Volume of Moduli Space of Vortices and Localization Formula

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Introduction

Topic:

- Calculation of the volume of the moduli space of BPS solitons

Moduli space?

- Kähler or hyper-Kähler quotient space

  {Solutions of F & D-term constraints} / {Gauge symmetry}

BPS soliton

- Abelian/non-Abelian vortex on a *compact* Riemann surface with genus h (Σ_h)

Introduction

BPS equations ($G=U(N_c)$ and N_f flavors):

$$\mu_r \equiv F - \frac{g^2}{2}(c - HH^\dagger)\omega = 0$$
$$\mu_{\bar z} \equiv \mathcal{D}_{\bar z} H = 0$$
$$\mu_z \equiv \mathcal{D}_z H^\dagger = 0$$

where g:gauge coupling, c: FI parameter, ω:Kähler 2-form on Σ_h, H:$N_c \times N_f$ matrix.

$$\mathcal{M}_k \equiv \frac{\{\text{Solutions of } \mu_r = \mu_z = \mu_{\bar z} = 0 \text{ with } \frac{1}{2\pi}\int F = k\}}{U(N_c)}$$

Volume of $\mathcal{M}_k$?

Introduction

The volume of moduli space of the BPS solitons relates to

- Non-perturbative corrections in supersymmetric gauge theory (Nekrasov's formula)

- Thermodynamical partition function of the BPS solitons (Manton et al.)

$$Z = \frac{1}{\hbar^{2k}} \int d^k p d^k x \, e^{-\frac{1}{2T} g^{ij} p_i p_j} = \left(\frac{2\pi^2 T}{\hbar^2} \right)^k \mathrm{Vol}(\mathcal{M}_k)$$

The volume of moduli space of the BPS solitons (supersymmetric systems) is a key to understand the non-perturbative dynamics and dualities in gauge/string theory

Method

Straightforwardly,

BPS eqs. ⇒ BPS solutions ⇒ effective action ⇒ metric ⇒ volume

Instead, we define the field theoretical partition function to obtain the volume of the moduli space [Moore-Nekrasov-Shatashvili (1997)]:

$$\mathcal{Z}_k^{N_c,N_f}(\Sigma_h) = \int \mathcal{D}\Phi\mathcal{D}^2 A\mathcal{D}^2\lambda\mathcal{D}^2 H\mathcal{D}^2\psi\mathcal{D}^2 Y\mathcal{D}^2\chi\, e^{-S_0-S_1}$$

where

$$S_0 = \int_{\Sigma_h} \mathrm{Tr}\left[i\Phi\left\{F - \frac{g^2}{2}(c - HH^\dagger)\omega\right\} + \frac{1}{2}\lambda\wedge\lambda + \frac{g^2}{2}\psi\psi^\dagger\omega\right]$$

$$S_1 = \int_{\Sigma_h} d^2z\,\mathrm{Tr}\left[g^{z\bar{z}}(Y_zY_{\bar{z}} + i\Phi\chi_z\chi_{\bar{z}})\right] + \cdots$$

Method

This is essentially a constrained system on the moduli space of the vortex

$$\mathcal{Z}_k^{N_c,N_f} = \int \mathcal{D}^2 A \mathcal{D}^2 H \, \delta(\mu_r) J_r \, \delta(\mu_z) J_z \, \delta(\mu_{\bar{z}}) J_{\bar{z}} \cdots$$

$$= \mathrm{Vol}(\mathcal{M}_k^{N_c,N_f})$$

We can perform the path integral, which reduces to residue integrals over zero modes of Lagrange multiplier field Φ

$$\mathcal{Z}_k^{N_c,N_f}(\Sigma_h) = \sum_{\sum_a k_a = k} (-1)^\sigma \int \prod_a \frac{d\phi_a}{2\pi} \frac{\prod_a (1 + \frac{N_f}{2\pi i \phi_a})^h \prod_{a<b} (i\phi_a - i\phi_b)^{2-2h}}{\prod_a (i\phi_a)^{N_f(1-h+k_a)}}$$

$$\times e^{2\pi i \sum_a \phi_a (\frac{g^2 c}{4\pi} \mathcal{A} - k_a)}$$

Integrals are localized at poles (Localization formula)

