Shear viscosity of a highly excited string and black hole membrane paradigm

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1. Introduction

Mysteries of black holes

- Microscopic origin of Bekenstein-Hawking entropy

$$S_{BH} = \frac{A}{4G}$$

(A: area of the event horizon)

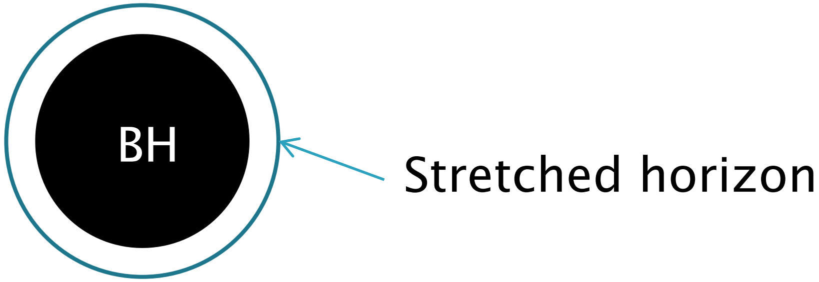
- Microscopic origin of membrane paradigm

"A certain fictitious viscous membrane seems to be sitting on a stretched horizon for a distant observer."

Thorne, Price, Macdonald (1986)

$$\eta_{BH} = \frac{1}{16\pi G}$$

$$\frac{\eta_{BH}}{s_{BH}} = \frac{1}{4\pi}$$



We need a consistent quantum theory which includes gravity.

⇒ String theory!

Bekenstein-Hawking entropy vs entropy of a fundamental string

Entropy of a fundamental string	$\equiv$	Logarithm of the number of degeneracy of the string states

For a highly excited free string,

$$S \sim \sqrt{N} \sim l_s M$$

On the other hand,

$$S_{BH} \sim G^{\frac{1}{d-2}} M^{\frac{d-1}{d-2}}$$

(d: Number of spatial dimensions)

$$\sim g_s^{\frac{2}{d-2}} (l_s M)^{\frac{d-1}{d-2}}$$

$$G \sim g_s^2 l_s^{d-1}$$

Clearly, in generic g_s

$$S \neq S_{BH}$$

String-black hole correspondence

Horowitz, Polchinski (1996)
Damour, Veneziano (1999)

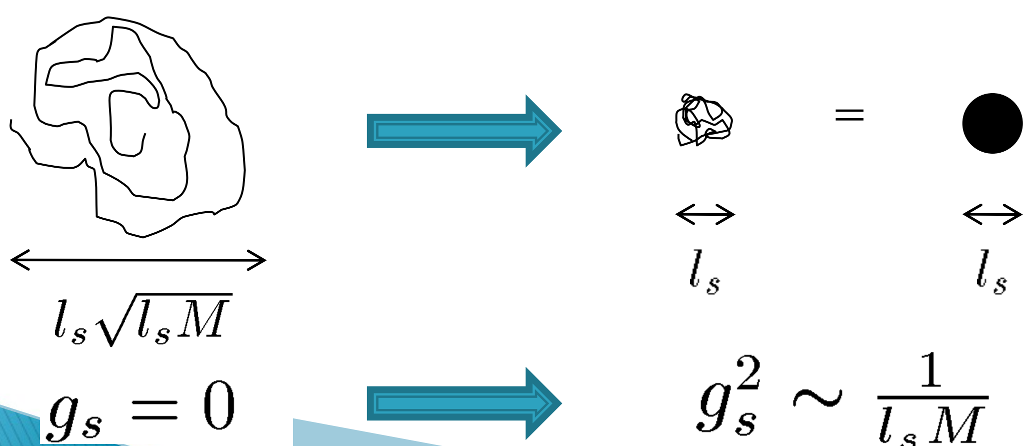
If $\quad g_s^2 \sim \frac{1}{l_s M}$,

$$S \sim S_{BH} \quad !$$

At this point, the horizon radius becomes

$$r_H \sim (GM)^{\frac{1}{d-2}} \sim l_s$$

> If we increase g_s adiabatically, a highly excited string becomes a black hole with $r_H \sim l_s$ at $g_s^2 \sim \frac{1}{l_s M}$.



Entropy of a macroscopic black hole from a fundamental string

Susskind (1993)

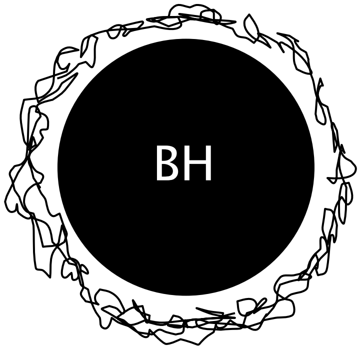
A large gravitational redshift of a black hole explains the difference between S and S_{BH} .

