E_6 GUT with 3 Generations from Strings

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Phenomenological Requirements

- E_6 Unification Group
- Adjoint Higgs Fields
- 3 Families
- 4D $\mathcal{N} = 1$ SUSY

"Translate each requirement into the string setup!!"

String Setup

- Lattice Engineering Technique
- A Higher Kac-Moody Level from Diagonal Embedding Method
- Level-3 with Shift Action ?
- Rotation of Right-Movers satisfying the SUSY Condition ($\boldsymbol{E_6}$ Coxeter Element, $\mathbb{Z}_{\mathbf{12}}$ Orbifold Theory for $\boldsymbol{E_6}$ Lattice)

Strategy

1. We prepare a $(\mathbf{22},\mathbf{6})$-dim lattice with 3-copies of $\boldsymbol{E_6}$ lattices.
   (Lattice Engineering Technique)
2. We perform an orbifold action which includes ...
   - Left-movers:
     Permutation of 3 $\boldsymbol{E_6}$'s with shift actions (Diagonal Embedding)
   - Right-movers:
     Rotation with the $\boldsymbol{E_6}$ Coxeter element
3. We find out #(Generations)!!

We shall explain these setups using partition functions.

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2. Partition Function
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4. Summary

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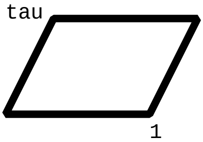
Partition Function

One-loop vacuum diagram of the closed strings

($\simeq$ generating function of the spectrum)

$$Z(\tau) = \mathrm{Tr}_{\mathcal{H}} q^H \qquad q = e^{2\pi i \tau}$$



Worldsheet = Torus

(= Parallelogram with two pairs of sides identified)

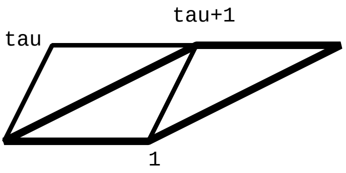
Moduli Transformation

Torus

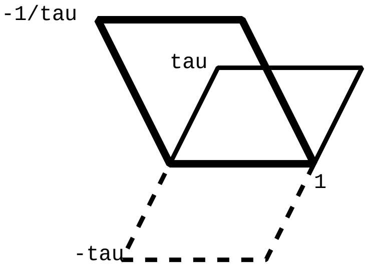
- Characterized by moduli τ.
- Different moduli may represent the same torus.

Modular transformation

$\mathcal{T}$: Shift one side by the other



$\mathcal{S}$: Exchange two sides



Invariance Under Modular Transformation

Modular transformation

$$\mathcal{T} : \tau \mapsto \tau + 1 \qquad \mathcal{S} : \tau \mapsto -1/\tau$$

Partition function should also be invariant under the modular transformation.

$$Z(\tau + 1) = Z(\tau) \qquad Z(-1/\tau) = Z(\tau)$$

Orbifold Theory (Symmetric or Asymmetric)

1. Project out the unwanted string states.
2. Recover the modular invariance by adding twisted strings.

Partition Function of Orbifold Theory

(α, β) Sector

α: Twisted boundary condition β: Projection

$$Z\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\tau) = \mathrm{Tr}_{\mathcal{H}_\alpha} q^H \theta^\beta$$

Expected modular tranformation:

$$Z\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\tau+1) = Z\begin{bmatrix}\alpha\\ \beta+\alpha\end{bmatrix}(\tau) \qquad Z\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(-1/\tau) = Z\begin{bmatrix}\beta\\ -\alpha\end{bmatrix}(\tau)$$

Spectrum of Bosonic String

$$\cdots$$

$$\alpha^i_{-3}|0\rangle \quad \alpha^i_{-2}\alpha^j_{-1}|0\rangle \quad \alpha^i_{-1}\alpha^j_{-1}\alpha^k_{-1}|0\rangle$$

$$\alpha^i_{-2}|0\rangle \quad \alpha^i_{-1}\alpha^j_{-1}|0\rangle$$

$$\alpha^i_{-1}|0\rangle$$

$$|0\rangle$$

Formal sum, for each direction

$$\begin{aligned} \text{Hilbert space} = &(1 \oplus \alpha_{-1} \oplus \alpha^2_{-1} \oplus \cdots) \\ &\otimes (1 \oplus \alpha_{-2} \oplus \alpha^2_{-2} \oplus \cdots) \otimes \cdots \\ &\otimes (1 \oplus \alpha_{-n} \oplus \alpha^2_{-n} \oplus \cdots) \otimes \cdots |0\rangle \end{aligned}$$