Results

N_c=N_f=1 (Abrikosov–Nielsen–Olesen vortex)

$$\mathcal{Z}_k^{1,1}(\Sigma_h) = (2\pi)^{k-h} \sum_{j=0}^{h} \frac{h!}{j!(k-j)!(h-j)!} \left(\frac{g^2 c}{4\pi} \mathcal{A} - k \right)^{k-j}$$

We find

$$\mathcal{A} \geq \frac{4\pi}{g^2 c} k \qquad \text{Bradlow limit}$$

For h=0 (sphere)

$$\mathcal{Z}_k(S^2) = \frac{(2\pi)^k}{k!} \left(\frac{g^2 c}{4\pi} \mathcal{A} - k \right)^k \xrightarrow{\mathcal{A}\to\infty} \mathrm{Vol}((S^2)^k/\mathcal{S}_k) \sim \frac{\mathcal{A}^k}{k!}$$

Eq of state: $P\left(\mathcal{A} - \frac{4\pi}{g^2 c} k\right) = kT$

Results

N_c=2, N_f (non-Abelian *semi-local* vortex) on the sphere

$$\mathcal{Z}_0^{2,N_f}(S^2) = \frac{2!}{(N_f-1)!(N_f-2)!}(2\pi\tilde{\mathcal{A}})^{2(N_f-2)}$$

$$\mathcal{Z}_1^{2,N_f}(S^2) = \frac{(2\pi)^{3N_f-4}}{(2N_f-1)(N_f-1)!(2N_f-3)!}\tilde{\mathcal{A}}^{N_f-3}(\tilde{\mathcal{A}}-1)^{2N_f-3} \times \left((N_f-2)\tilde{\mathcal{A}}^2 + 2(N_f+1)\tilde{\mathcal{A}} + (N_f-2)\right)$$

$$\mathcal{Z}_2^{2,N_f}(S^2) = 2(2\pi)^{4N_f-4}\left[\frac{-2}{(N_f-1)!(3N_f-1)!}\tilde{\mathcal{A}}^{N_f-3}(\tilde{\mathcal{A}}-2)^{3N_f-3} \times \left((2N_f^2-2N_f+1)\tilde{\mathcal{A}}^2 + 2(2N_f+1)(N_f-1)\tilde{\mathcal{A}} + 2(N_f-1)(N_f-2)\right) + \frac{1}{(2N_f-1)!(2N_f-2)!}(\tilde{\mathcal{A}}-1)^{4N_f-4}\right]$$

where $\tilde{\mathcal{A}} \equiv \frac{g^2 c}{4\pi}\mathcal{A}$

Comments

On the sphere, k=0 (perturbative part) gives the volume of the vacuum moduli space

$$\mathcal{Z}_0^{N_c,N_f}(S^2) = N_c! \times \mathrm{Vol}(G_{N_c,N_f})\tilde{\mathcal{A}}^{N_c(N_f-N_c)}$$

Vol. of Grassmannian

The case of N_c=N_f (non-Abelian *local* vortex) is rather special

$$\mathcal{Z}_0^{2,2}(S^2) = 2$$

$$\mathcal{Z}_1^{2,2}(S^2) = 2 \times (2\pi)^2(\tilde{\mathcal{A}} - 1)$$

$$\mathcal{Z}_2^{2,2}(S^2) = 2 \times \frac{(2\pi)^4}{2!}\left(\tilde{\mathcal{A}}^2 - \frac{20}{6}\tilde{\mathcal{A}} + \frac{17}{6}\right)$$

In the limit of $\mathcal{A} \to \infty$, the divergence comes from the position moduli only

Conclusion

- We evaluate the volume of the moduli space of BPS vortices on the Riemann surface by using the localization formula
- We can see not only the volume itself but also the geometrical structure of the moduli space
- Using the deconstruction, we can also apply the localization method to the solitons in the Chern-Simons-Higgs system, etc.
- The volume of the moduli space (aka localization formula) is a key to understand the dualities between gauge theories, string theories, matrix models, and more!