **[arXiv:0810.3003 [hep-ph]]**

interlude: Non-Abelian global vortices

1. **high temperature QCD (chiral phase transition)**

   $U(1)_{\rm A} \times SU(3)_{\rm L} \times SU(3)_{\rm R} \to SU(3)_{\rm L+R}$ **($\leftarrow$ all global symmetry)**

   Balachandran and Digal('02), MN and Shiiki('07)

   CP^2-dependent repulsion

   Nakano, MN and Matsuura, PLB [arXiv:0708.4092 [hep-ph]]

2. **superfluid of ^{3}He in the B-phase**

   $U(1)_\Phi \times SO(3)_{\rm S} \times SO(3)_{\rm L} \to SO(3)_{\rm S+L}$ **(See Volovik's book)**

$$\frac{G}{H} = \frac{U(1)_\Phi \times SO(3)_{\rm S} \times SO(3)_{\rm L}}{SO(3)_{\rm S+L}} \simeq SO(3) \times U(1) \qquad (23)$$

$$\pi_1(G/H) = \mathbf{Z} \oplus \mathbf{Z}_2 \qquad (24)$$

§3 Moduli Matrix Formalism

PRL[hep-th/0511088], J.Phys.A [hep-th/0602170]

Solving the vortex eqs: $0 = (\mathcal{D}_1 + i\mathcal{D}_2)H, \quad 0 = F_{12} + \frac{g^2}{2}(c\mathbf{1}_N - HH^\dagger).$

The 1st eq. can be solved: $(z \equiv x^1 + ix^2)$

$$H = S^{-1}H_0(z), \quad A_1 + iA_2 = -i2S^{-1}\bar{\partial}_z S, \tag{25}$$

$$S = S(z,\bar{z}) \in GL(N_{\rm C}, \mathbf{C}). \tag{26}$$

The 2nd eq. $\Rightarrow$

$$\partial_z(\Omega^{-1}\bar{\partial}_z\Omega) = \frac{g^2}{4}(c\mathbf{1}_{N_{\rm C}} - \Omega^{-1}H_0H_0^\dagger), \tag{27}$$

$$\Omega(z,\bar{z}) \equiv S(z,\bar{z})S^\dagger(z,\bar{z}) \tag{28}$$

The V-transformations $[V(z) \in GL(N_{\rm C}, \mathbf{C})$ **for** $^\forall z \in \mathbf{C}]$**:**

$$H_0(z) \to H_0'(z) = V(z)H_0(z), \quad S(z,\bar{z}) \to S'(z,\bar{z}) = V(z)S(z,\bar{z}), \tag{29}$$

$H_0(z)$: the moduli matrix, **(27): the master equation.**

For $U(1)$ ($N = 1$) the master eq. $\rightarrow$ the Taubes equation:

$$\text{by } c\Omega(z,\bar{z}) = |H_0|^2 e^{-\xi(z,\bar{z})} \text{ with } H_0 = \prod_i (z - z_i).$$

The equation admits the unique solution. Taubes ('80)

We assume that the master equation admits the unique solution. This

- is consistent with the index theorem (Hanany-Tong),
- was rigorously proven for vortices in arbitrary gauge group on compact Riemann surfaces. (the Hitchin-Kobayashi correspondence).

  Mundet i reira, Cieliebak-Gaito-Salamon ('00)
- has been checked for our $U(N)$ vortices on compact Riemann surfaces. Baptista ('08: arXiv:0810.3220 [hep-th])

All moduli parameters are encoded in $H_0(z)$

interlude: Non-integrability of the master eq., Inami-Minakami-MN('06)
"half integrability" $\rightarrow$ half integrable hierarchy?

The conditions on H_0 for vortex number k:

$$k = \frac{1}{2\pi}\mathrm{Im}\oint dz\ \partial \log(\det H_0). \tag{30}$$

$$\Rightarrow \det(H_0) \sim z^k \ \ (\textbf{for } z \to \infty) \quad \Rightarrow \quad \det H_0(z) = \prod_{i=1}^{k}(z - z_i), \tag{31}$$

The moduli space of k-vortices in $U(N)$ gauge theory:

$$\mathcal{M}_{N,k} = \frac{\{H_0(z)|\deg\left(\det(H_0(z))\right) = k\}}{\{V(z)|\det V(z) = 1\}} \tag{32}$$

This is equivalent to one obtained in string theory:
PRL[hep-th/0511088], J.Phys.A [hep-th/0602170]

$$\mathcal{M}_{N,k} \simeq \left\{Z, \Psi\right\} /\!/ GL(k,\mathbb{C})$$

Z **adjoint** $(k \times k)$ **and** Ψ **fundamental** $(N \times k)$

Caution: This is topologically correct. The flat metric on Z, ψ does not give correct metric on the moduli space.

$U(2), k = 1$ (single vortex in $U(2)$ gauge theory):

$$\mathcal{M}_{N=2,k=1} \simeq \mathbf{C} \times \mathbf{C}P^1 \tag{33}$$

The moduli matrices for $\mathcal{M}_{N=2,k=1}$:

$$H_0^{(1,0)}(z) = \begin{pmatrix} z - z_0 & 0 \\ -b' & 1 \end{pmatrix}, \qquad H_0^{(0,1)}(z) = \begin{pmatrix} 1 & -b \\ 0 & z - z_0 \end{pmatrix} \tag{34}$$

z_0**: vortex position on z.** $(\det H_0 = z - z_0)$
b, b'**: vortex orientation $\mathbf{C}P^1$.**

In general, a V-tr. gives transition functions:

$$V = \begin{pmatrix} 0 & -1/b' \\ b' & z - z_0 \end{pmatrix} \in GL(2, \mathbf{C}) \rightarrow b = 1/b'. \tag{35}$$

$U(2), k=2$ (2-vortices in $U(2)$ gauge) PRD [hep-th/0607070]

$$\mathcal{M}_{N=2,k=2} \leftarrow \left(\mathbf{C}\times \mathbf{C}P^1\right)^2/\mathfrak{S}_2 \tag{36}$$

general $k=2$, $\det H_0 \sim z^2$	$\Rightarrow$	**coincident** $k=2$, $\det H_0 = z^2$
$\mathcal{M}_{N=2,k=2}$	$\supset$	$W\mathbf{C}P^2{}_{(2,1,1)} \simeq \mathbf{C}P^2/\mathbf{Z}_2$

$$H_0^{(2,0)} = \begin{pmatrix} z^2 - \alpha' z - \beta' & 0 \\ -a' z - b' & 1 \end{pmatrix}$$
$$H_0^{(1,1)} = \begin{pmatrix} z-\phi & -\eta \\ -\tilde{\eta} & z-\tilde{\phi} \end{pmatrix}$$
$$H_0^{(0,2)} = \begin{pmatrix} 1 & -a z - b \\ 0 & z^2 - \alpha z - \beta \end{pmatrix}$$