Partition Function

Partition function of bosonic strings without the zero modes

$$Z'(\tau) = \frac{1}{\eta(\tau)^{24}}$$

are given in terms of Dedekind eta function $\eta(\tau)$:

$$\eta(\tau) = q^{1/12} \prod_{n=1}^{\infty} (1 - q^n) = q^{1/12} \sum_n q^{\pi(n)}$$

Example: $3 = 2 + 1 = 1 + 1 + 1 \Rightarrow \pi(3) = 3$

Beautiful Moduli Transformation Property

$$\eta(\tau+1) = e^{2\pi i(1/24)}\eta(\tau)$$
$$\eta(-1/\tau) = \sqrt{-i\tau}\,\eta(\tau)$$

Interpretation

- $e^{2\pi i(1/24)}$: 24-Dim of the light-cone bosonic string
- $\sqrt{-i\tau}$: Cancellation with the zero-mode contribution

Beautiful because ...

Construction of the modular invariant partition function is possible.

Jacobi theta Function

Definition

$$\vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \sum_n \exp(-\pi(n+\alpha)(-i\tau)(n+\alpha) + 2\pi i(n+\alpha)\beta)$$

$$= \sum_n q^{(n+\alpha)^2/2} e^{2\pi i(n+\alpha)\beta}$$

$$= \eta(\tau) q^{\cdots} e^{2\pi i \cdots} \prod_{n=1}^{\infty} (1 + q^{n+\alpha-\frac{1}{2}} e^{2\pi i\beta})(1 + q^{n-\alpha-\frac{1}{2}} e^{-2\pi i\beta})$$

α: Shifts from integers β: Projections

Modular Transformation

Modular transformation properties are not so beautiful:

$$\vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\tau+1)=e^{\pi i\alpha(1-\alpha)}\vartheta\begin{bmatrix}\alpha\\ \beta+\alpha-1/2\end{bmatrix}(\tau)$$

$$\vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(-1/\tau)=\sqrt{-i\tau}e^{2\pi i\alpha\beta}\vartheta\begin{bmatrix}\beta\\ -\alpha\end{bmatrix}$$

Not-so-beautiful because ...

Interpretation of factors is unclear.

Attempt at Beauty

Redefinition

$$\Theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = e^{-\pi i\alpha\beta}\vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}$$

Moduli transformation properties are improved:

$$\Theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\tau+1) = e^{\pi i\alpha/2}\Theta\begin{bmatrix}\alpha\\ \beta+\alpha-1/2\end{bmatrix}(\tau)$$

$$\Theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(-1/\tau) = \sqrt{-i\tau}\,\Theta\begin{bmatrix}\beta\\ -\alpha\end{bmatrix}$$

However, they are still not beautiful enough!

Further Attempt

Applying formula valid for $n \in \mathbb{Z}$

$$\Theta\begin{bmatrix}\alpha\\ \beta+n\end{bmatrix} = e^{\pi i \alpha n}\Theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}$$

to a non-integral case $n = 1/2$ (!),

we end up with beautiful BUT incorrect modular transformation properties:

$$\Theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\tau+1) \simeq \Theta\begin{bmatrix}\alpha\\ \beta+\alpha\end{bmatrix}(\tau)$$

$$\Theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(-1/\tau) = \sqrt{-i\tau}\,\Theta\begin{bmatrix}\beta\\ -\alpha\end{bmatrix}$$

To Justify The Cheating

- Fermion Partition Function
- Twisted Boson Partition Function

Fermion Partition Function

Let's add the 1/2-shifted terms from the beginning.