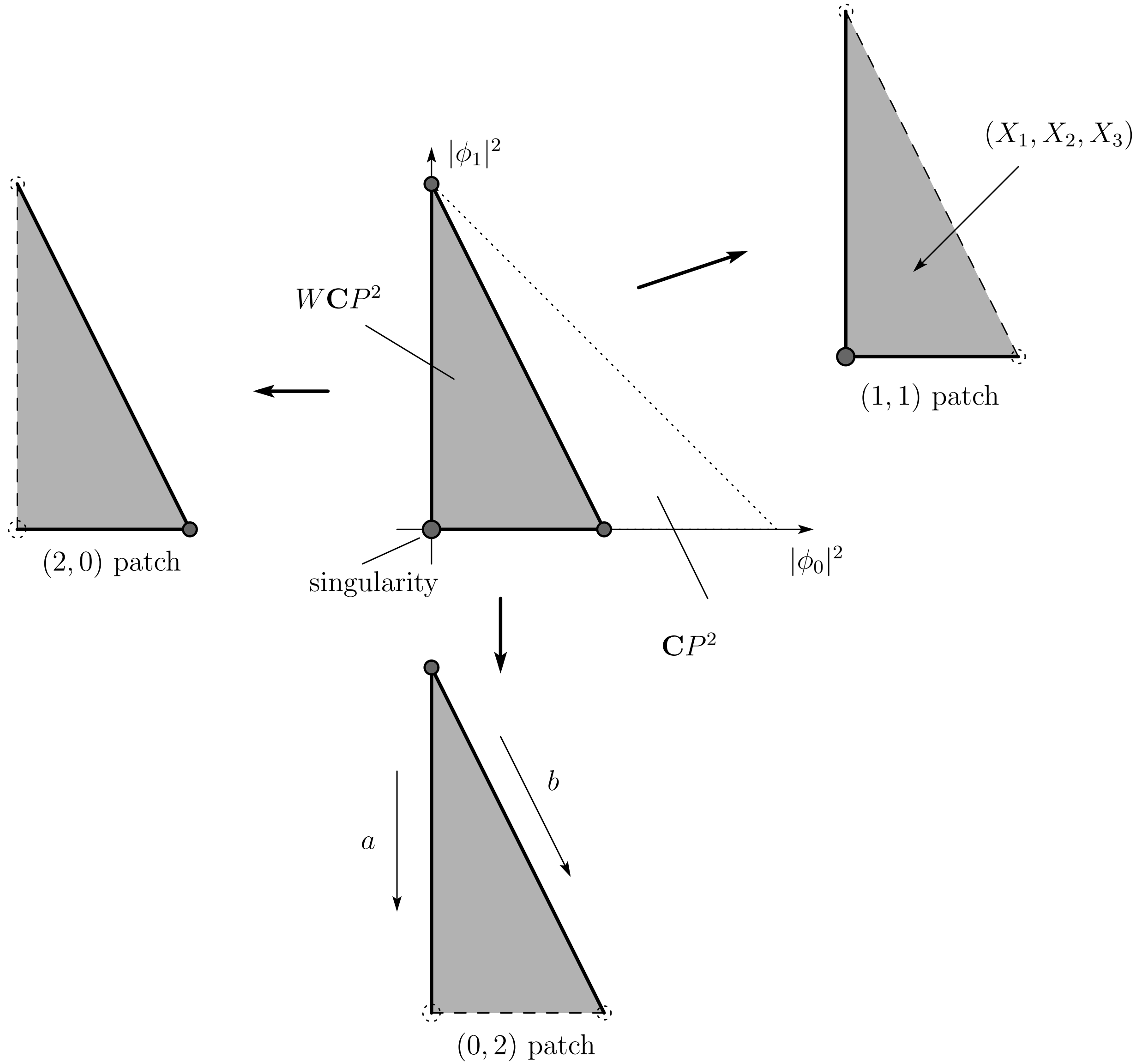
$\Rightarrow$

$$\tilde{H}_0^{(2,0)} = \begin{pmatrix} z^2 & 0 \\ -a' z - b' & 1 \end{pmatrix}$$
$$\tilde{H}_0^{(1,1)} = \begin{pmatrix} z-\phi & -\eta \\ -\tilde{\eta} & z+\phi \end{pmatrix}$$
with $\phi^2 + \eta\tilde{\eta} = 0$,
$$\tilde{H}_0^{(0,2)} = \begin{pmatrix} 1 & -a z - b \\ 0 & z^2 \end{pmatrix}$$

three patches $\mathcal{U}^{(2,0)} = \{a', b', \alpha', \beta'\}$ $\mathcal{U}^{(1,1)} = \{\phi, \tilde{\phi}, \eta, \tilde{\eta}\}$, $\mathcal{U}^{(0,2)} = \{a, b, \alpha, \beta\}$.

$XY \equiv -\phi, X^2 \equiv \eta, Y^2 \equiv -\tilde{\eta}$

$(X,Y) \sim (-X,-Y)$ $\mathbf{Z}_2$ **sing**



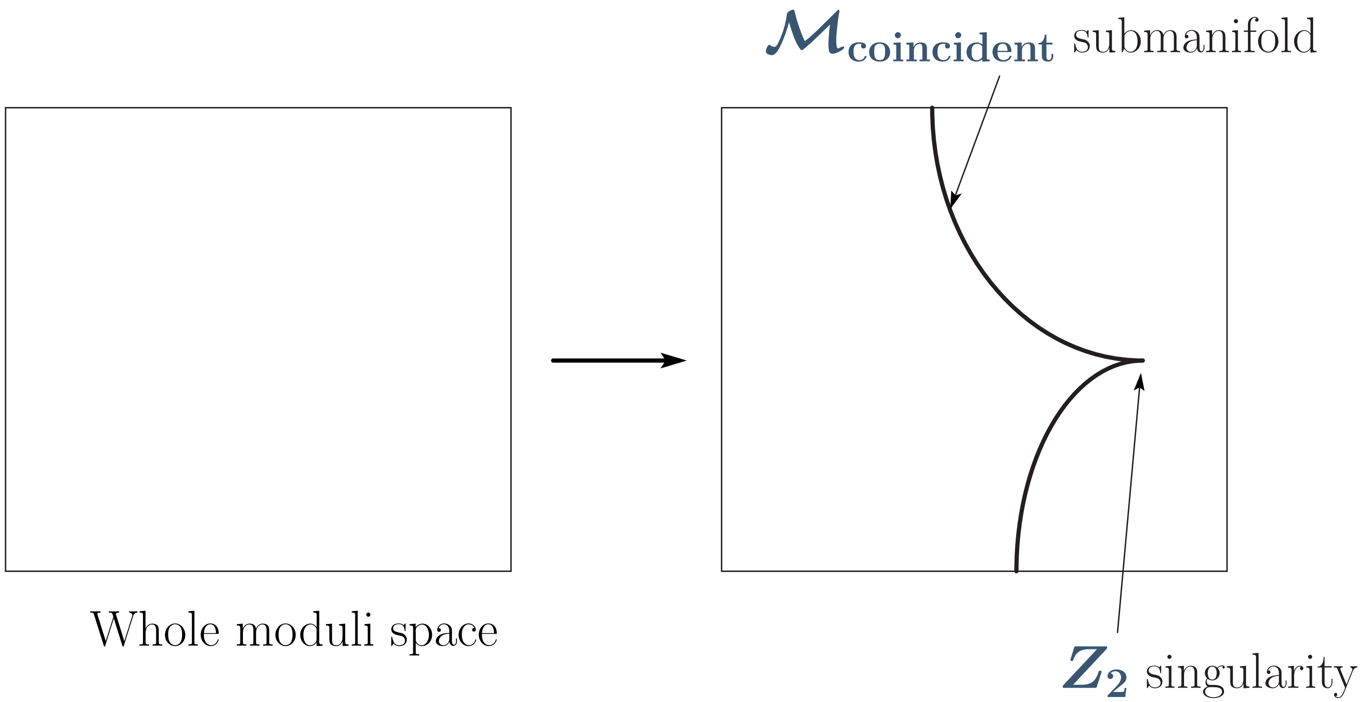
$$\tilde{\mathcal{U}}^{(2,0)} \simeq \mathbf{C}^2, \quad \tilde{\mathcal{U}}^{(1,1)} \simeq \mathbf{C}^2/\mathbf{Z}_2, \quad \tilde{\mathcal{U}}^{(0,2)} \simeq \mathbf{C}^2.$$

Solving the master eq. at the $\mathbf{Z}_2$ sing. PRL [hep-th/0609214]

$$K = 2\pi c(|\phi|^2 + |\tilde{\phi}|^2 + |\eta|^2 + |\tilde{\eta}|^2) + \textbf{higher} \Longrightarrow \text{smooth} \tag{37}$$

$$\mathcal{M}_{N=2,k=2} \simeq \left(\mathbf{C} \times \mathbf{C}P^1\right)^2 / \mathfrak{S}_2 \quad \cup \quad \mathbf{C} \times W\mathbf{C}P^2_{(2,1,1)} \tag{38}$$

↑ **smooth** ↑ **very singular** ↑ **$\mathbf{Z}_2$ singular**



interlude: Kähler metric of vortex eff.th. PRD [hep-th/0602289]

general formula for the Kähler potential

$$K = \underbrace{\int d^2 z}_{\text{integral over codim}} \quad \mathrm{Tr}\Big[-2c\mathbf{V} + e^{2\mathbf{V}} \mathbf{H}_0 \mathbf{H}_0^\dagger + \underbrace{\frac{16}{g^2} \int_0^1 dx \int_0^x dy \bar{\partial} \mathbf{V} e^{2y L_{\mathbf{V}}} \partial \mathbf{V}}_{\textbf{WZ-like term}} \Big], \quad (39)$$

Elimination of V gives the result.

- **infinite dimensional Kähler quotient** $\mathbf{V}(x, \theta, \bar{\theta})$
- **EOM of V = the master equation (miracle)**

The Kähler metric

$$\delta^{\dagger\mu} \delta_\mu K \Big|_{\Omega=\Omega_{\mathrm{sol}}}$$

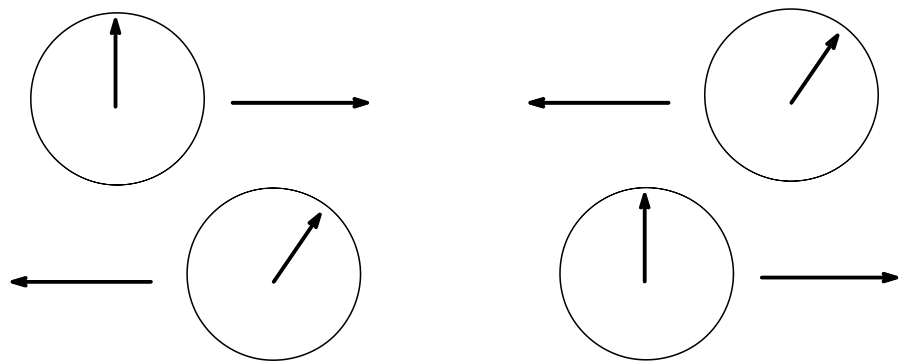
$$= \int d^2 z \mathrm{Tr}\left[\delta^{\dagger\mu} \delta_\mu c \log \Omega \right.$$

$$\left. + \frac{4}{g^2} \left\{ \partial \left(\delta^\mu \Omega \Omega^{-1} \right) \delta_\mu^\dagger \left(\bar{\partial} \Omega \Omega^{-1} \right) - \partial (\bar{\partial} \Omega \Omega^{-1}) \delta_\mu^\dagger \left(\delta^\mu \Omega \Omega^{-1} \right) \right\} \right] \Big|_{\Omega=\Omega_{\mathrm{sol}}}, \quad (40)$$

Dynamics (Scattering/Reconnection) PRL [hep-th/0609214]

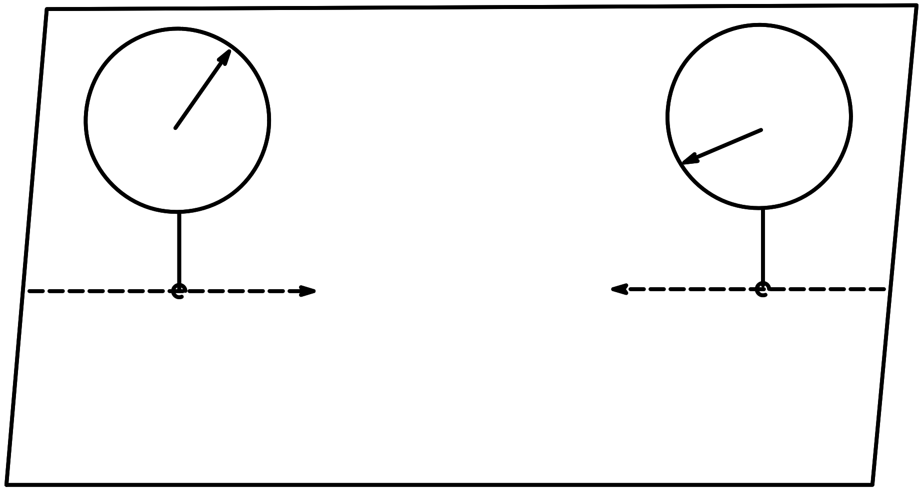
1. Do they pass through or scatter at right angles, when two vortices collide in head-on collisions??
2. What are roles of orientation moduli?

1. When two orientations are aligned ($\sim$ Abelian case).
   $\Rightarrow$ they would scatter at right angles
2. When two orientations are not aligned
   $\Rightarrow$ they would pass through

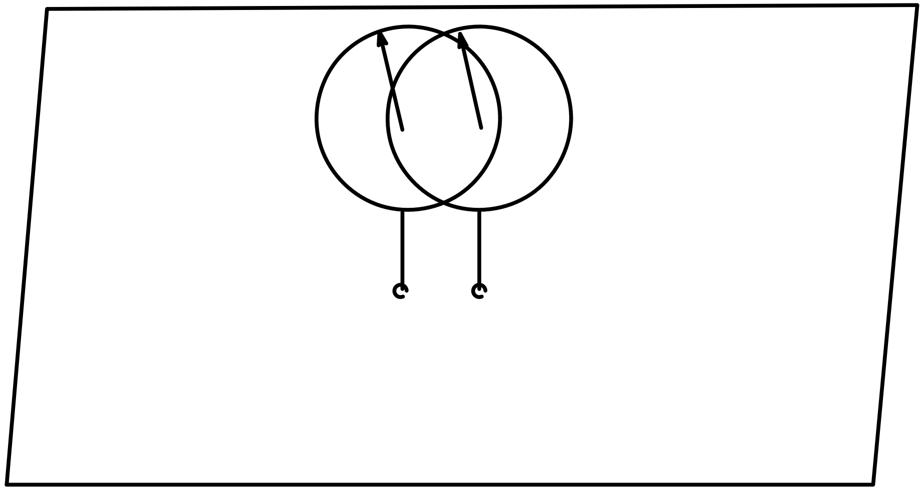


Naively thinking, the 2nd occurs for generic initial cond.

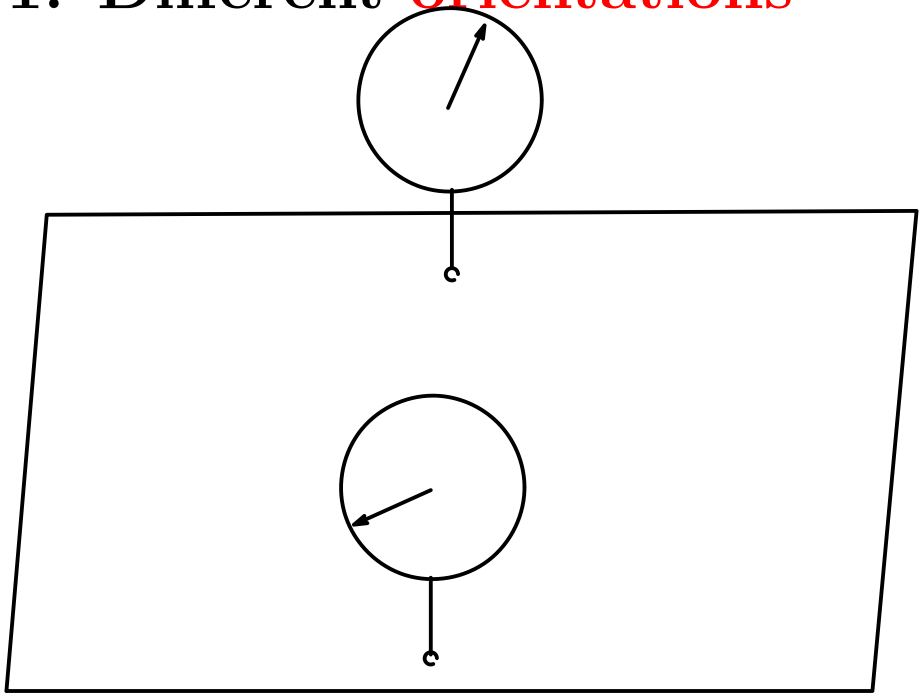
Approximate geodesics by straight lines linearly before and after the collision moment $t = 0$. A short time behavior is OK (a long time is difficult).