$$\Theta^{\mathrm{F}}_{(\phi)}\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \frac{e^{-\pi i(\alpha\phi)\cdot(\beta\phi)}}{2\eta^4}\times$$

$$\left\{\prod_{i=0}^{3}\vartheta\begin{bmatrix}\alpha\phi^i\\ \beta\phi^i\end{bmatrix} - \prod_{i=0}^{3}e^{-\pi i\alpha\phi^i}\vartheta\begin{bmatrix}\alpha\phi^i\\ \beta\phi^i+1/2\end{bmatrix} - \prod_{i=0}^{3}\vartheta\begin{bmatrix}\alpha\phi^i+1/2\\ \beta\phi^i\end{bmatrix}\right\}$$

ϕ^i: Rotation angles of the orbifold action

The first two terms: NS-sector The last term: R-sector

Modular Transformation

Beautiful and correct modular transformation properties:

$$\Theta^{\mathbf{F}}_{(\phi)}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\tau+1) = e^{2\pi i(16/24)}\Theta^{\mathbf{F}}_{(\phi)}\begin{bmatrix}\alpha\\ \beta+\alpha\end{bmatrix}(\tau)$$

$$\Theta^{\mathbf{F}}_{(\phi)}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(-1/\tau) = \Theta^{\mathbf{F}}_{(\phi)}\begin{bmatrix}\beta\\ -\alpha\end{bmatrix}(\tau)$$

Partition functions lead to understanding of the GSO projection.

Similarly, Twisted Boson Partition Function

$$\Theta^{\mathrm{B}}_{(\phi)}\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \frac{\eta}{\vartheta\begin{bmatrix}\alpha\phi + 1/2\\ \beta\phi - 1/2\end{bmatrix}}$$

Modular transformation properties:

$$\Theta^{\mathrm{B}}_{(\phi)}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\tau+1) = e^{2\pi i(2/24)}\Theta^{\mathrm{B}}_{(\phi)}\begin{bmatrix}\alpha\\ \beta+\alpha\end{bmatrix}(\tau)$$

$$\Theta^{\mathrm{B}}_{(\phi)}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(-1/\tau) = i\sqrt{-i\tau}\,\Theta^{\mathrm{B}}_{(\phi)}\begin{bmatrix}\beta\\ -\alpha\end{bmatrix}(\tau)$$

Mostly beautiful except the extra factor i in the S-transformation.

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Lattice Partition Function

Before orbifolding

$$\vartheta_M(\tau) = \sum_{n_i} \exp\left(2\pi i\tau (n_i e^i)^2/2\right) = \sum_{\vec{n}} \exp\left(-\pi \vec{n}\cdot(-i\tau M)\vec{n}\right)$$

$M^{ij} = \langle e^i, e^j \rangle$: Inner product of the lattice basis e^i

To satisfy the modular invariance

$$\vartheta_M(\tau+1) = \vartheta_M(\tau)$$
$$\vartheta_M(-1/\tau) = \sqrt{-i\tau}^{\dim M} \vartheta_M(\tau)$$

we have to require the following conditions for the lattice:

- Even $(n_i e^i)^2 = \vec{n}\cdot M\vec{n} \in 2\mathbb{Z}$
- Self-Dual $(\widetilde{e}^i \cdot e^j = \delta^{ij})$

$$\{n_i e^i | n_i \in \mathbb{Z}\} = \{m_i \widetilde{e}^i | m_i \in \mathbb{Z}\}$$

Lie Lattice

In the case of the root lattice for Lie algebra

- $e^i = \alpha^i$ (Root Vector)
- $M = C$ (Cartan Matrix)

Modular invariance conditions

- Even: Valid for every simply-laced Lie algebra
- Even and Self-Dual: Valid only for E_8

E_8 Lattice

Modular invariant partition function

$$\mathcal{E}(\tau+1) = \mathcal{E}(\tau) \qquad \mathcal{E}(-1/\tau) = \sqrt{-i\tau}^{\,8}\,\mathcal{E}(\tau)$$

Let's decompose it into the $E_6 \times A_2$ lattice:

$$\mathcal{E}(\tau) = E(\tau)A(\tau) + e(\tau)a(\tau) + e(\tau)a(\tau)$$

- A similar decomposition in the representation theory.

$$248 = (78 \times 1 + 1 \times 8) + 27 \times 3 + \overline{27} \times \overline{3}$$