1. Different orientations



2. Orientations become parallel in the collision.



3. Scatter with right angle!!

The (0,2) patch:

$$H_0^{(0,2)} = \begin{pmatrix} 1 & -a\,z - b \\ 0 & z^2 - \alpha\,z - \beta \end{pmatrix}. \tag{41}$$

Free motion:

$$a = a_0 + \epsilon_1 t + \mathcal{O}(t^2), \qquad b = b_0 + \epsilon_2 t + \mathcal{O}(t^2), \tag{42}$$

$$\alpha = 0 + \mathcal{O}(t^2), \qquad \beta = \epsilon_3 t + \mathcal{O}(t^2), \tag{43}$$

Relations to positions z_i, orientations b_i are:

$$a = \frac{b_1 - b_2}{z_1 - z_2},\ b = \frac{b_2 z_1 - b_1 z_2}{z_1 - z_2},\ \alpha = z_1 + z_2,\ \beta = -z_1 z_2. \tag{44}$$

$$z_1 = -z_2 = \sqrt{\epsilon_3 t} + \mathcal{O}(t^{3/2}), \tag{45}$$

$$b_i = b_0 + (-1)^{i-1} a_0 \sqrt{\epsilon_3 t} + \mathcal{O}(t), \qquad (i = 1, 2). \tag{46}$$

The 1st: the right-angle scattering.
The 2nd: as vortices approach each other in the real space, the orientations b_i approach each other b_0!!

The (1,1) patch:

$$H_0^{(1,1)} = \begin{pmatrix} z-\phi & -\eta \\ -\tilde{\eta} & z-\tilde{\phi} \end{pmatrix}. \tag{47}$$

$$\phi = -\tilde{\phi} = -XY + s_1 t + \mathcal{O}(t^2), \tag{48}$$

$$\eta = X^2 + s_2 t + \mathcal{O}(t^2), \qquad \tilde{\eta} = -Y^2 + s_3 t + \mathcal{O}(t^2), \tag{49}$$

1) $(X,Y) \neq 0$ **(generic; the same result with the (0,2) patch)**

$$z_1 = -z_2 = \sqrt{\phi^2 + \eta\tilde{\eta}} = \sqrt{st} + \mathcal{O}(t^{3/2}), \tag{50}$$

$$b_i = XY^{-1} + (-1)^i Y^{-2}\sqrt{st} + \mathcal{O}(t), \tag{51}$$

2) $(X,Y) = 0$ **(fine tuned collision)**

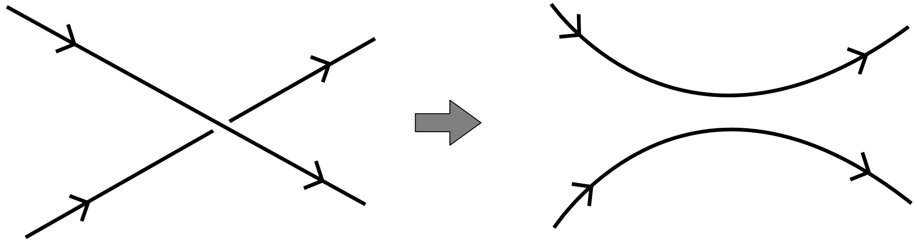
$$z_1 = -z_2 = \sqrt{s_1^2 + s_2 s_3}\;t + \mathcal{O}(t^{3/2}), \tag{52}$$

$$b_i = s_1 s_3^{-1} + (-1)^{i-1} s_3^{-1}\sqrt{s_1^2 + s_2 s_3} + \mathcal{O}(t^{1/2}), \tag{53}$$

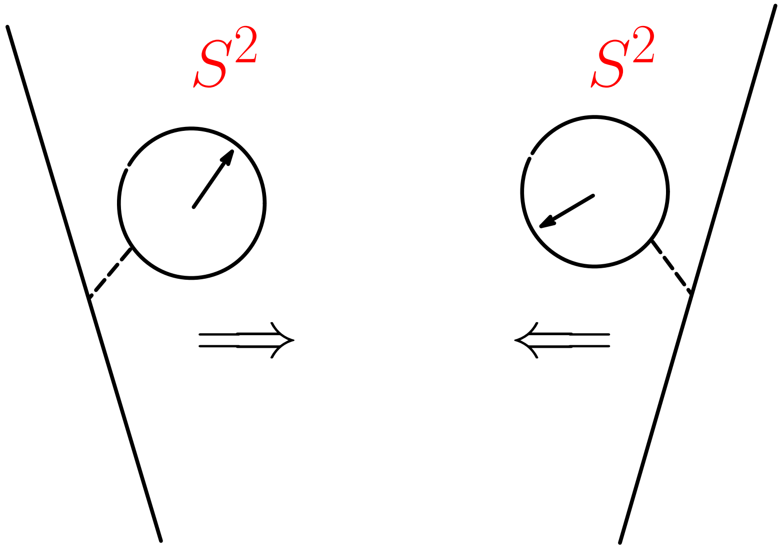
They pass through with arbitrary orientations $b_1 \neq b_2$**.**

Non-Abelian Cosmic Strings PRL [hep-th/0609214]

Abelian cosmic strings reconnect ⇒ no cosmic string problem



Do two non-Abelian strings reconnect?



no reconnection? ⇒ cosmic string problem?? (Polchinski)

The reconnection always occurs

Representation Theory in preparation

$$\mathbf{C}P^{N-1} \Leftrightarrow \mathbf{N}$$

$U(2), k = 2$ **collision:** $\mathbf{2} \otimes \mathbf{2} = \mathbf{3} \oplus \mathbf{1}$**?**

Promote color-flavor symmetry z-dependent (*loop group*)

1. Separated: all orientation moduli are *connected*

2. Coincident: orientation moduli are *decomposed* $\mathbf{2} \otimes \mathbf{2} = \mathbf{3} \oplus \mathbf{1}$

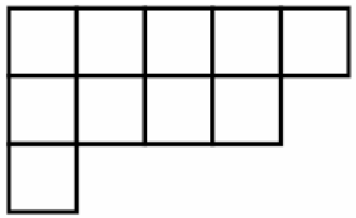
$$H_0 = \underset{\mathbf{3}}{\begin{pmatrix} z^2 & 0 \\ 0 & 1 \end{pmatrix}} \quad \underset{\oplus}{\text{or}} \quad \underset{\mathbf{1}}{\begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}} \tag{54}$$

$U(N), k:$

$$H_0 = \begin{pmatrix} z^{k_1} & 0 & \cdots & \\ 0 & z^{k_2} & & \\ \vdots & & \ddots & \\ & & & z^{k_N} \end{pmatrix} \tag{55}$$

$$k = \sum_i^N k_i, \quad k_1 \geq k_2 \geq \cdots \geq k_N$$

$\Longleftrightarrow$



Young diagram

as if YM instantons

Arbitrary Gauge Groups PLB [arXiv:0802.1020]

Condition on local vortices for $SO(2M), USp(2M)$

(all invariants must have common zeros)

$$H^T_{0,\mathrm{local}}(z) J H_{0,\mathrm{local}}(z) = \prod_{\ell=1}^{k}(z - z_{0\ell})\, J. \tag{56}$$

$$J = \begin{pmatrix} \mathbf{0}_M & \mathbf{1}_M \\ \epsilon \mathbf{1}_M & \mathbf{0}_M \end{pmatrix}, \tag{57}$$

$\epsilon = +1$ **for** $SO(2M)$
$\epsilon = -1$ **for** $USp(2M)$

$$\Downarrow$$

$$H_{0,\mathrm{local}} = \begin{pmatrix} (z-a)\mathbf{1}_M & 0 \\ \mathbf{B}_{A/S} & \mathbf{1}_M \end{pmatrix}, \quad \frac{SO(2M)}{U(M)}, \quad \frac{USp(2M)}{U(M)} \tag{58}$$

We have also constructed multiple vortices.