- (Conjugacy Class) = (Weight Lattice)/(Root Lattice)
  In both cases of E_6 and A_2, #(Conjugacy Class) = 3
- E and A: Partition functions of root lattices
  e and a: Partition functions of root lattices shifted by a weight

Modular Transformation Properties

$$E(\tau+1) = E(\tau) \qquad E(-1/\tau) = \sqrt{-i\tau}^{6}\frac{1}{\sqrt{3}}(E+2e)(\tau)$$

$$e(\tau+1) = \omega^{-1}e(\tau) \qquad e(-1/\tau) = \sqrt{-i\tau}^{6}\frac{1}{\sqrt{3}}(E-e)(\tau)$$

$$A(\tau+1) = A(\tau) \qquad A(-1/\tau) = \sqrt{-i\tau}^{2}\frac{1}{\sqrt{3}}(A+2a)(\tau)$$

$$a(\tau+1) = \omega a(\tau) \qquad a(-1/\tau) = \sqrt{-i\tau}^{2}\frac{1}{\sqrt{3}}(A-a)(\tau)$$

$$(\omega = \exp(2\pi i/3))$$

Matrix Form

$$\mathcal{E}(\tau) = \begin{pmatrix} E & e & e \end{pmatrix} \begin{pmatrix} A \\ a \\ a \end{pmatrix}$$

$\mathcal{S}$-transformation

$$\mathcal{S} \cdot \begin{pmatrix} E \\ e \\ e \end{pmatrix} = \sqrt{-i\tau}^{6} \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} \begin{pmatrix} E \\ e \\ e \end{pmatrix}$$

$$\mathcal{S} \cdot \begin{pmatrix} A \\ a \\ a \end{pmatrix} = \sqrt{-i\tau}^{2} \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{pmatrix} \begin{pmatrix} A \\ a \\ a \end{pmatrix}$$

Lessons

E_6 lattice and A_2 lattice transform oppositely.

More generally, a lattice transforms oppositely as its complement lattice in the even self-dual E_8 lattice.

Conjugacy classes also relate to those in the complement lattice.

Lattice Engineering Technique

We can always replace a left-moving lattice by the complement right-moving lattice (denoted with an asterisk).

$$A_2 \Rightarrow [E_6]^* \xrightarrow{\text{decomp}} [(A_2)^3]^* \Rightarrow (E_6)^3$$

Hence, $(E_6)^3$ lattice can be obtained from any A_2 lattice!! Explicit correspondence of conjugacy classs reads

$$\begin{aligned} 0 &\longrightarrow (0,0,0) \oplus (1,1,1) \oplus (2,2,2) \\ 1 &\longrightarrow (0,1,2) \oplus (1,2,0) \oplus (2,0,1) \\ 2 &\longrightarrow (0,2,1) \oplus (2,1,0) \oplus (1,0,2) \end{aligned}$$

Setup

1. Starting point: $\boldsymbol{E_6} \times [\boldsymbol{E_6}]^*$
2. Decomposition: $(\boldsymbol{A_2})^{\mathbf{3}} \times [\boldsymbol{E_6}]^*$
3. Finally: $[(\boldsymbol{A_2})^{\mathbf{2}} \times (\boldsymbol{E_6})^{\mathbf{3}}] \times [\boldsymbol{E_6}]^*$

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Diagonal Embedding Method

- A higher Kac-Moody level is necessary for adjoint Higgs fields.
- We raise the level by the diagonal embedding method (an orbifold action which permutes 3 copies of E_6's) so that only the diagonal E_6 remains phaseless.

Besides, shift action has to be introduced simultaneously to break symmetry between chiral matters and anti-chiral matters.

Study from Partition Function

Orbifold projection by permutation

- States in the $(E_6)^3$ lattice take the form $|p, p', p''\rangle$.
- $\mathbb{Z}_3$ orbifold action by permuting 3 copies of E_6's reduces to the states $|p, p, p\rangle$.

In terms of the partition function

- Originally

$$(0,0,0) \oplus (1,1,1) \oplus (2,2,2) \Rightarrow [E(\tau)]^3 + 2[e(\tau)]^3$$
$$(0,1,2) \oplus (1,2,0) \oplus (2,0,1) \Rightarrow 3E(\tau)e(\tau)^2$$
$$(0,2,1) \oplus (2,1,0) \oplus (1,0,2) \Rightarrow 3E(\tau)e(\tau)^2$$

- After orbifold projection: $E(3\tau) + 2e(3\tau)$

Matrix Form

In the matrix form $\begin{bmatrix} & {}^{0}\square_{1} & {}^{0}\square_{2} \\ {}^{1}\square_{0} & {}^{1}\square_{1} & {}^{1}\square_{2} \\ {}^{2}\square_{0} & {}^{2}\square_{1} & {}^{2}\square_{2} \end{bmatrix}$

$$\begin{bmatrix} & (-i)^3\frac{\sqrt{3}}{27}E\left(\frac{\tau}{3}\right) & i^3\frac{\sqrt{3}}{27}E\left(\frac{\tau}{3}\right) \\ (E+2e)(3\tau) & (-i)^3\frac{\sqrt{3}}{27}E\left(\frac{\tau+1}{3}\right) & i^3\frac{\sqrt{3}}{27}E\left(\frac{\tau+2}{3}\right) \\ (-1)^3(E+2e)(3\tau) & (-i)^3\frac{\sqrt{3}}{27}E\left(\frac{\tau+2}{3}\right) & i^3\frac{\sqrt{3}}{27}E\left(\frac{\tau+1}{3}\right) \end{bmatrix}$$

Not e but E appears in the twisted sectors. Reason?

Nothing to break to symmetry between chiral and antichiral matters.

Shift

If we introduce the shift action, conjugacy class 1 & 2 acquire phases. Then, we find $(E+\omega e+\omega^2 e)(3\tau)=(E-e)(3\tau)$ instead of $(E+e+e)(3\tau)=(E+2e)(3\tau)$ in the untwisted sector.

$$\begin{bmatrix} & (-i)^3\frac{\sqrt{3}}{27}e\left(\frac{\tau}{3}\right) & i^3\frac{\sqrt{3}}{27}e\left(\frac{\tau}{3}\right) \\ (E-e)(3\tau) & (-i)^3\frac{\sqrt{3}}{27}e\left(\frac{\tau+1}{3}\right) & i^3\omega^2\frac{\sqrt{3}}{27}e\left(\frac{\tau+2}{3}\right) \\ (-1)^3(E-e)(3\tau) & (-i)^3\frac{\sqrt{3}}{27}e\left(\frac{\tau+2}{3}\right) & i^3\frac{\sqrt{3}}{27}e\left(\frac{\tau+1}{3}\right) \end{bmatrix}$$

Thus, asymmetry between chiral and antichiral matters apears.

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Generalization

To incooperate the shift transformation, let us define a generalized theta function.

$$\vartheta_C \begin{bmatrix} \vec{\alpha} \\ \vec{\beta} \end{bmatrix} (\tau) = \sum_{\vec{n}} \exp\Big(-\pi (\vec{n} + \vec{\alpha}) \cdot (-i\tau C)(\vec{n} + \vec{\alpha}) + 2\pi i (\vec{n} + \vec{\alpha}) \cdot C\vec{\beta} \Big)$$

Modular transformation properties:

$$\vartheta_C \begin{bmatrix} \vec{\alpha} \\ \vec{\beta} \end{bmatrix} (\tau + 1) = e^{-\pi i \vec{\alpha} \cdot C \vec{\alpha}} \vartheta_C \begin{bmatrix} \vec{\alpha} \\ \vec{\beta} + \vec{\alpha} \end{bmatrix} (\tau)$$

$$\vartheta_C \begin{bmatrix} \vec{\alpha} \\ \vec{\beta} \end{bmatrix} (-1/\tau) = \frac{\sqrt{-i\tau}^{\dim C}}{\sqrt{\det C}} e^{2\pi i \vec{\alpha} \cdot C \vec{\beta}} \vartheta_{C^{-1}} \begin{bmatrix} C\vec{\beta} \\ -C\vec{\alpha} \end{bmatrix} (\tau)$$