Arbitrary groups, including exceptional: E_6, E_7, E_8, F_4, G_2

G'	$SU(N)$	$SO(2M+1)$	$USp(2M), SO(2M)$	E_6	E_7	E_8	F_4	G_2
N	N	$2M+1$	$2M$	27	56	248	26	7
$C_{G'}$	$\mathbf{Z}_N$	$\mathbf{1}$	$\mathbf{Z}_2$	$\mathbf{Z}_3$	$\mathbf{Z}_2$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
ν	k/N	k	$k/2$	$k/3$	$k/2$	k	k	k

(cf: ADHM of YM instantons exists only for SU, SO, USp**)**

Many extensions

1. Composite solitons Hanany-Tong, Shifman-Yung, our group
2. 4D/2D correspondence Hanany-Tong, Shifman-Yung
3. dyonic NA vortices our group, Collie
4. semi-local NA strings Shifman-Yung, our group
5. $\mathcal{N} = 1$ theory Shifman-Yung, Eto-Hashimoto-Terashima, Tong
6. superconformal theory Tong
7. non-BPS NA vortices Auzzi-Eto-Vinci('07), Auzzi-Eto-Konishi et.al('08)
8. Chern-Simons coupling Schaposnik et.al, Collie-Tong('07)
9. gravity coupling Aldrovandi
10. Changing geometry
    (a) on a cylinder $\Rightarrow$ T-duality to walls our group
    (b) on T^2 $\Rightarrow$ statistical mechanics our group, Schaposnik et.al
    (c) on compact Riemann surface Popov('07), Baptista('08)
    (d) on a discrete space Ikemori-Kitakado-Otsu-Sato('08)

§4. Conclusion / Discussion

1. **$U(N)$ vortices in color-flavor locked phase,**
   - (a) **carry color flux and CP^{N-1} moduli,** Hanany-Tong, Konishi *et.al*
   - (b) **confine a monopole if Higgs masses are added,** Tong, Shifman-Yung
   - (c) **allow k-vortex moduli *conjectured* by D-branes** Hanany-Tong.
2. **The moduli matrix offers all necessary tools:**
   - (a) **general k-vortex solution and moduli space,**
   - (b) **equivalence to Kähler quotient (D-brane),**
   - (c) **general formula for Kähler metric on the moduli space,**
   - (d) **a detailed structure of $k = 2$ vortex moduli space**
     ($k = 2$ coincident moduli, resolution of orbifold singularity),
   - (e) **dynamics of $k = 2$ vortex,** reconnection of $U(N)$ cosmic strings,
   - (f) **(non-)normalizability of semi-local vortex moduli,**
   - (g) **1/4, 1/8 BPS composite solitons,**
   - (h) **the partition function of $U(N)$ vortices,**

3. **The moduli matrix also offers all necessary tools to construct vortices in $U(1) \times G'$ with arbitrary simple group G':**

   (a) **semi-local vortices for general G' (smaller than $SU(N)$),**

   (b) **single local vortex moduli spaces:**
   $\frac{SU(N)}{SU(N-1)\times U(1)}, \frac{SO(2M)}{U(M)}, \frac{USp(2M)}{U(M)}$

Discussion

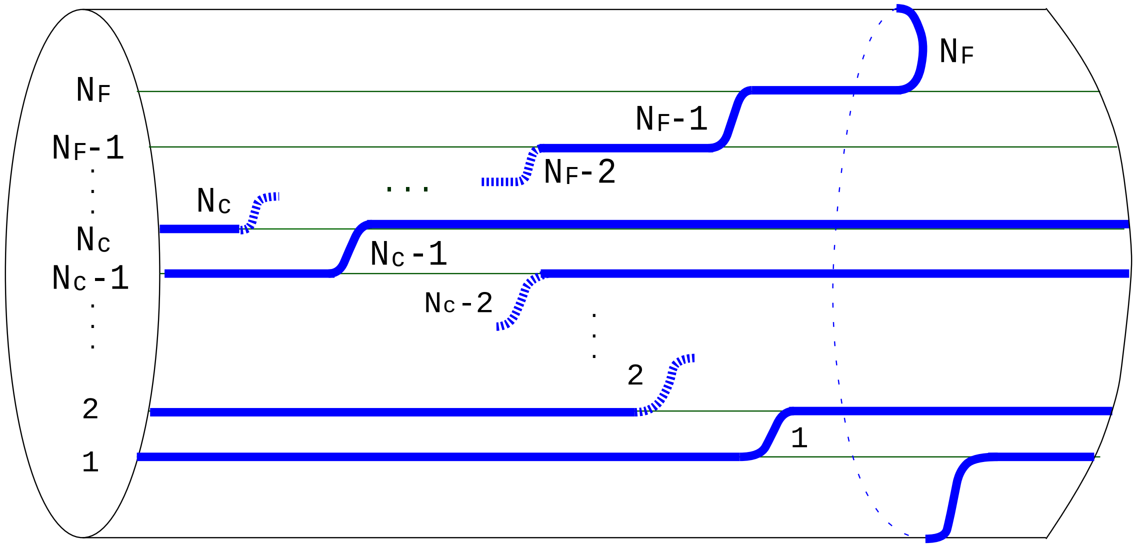
1. **Relation to SO, USp lumps arXiv:0809.2014 [hep-th]**
2. **More detailed study of SO, USp (multi,...), in preparation**
3. **Monopoles in the Higgs phase (1/4 BPS), wall-vortex comp. for general G'**
4. **toward a proof of GNO duality, in preparation**
5. **New kind of vortices = "fractional" vortices, in preparation**
6. **D-brane construction for SO, USp?**
   Kähler quotient (ADHM) for moduli

§App. T-Duality to Domain Walls and Partition Function

K.Ohta+TiTech, PRD [hep-th/0601181]

Vortices on a cylinder
T-dual ⇓
Domain walls

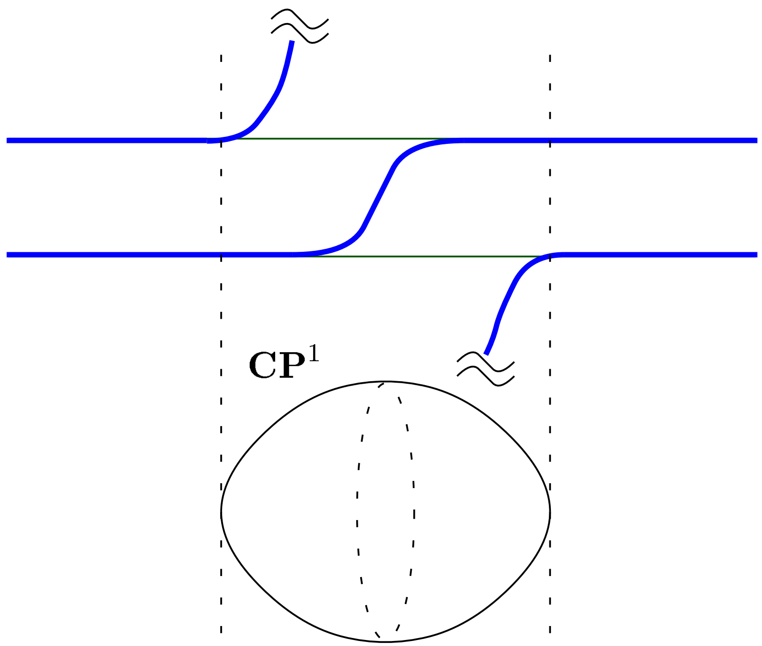
In a D-brane picture, vortices are D1-branes wrapping the cycle.



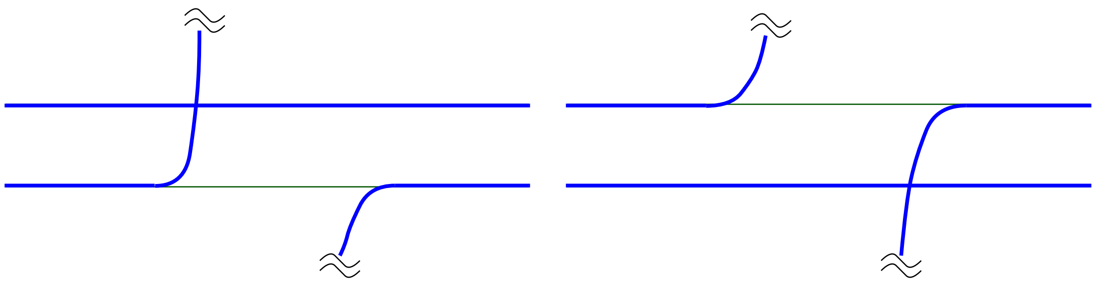
This picture is very nice to understand moduli space of vortices !