Orbifold Partition Function

Using the mudular transformation properties, let us define the orbifold partition function as

$$\Theta_A\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \begin{bmatrix} & \frac{-i}{\sqrt{3}}\widehat{\Theta}_A\begin{bmatrix}1\\0\end{bmatrix} & \frac{i}{\sqrt{3}}\widehat{\Theta}_A\begin{bmatrix}2\\0\end{bmatrix} \\ \widehat{\Theta}_A\begin{bmatrix}0\\1\end{bmatrix} & \frac{-i}{\sqrt{3}}\widehat{\Theta}_A\begin{bmatrix}1\\1\end{bmatrix} & \frac{i}{\sqrt{3}}\widehat{\Theta}_A\begin{bmatrix}2\\1\end{bmatrix} \\ -\widehat{\Theta}_A\begin{bmatrix}0\\2\end{bmatrix} & \frac{-i}{\sqrt{3}}\widehat{\Theta}_A\begin{bmatrix}1\\2\end{bmatrix} & \frac{i}{\sqrt{3}}\widehat{\Theta}_A\begin{bmatrix}2\\2\end{bmatrix} \end{bmatrix}$$

Various Quantities

$$\widehat{\Theta}_A \begin{bmatrix} 0 \\ \beta_{\neq 0} \end{bmatrix} = \vartheta_A \begin{bmatrix} \vec{0} \\ \beta \vec{s} \end{bmatrix}$$

$$\widehat{\Theta}_A \begin{bmatrix} \alpha_{\neq 0} \\ \beta \end{bmatrix} = \sum_{k=0}^{2} e^{\pi i \beta (\alpha^{-1} f k^2 - \alpha \vec{s} \cdot A \vec{s})} \vartheta_A \begin{bmatrix} \alpha \vec{s} + k \vec{F} \\ \beta \vec{s} \end{bmatrix}$$

$$A = \frac{1}{3} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \quad \vec{F} = \frac{1}{3} \begin{pmatrix} 2 \\ -1 \end{pmatrix} \quad f = \frac{2}{3}$$

$\vec{s}$: Shift vector

- Introducing phases in the untwisted sectors
- Raising shift actions in the twised sectors

Modular Transformation

Modular transformation

$$\Theta_A\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\tau+1)=\Theta_A\begin{bmatrix}\alpha\\ \beta+\alpha\end{bmatrix}(\tau)$$

$$\Theta_A\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(-1/\tau)=i\sqrt{-i\tau}^2\,\Theta_A\begin{bmatrix}\beta\\ -\alpha\end{bmatrix}(\tau)$$

Lesson: Shift action does not affect the modular transformation.

Due to periodic condition in $\mathbb{Z}_N$ orbifold theories ($N=12$)

$$Z^{\mathrm{total}}\begin{bmatrix}\alpha\\ \beta+N\end{bmatrix}=Z^{\mathrm{total}}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}$$

$$12\sum_{i=1}^{2}(\mathrm{norm})_i-\frac{12/3}{3}\in\mathbb{Z}$$

Shift Action

Shift satisfying

$$12\vec{s} = 0 \qquad (\text{mod Root Lattice})$$

can be expressed as

$$\vec{s} = \frac{m_1\alpha_1 + m_2\alpha_2}{12} \qquad (m_1, m_2 \in \mathbb{Z})$$

Periodic condition requires the sum of norms

$$\text{norm} = \frac{1}{2}||m_1\alpha_1 + m_2\alpha_2||^2 = m_1^2 - m_1m_2 + m_2^2$$

in two A_2 lattices to be 4 mod 12.

Classification of Shifts

Since possible values of norms are

$$\text{norm} = \mathbf{0, 1, 3, 4, 7, 9} \qquad (\text{mod } 12)$$

the combinations can be $\{\mathbf{4, 0}\}$, $\{\mathbf{1, 3}\}$, $\{\mathbf{7, 9}\}$.