The moduli of a single vortex in $U(2)$ $N_{\mathrm{F}} = 2$

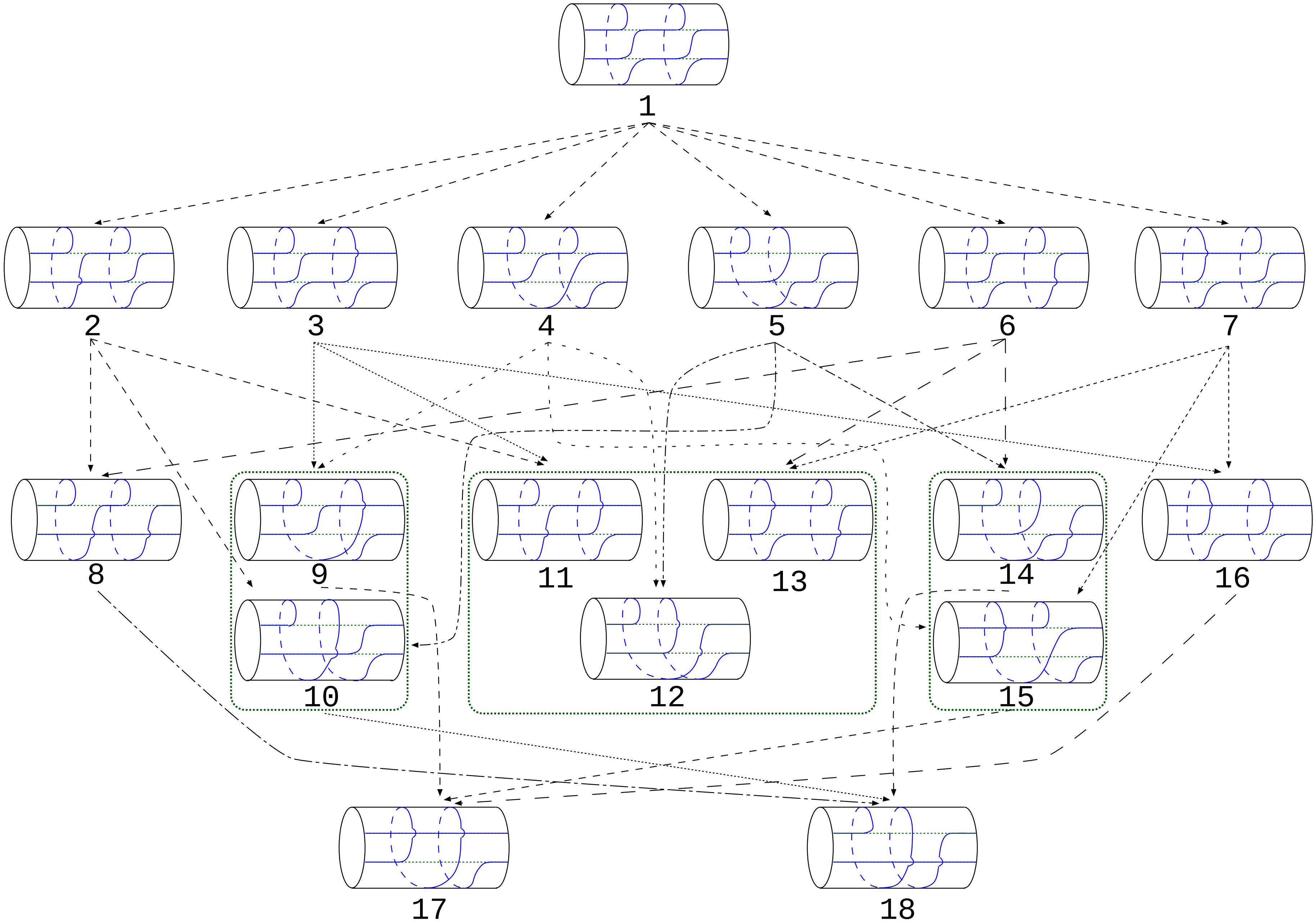
$\mathcal{M} \simeq \mathbf{R} \times S^1 \times \mathbf{C}P^1$



Two limits reduce to an Abrikosov-Nielsen-Olesen vortex;

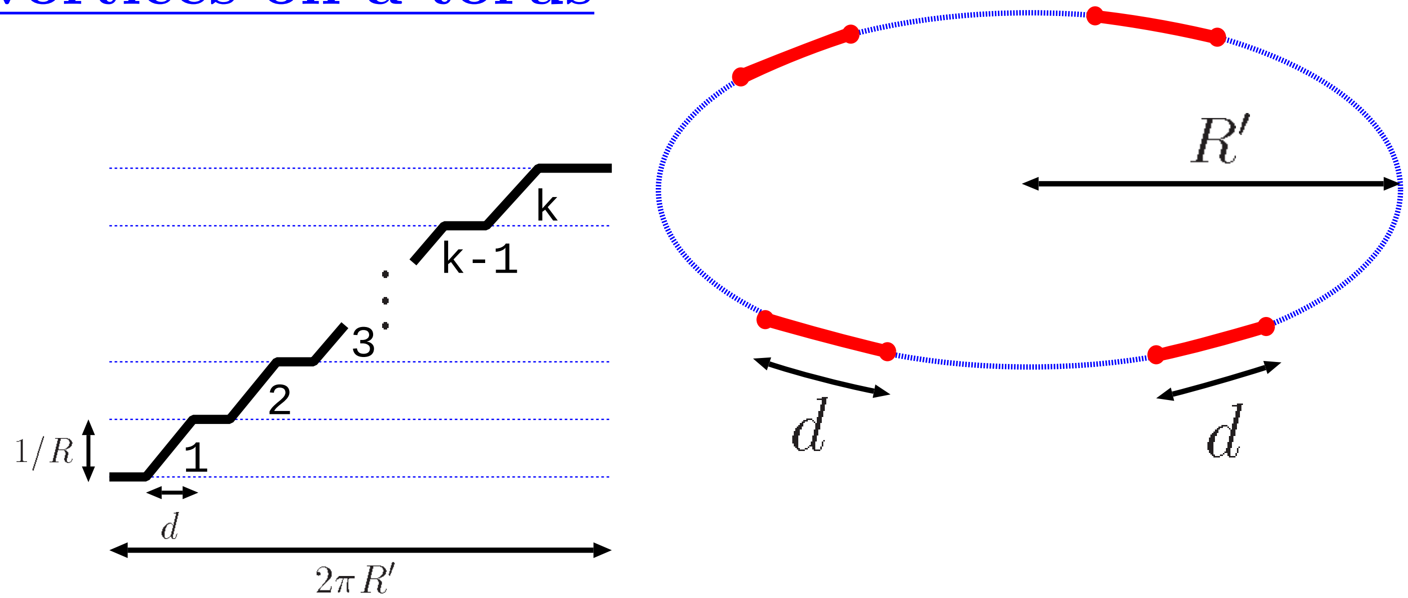


The moduli of double ($k = 2$) vortex in $U(2)$ $N_F = 2$



Partition function K.Ohta+TiTech, NPB[hep-th/0703197]

Abelian k vortices on a torus



gas of 1D hard rods

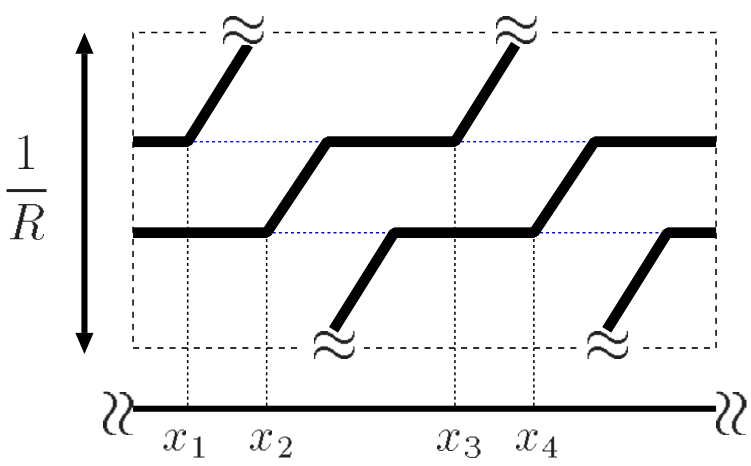
Patition function:

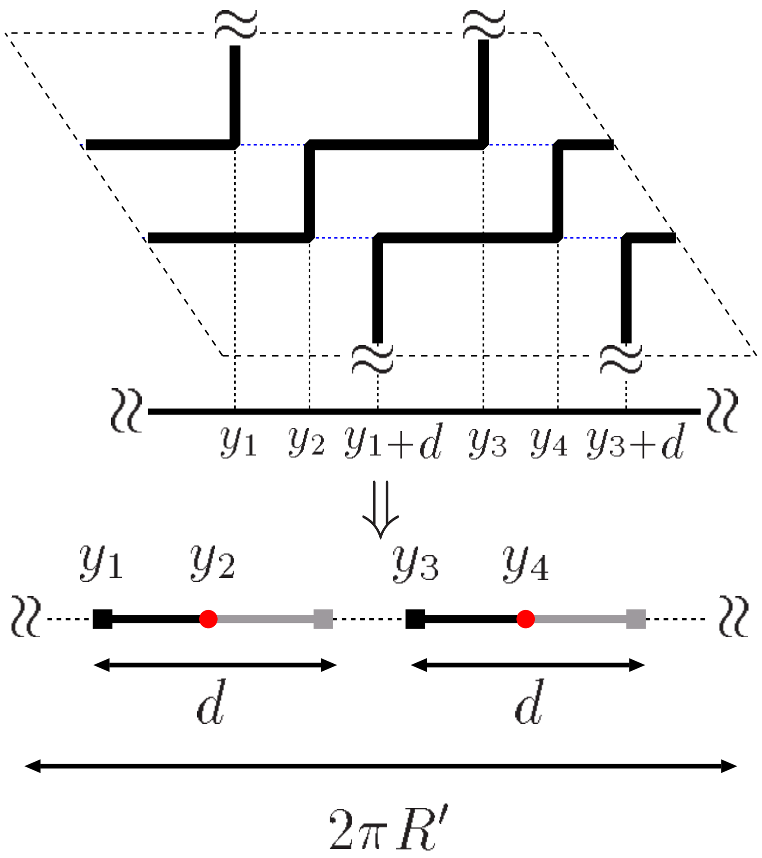
$$Z_{k,T^2}^{N_C=N_F=1} = \frac{1}{k!}(cT)^k A \left(A - \frac{4\pi k}{g^2 c} \right)^{k-1}, \qquad (59)$$

A: **Area of the torus**

$\Rightarrow$ **coinciding with the Manton's result, explaining why 1D.**

Non-Abelian Vortices on a torus ($N_{\rm C}=N_{\rm F}=2,\ k=2$)





1D soft rods with hard pieces

$$Z^{N_{\rm C}=2,N_{\rm F}=2}_{k=2,T^2}=\begin{cases}\frac{1}{2}(cT)^4\left(\frac{4\pi}{g^2c}\right)^2 A\left(A-\frac{2}{3}\frac{8\pi}{g^2c}\right) & \textbf{for}\quad \frac{8\pi}{g^2c}\le A\\ \frac{1}{6}(cT)^4\left(A-\frac{4\pi}{g^2c}\right)^2 A\left(\frac{16\pi}{g^2c}-A\right) & \textbf{for}\quad \frac{4\pi}{g^2c}\le A\le\frac{8\pi}{g^2c}\end{cases}. \tag{60}$$

§App. Arbitrary Gauge Groups PLB [arXiv:0802.1020]

Lagrangian

$$\mathcal{L} = -\frac{1}{4e^2}F^0_{\mu\nu}F^{0\mu\nu}(W^0) - \frac{1}{4g^2}F^a_{\mu\nu}F^{a\mu\nu}(W^a) + \left(\mathcal{D}_\mu H_A\right)^\dagger \mathcal{D}^\mu H_A$$
$$-\frac{e^2}{2}\left|H_A^\dagger t^0 H_A - \frac{v^2}{\sqrt{2N}}\right|^2 - \frac{g^2}{2}|H_A^\dagger t^a H_A|^2, \quad (61)$$

gauge group $G = G' \times U(1)$ **(indices:** $0 \cdots U(1)$**,** $a \cdots G'$**)**

G' **arbitrary simple group**

e**:** $U(1)$ **gauge coupling,** g**:** G' **gauge coupling**

BPS vortex equations

$$\mathcal{D}_{\bar{z}}H = 0, \quad (62)$$
$$F^0_{12} - \frac{e^2}{\sqrt{2N}}\left(\mathrm{tr}\,(HH^\dagger) - v^2\right) = 0, \quad (63)$$
$$F^a_{12} - \frac{g^2}{4}\left(H^\dagger t_a H\right) = 0, \quad (64)$$

Boundary conditions at $\theta = (0 \sim 2\pi) \in S^1_\infty$

$$H \sim e^{i\alpha(\theta)} U(\theta) \langle H \rangle , \quad e^{i\alpha(\theta)} \in U(1), \quad U(\theta) \in G' \tag{65}$$

$$e^{i\alpha(\theta=2\pi)} = e^{2\pi i \nu} e^{i\alpha(\theta=0)}, \quad U(\theta = 2\pi) = e^{-2\pi i \nu} U(\theta = 0) \tag{66}$$

$e^{2\pi i \nu} \mathbf{1}_N \in C_{G'}$: **the center of** G

G'	$SU(N)$	$SO(2M+1)$	$USp(2M), SO(2M)$	E_6	E_7	E_8	F_4	G_2
N	N	$2M+1$	$2M$	27	56	248	26	7
$C_{G'}$	$\mathbf{Z}_N$	$\mathbf{1}$	$\mathbf{Z}_2$	$\mathbf{Z}_3$	$\mathbf{Z}_2$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
ν	k/N	k	$k/2$	$k/3$	$k/2$	k	k	k

$$S^1_\infty \to \frac{U(1) \times G'}{C_{G'}} \quad \Leftrightarrow \quad \pi_1 \left(\frac{U(1) \times G'}{C_{G'}} \right) \tag{67}$$

The tension of BPS vortices

$$T = -\frac{v^2}{\sqrt{2N}} \int d^2x \, F^0_{12} = v^2[\alpha(2\pi) - \alpha(0)] = 2\pi v^2 \nu = 2\pi v^2 \frac{k}{C_{G'}} \tag{68}$$

The Moduli Matrix Formalism

$$S(z,\bar{z}) = S_e(z,\bar{z})S'(z,\bar{z}) \in U(1)^{\mathbf{C}} \times G'^{\mathbf{C}} \tag{69}$$

$$W_1 + iW_2 = -2iS^{-1}(z,\bar{z})\bar{\partial}S(z,\bar{z}) \tag{70}$$

$$H = S^{-1}H_0(z) = S_e^{-1}S'^{-1}H_0(z), \tag{71}$$

Then the 1st BPS eq:

$$\mathcal{D}_{\bar{z}}H = 0 \Rightarrow \partial_{\bar{z}}H_0 = 0 \tag{72}$$

H_0: **holomorphic matrix called the moduli matrix**

The other BPS eqs: $e^{\psi} \equiv S_e S_e^{\dagger}, \qquad \Omega \equiv S' S'^{\dagger}$

$$\bar{\partial}\partial\psi = -\frac{e^2}{4N}\left(\mathrm{tr}\,(\Omega_0\Omega'^{-1})e^{-\psi} - v^2\right), \tag{73}$$

$$\bar{\partial}(\Omega'\partial\Omega'^{-1}) = \frac{g^2}{8}\mathrm{Tr}\,\left(H_0H_0^{\dagger}\Omega'^{-1}t_a\right)e^{-\psi}t_a, \tag{74}$$

the master equations

Constraints

Prepare $G^{\mathbf{C}\prime}$ invariants I^i (with $U(1)$ charge n_i)

$$I^i_{G'}(H) = I^i_{G'}\left(S_e^{-1}S'^{-1}H_0\right) = S_e^{-n_i}I^i_{G'}(H_0(z)) \tag{75}$$

$$I^i_{G'}(H_0) = S_e^{n_i}I^i_{G'}(H) \sim I^i_{\mathrm{vev}}\, z^{\nu\, n_i} = I^i_{\mathrm{vev}} z^{kn_i/n_0} \tag{76}$$

$$\nu = k/n_0, \qquad n_0 \equiv \mathrm{GCD}\{n_i \mid I^i_{\mathrm{vev}} \neq 0\}. \tag{77}$$