There seem lots of possibilities. However, many of them are related by Weyl group.

$$\text{norm-4: } (\mathbf{2, 0}), (\mathbf{6, 8}), (\mathbf{4, 0})$$

$$\text{norm-0: } (\mathbf{0, 0}), (\mathbf{4, 8}), (\mathbf{6, 0}), (\mathbf{10, 2})$$

$$\text{norm-1: } (\mathbf{1, 0}), (\mathbf{5, 8}), (\mathbf{5, 0}), (\mathbf{9, 8})$$

$$\text{norm-3: } (\mathbf{3, 6}), (\mathbf{7, 2}), (\mathbf{11, 10})$$

$$\text{norm-7: } (\mathbf{1, 6}), (\mathbf{5, 2}), (\mathbf{9, 10})$$

$$\text{norm-9: } (\mathbf{3, 0}), (\mathbf{7, 8})$$

Moreover

- Two shifts with the difference belonging to the weight lattice should also be identified because ...
  - $\boldsymbol{\alpha = 0}$ mod 3 **:** 3 x Shift vectors $\in$ Root lattice
  - $\boldsymbol{\alpha \neq 0}$ mod 3 **:** Right-movers are twisted nontrivially.
- (1,0) and (5,0) should also be identified because the same spectrum only comes from different sectors.

Finally, combining with the possibility of rotation, we are left with

$$\{(2,0),(4,0),\text{"rot"}\} \otimes \{(0,0),(6,0)\},$$
$$(1,0) \otimes (3,6), \qquad (1,6) \otimes (3,0)$$

$\mathbf{3 \times 2 + 1 \times 1 + 1 \times 1 = 8}$ consistent models.

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Setup

Lattice

$$E_6 \times [E_6]^* \Rightarrow (A_2)^3 \times [E_6]^* \Rightarrow [(A_2)^2 \times (E_6)^3] \times [E_6]^*$$

Orbifold Action

- Permutation of three copies of E_6's
- Coxeter element of $[E_6]^*$
- Shift action of A_2

“Modular Multiplets”

Different coefficients depending on GCD(α,β,12):

$$\begin{bmatrix}
 & \circ & \heartsuit & \triangle & \square & \circ & \bigcirc & \circ & \square & \cdots \\
\circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \cdots \\
\heartsuit & \circ & \heartsuit & \circ & \heartsuit & \circ & \heartsuit & \circ & \heartsuit & \cdots \\
\triangle & \circ & \circ & \triangle & \circ & \circ & \triangle & \circ & \circ & \cdots \\
\square & \circ & \heartsuit & \circ & \square & \circ & \heartsuit & \circ & \square & \cdots \\
\circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \cdots \\
\bigcirc & \circ & \heartsuit & \triangle & \heartsuit & \circ & \bigcirc & \circ & \heartsuit & \cdots \\
\circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \cdots \\
\square & \circ & \heartsuit & \circ & \square & \circ & \heartsuit & \circ & \square & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}$$

Phases of Fixed Points

1. We rewrite the partition functions in terms of the modular-invariant theta functions into "physical" theta functions (which exhibit the state counting explicitly).
2. Coefficients

$\alpha \setminus \beta$	0	1	2	3	4	5	6
$\mathbf{2}$	$\mathbf{1}$	$\mathbf{-1}$	$\mathbf{1}$	$\mathbf{-1}$	$\mathbf{1}$	$\mathbf{-1}$	$\mathbf{1}$
$\mathbf{4}$	$\mathbf{3}$	$\mathbf{1}$	$\mathbf{-1}$	$\mathbf{1}$	$\mathbf{3}$	$\mathbf{1}$	$\mathbf{-1}$
$\mathbf{6}$	$\mathbf{2}$	$\mathbf{1}$	$\mathbf{-1}$	$\mathbf{-2}$	$\mathbf{-1}$	$\mathbf{1}$	$\mathbf{2}$

3. Interpretation as phases of fixed points $\sum_k e^{2\pi i\beta\varphi_k}$

α	phases of fixed points
$\mathbf{2}$	$\mathbf{1/2}$
$\mathbf{4}$	$\mathbf{0, \pm 1/4}$
$\mathbf{6}$	$\mathbf{0, \pm 1/6}$

States Without Phases

Spectrum comes from states with various contributions of phases canceled.

- fermion
- twisted bosons
- lattice
- fixed points

Generations

Out of 8 models

- 3 : #(generations) $= 0$
- 3 : #(generations) $= 3$
- 2 : #(generations) $= 9$

To summarize

- Two New $\boldsymbol{E_6}$ Models As A Result
- Systematic Construction Of String Phenomenology