(GCD = the greatest common divisor)

Condition on H_0

$$\begin{aligned}
SU(N): \quad \det H_0(z) &= z^k + \mathcal{O}(z^{k-1}), \quad \nu = k/N, \\
SO(2M), USp(2M): \quad H_0^T(z) J H_0(z) &= z^k J + \mathcal{O}(z^{k-1}), \ \nu = k/2, \\
SO(2M+1): \quad H_0^T(z) J H_0(z) &= z^{2k} J + \mathcal{O}(z^{2k-1}), \ \nu = k, \\
E_6: \quad \Gamma_{i_1 i_2 i_3} (H_0)^{i_1}{}_{j_1} (H_0)^{i_2}{}_{j_2} (H_0)^{i_3}{}_{j_3} &= z^k \Gamma_{j_1 j_2 j_3} + \mathcal{O}(z^{k-1}), \\
E_7: \quad d_{i_1 i_2 i_3 i_4} (H_0)^{i_1}{}_{j_1} (H_0)^{i_2}{}_{j_2} (H_0)^{i_3}{}_{j_3} (H_0)^{i_4}{}_{j_4} &= z^{2k} d_{j_1 j_2 j_3 j_4} + \mathcal{O}(z^{k-1}), \\
f_{i_1 i_2} (H_0)^{i_1}{}_{j_1} (H_0)^{i_2}{}_{j_2} &= z^k f_{j_1 j_2} + \mathcal{O}(z^{k-1}),
\end{aligned} \tag{78}$$

G'	$SU(N)$	$SO(2M+1)$	$USp(2M), SO(2M)$	E_6	E_7	E_8	F_4	G_2
N	N	$2M+1$	$2M$	27	56	248	26	7
rank inv	–	2	2	3	2, 4	2, 3, 8	2, 3	2, 3
n_0	N	1	2	3	2	1	1	1

$$J = \begin{pmatrix} \mathbf{0}_M & \mathbf{1}_M \\ \epsilon \mathbf{1}_M & \mathbf{0}_M \end{pmatrix}, \quad \begin{pmatrix} J_{SO(2M)} & 0 \\ 0 & 1 \end{pmatrix}, \tag{79}$$

$\epsilon = +1$ **for** $SO(2M)$

$\epsilon = -1$ **for** $USp(2M)$

Examples of $k = 1$ (minimum)

$$SU(N): \qquad H_0 = \begin{pmatrix} z-a & 0 \\ \mathbf{b} & \mathbf{1}_{N-1} \end{pmatrix}, \tag{80}$$

$$SO(2M), USp(2M): \qquad H_0 = \begin{pmatrix} z\mathbf{1}_M - \mathbf{A} & \mathbf{C}_{S/A} \\ \mathbf{B}_{A/S} & \mathbf{1}_M \end{pmatrix}. \tag{81}$$

Condition on local vortices

(all invariants must have common zeros)

$$H_{0,\mathrm{local}}^T(z) J H_{0,\mathrm{local}}(z) = \prod_{\ell=1}^{k} (z - z_{0\ell})\, J. \tag{82}$$

$$H_{0,\mathrm{local}} = \begin{pmatrix} z-a & 0 \\ \mathbf{b} & \mathbf{1}_{N-1} \end{pmatrix}, \qquad \frac{SU(N)}{SU(N-1)\times U(1)} \tag{83}$$

$$H_{0,\mathrm{local}} = \begin{pmatrix} (z-a)\mathbf{1}_M & 0 \\ \mathbf{B}_{A/S} & \mathbf{1}_M \end{pmatrix}, \qquad \frac{SO(2M)}{U(M)}, \qquad \frac{USp(2M)}{U(M)} \tag{84}$$

Exceptional groups (in preparation)

1. E_6

(a) $\nu = 1/3$ **(non-BPS):** $E_6/SO(10) \times U(1)$

(b) $\nu = 2/3$ **(BPS):** $E_6/SO(10) \times U(1)$

2. E_7

(a) $\nu = 1/2$ **(non-BPS):** $E_7/E_6 \times U(1)$

(b) $\nu = 1$ **(BPS):** $E_7/SO(12) \times U(1)$

3. F_4

(a) $\nu = 1$ **(BPS):** $F_4/USp(6) \times U(1)$

§App. D-brane Configurations

Solitons	codim.	Solutions/Moduli	D-brane Construction
Instanton	4	ADHM ('78)	Dp-D(p+4) Douglas/Witten ('95)
Monopole	3	Nahm ('80)	D(p+1)-D(p+3) Green-Gutpele, Diaconescu ('96)
Vortex	2	EINOS ('05)	Dp-D(p+2)-D(p+4)-NS5 Hanany-Tong ('03)
Wall	1	INOS ('04)	[kinky Dp]-D(p+4) EINOO′S ('04)

Vortices ∼ "half" of instantons ('03 Hanany-Tong).
Walls ∼ "half" of monopoles ('05 Hanany-Tong).

(The former moduli space is a special Lagrangian submfd. of the latter moduli space.)

§App. Semi-local Vortices

The original meaning

Vortex in symm. breaking of both global and local symmetries.

$$\Phi = (\phi^1, \phi^2) \to e^{i\alpha}\Phi\, g, \qquad e^{i\alpha} \in U(1)_{\rm L}, \quad g \in SU(2)_{\rm F} \tag{85}$$

$$\langle \Phi \rangle \sim (1,0) : \qquad U(1)_{\rm L} \times SU(2)_{\rm F} \to U(1)_{\rm L+F} \tag{86}$$

1. non-topological:

$$\text{OPS} : \ \frac{U(1)_{\rm L} \times SU(2)_{\rm F}}{U(1)_{\rm L+F}} \simeq S^3, \qquad \pi_1(S^3) = 0. \tag{87}$$

2. The size(width) of a vortex can be arbitrary. It is non-normalizable, heavy and frozen in dynamics.

3. It is reduced to a skyrmion in strong gauge coupling limit.

$$S^3/U(1)_{\rm L} \simeq S^2, \quad \pi_2(S^2) \simeq \mathbf{Z} \tag{88}$$

The current definition $\pi_1(\text{OPS}) = 0, \quad \pi_1(G_{\rm L}/H_{\rm L}) \neq 0$

Semi-local Strings ($N_{\rm F} \geq 2, N_{\rm C} = 1$)

1. Their relative size can vary (moduli), while their total size is a non-normalizable mode, which is heavy and frozen in dynamics.
2. Their reconnection was shown by a computer simulation. Laguna, Natchu, Matzner and Vachaspati, hep-th/0604177

Non-Abelian Semi-local strings ($N_{\rm F} > N_{\rm C} \geq 2$)

1. The internal moduli $\mathbf{C}P^{N-1}$ of single vortex is non-normalizable. Shifman and Yung('06)
2. "relative orientation" and "relative size" are normalizable PRD [arXiv:0704.2218]
3. In collision, their sizes become the same and relative orientation goes to zero, resulting in reconnection!!

§App Solitons on solitons

Eto-MN-Ohashi-Tong PRL('05)

1)	kink	on	vortex	(in $D = 3 + 1$)	=	monopole
	1	+	2		=	3
2)	vortex	on	vortex	(in $D = 4 + 1$)	=	instanton
	2	+	2		=	4
3)	vortex	on	wall	(in $D = 3 + 1$)	=	boojum
	2	+	1		=	3
4)	Skyrmion	on	wall	(in $D = 4 + 1$)	=	instanton
	3	+	1		=	4

(#'s are codimensions)