Consider a highly excited string on a stretched horizon of a Schwarzshild black hole.

Due to the redshift, the energy for an observer at the stretched horizon is not the same as the energy for an asymptotic observer.

$$E_{sh} \sim \frac{G^{\frac{1}{d-2}} M^{\frac{d-1}{d-2}}}{l_s}$$

$$S_{sh} \sim l_s E_{sh} \sim G^{\frac{1}{d-2}} M^{\frac{d-1}{d-2}} \sim S_{BH}$$



Derivation of E_{sh}

Susskind (1993)

(d+1)-dimensional Schwarzshild metric

$$ds^2 = -h(r)dt^2 + \frac{1}{h(r)}dr^2 + r^2 d\Omega_{d-1}^2$$

$$\begin{cases} h(r) = 1 - \left(\frac{r_H}{r}\right)^{d-2} \\ r_H^{d-2} = \frac{16\pi MG}{(d-1)\Omega_{d-1}} \end{cases}$$

Ω_{d-1} : Volume of a unit (d-1) sphere

Near horizon geometry $r \sim r_H$

$$ds^2 \sim \underline{-\kappa^2 \rho^2 dt^2 + d\rho^2} + r_H^2 d\Omega_{d-1}^2$$

2 dim. Rindler spacetime

$$\begin{cases} r^{d-2} - r_H^{d-2} = \frac{(d-2)^2}{4} r_H^{d-4} \rho^2 \\ \kappa = \frac{d-2}{2r_H} \end{cases}$$

$\kappa = \frac{d-2}{2r_H}$: Surface gravity

$$T_H = \frac{\kappa}{2\pi}$$

To find the energy and temperature for an observer at ρ , it is convenient to introduce dimensionless Rindler quantities.

Define the Rindler time $\theta \equiv \kappa t$

Rindler energy E_R is conjugate to θ

$$[E_R, \theta] = i = \kappa [E_R, t]$$

M is conjugate to t

$$[M, t] = i \quad \Longrightarrow \quad t = -i \frac{\partial}{\partial M}$$

Thus,

$$\frac{\partial E_R}{\partial M} = \frac{1}{\kappa} \quad \Longrightarrow \quad \boxed{E_R = \frac{A_{d-1}}{8\pi G}}$$

By using the first law of thermodynamics, we obtain the Rindler temperature

$$\boxed{T_R = \frac{1}{2\pi}}$$

The proper time at ρ is $\tau|_\rho = \rho\theta$

$$E_\rho = \frac{E_R}{\rho} = \frac{A_{d-1}}{8\pi G\rho}$$

$$T_\rho = \frac{T_R}{\rho} = \frac{1}{2\pi\rho}$$

The stretched horizon is defined by the place where the local Unruh temperature is given by the Hagedorn temperature $T \sim \frac{1}{l_s}$

The stretched horizon is located at $\rho \sim l_s$.

$$E_{sh} \sim \frac{E_R}{l_s} = \frac{A_{d-1}}{8\pi G l_s} \sim \frac{G^{\frac{1}{d-2}} M^{\frac{d-1}{d-2}}}{l_s}$$

$$T_{sh} \sim \frac{T_R}{l_s} = \frac{1}{2\pi l_s}$$

Consistency with the string-black hole correspondence

If the redshift factor is of the order of one,

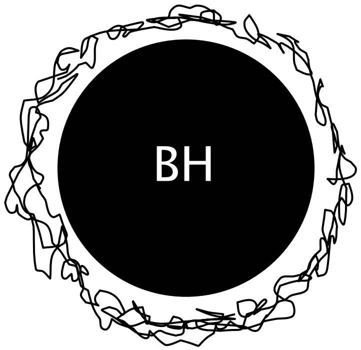
$$\frac{\tau|_{r=\infty}}{\tau|_{\rho\sim l_s}} = \frac{t}{\kappa l_s t} = \frac{1}{\kappa l_s} \sim \mathcal{O}(1)$$

$$\Rightarrow \begin{cases} r_H \sim l_s \\ g_s^2 \sim \frac{1}{l_s M} \end{cases}$$

This is the same situation as the string-black hole correspondence.

In this case, the energy and temperature for an observer on the stretched horizon are of the same order as those for the asymptotic observer.

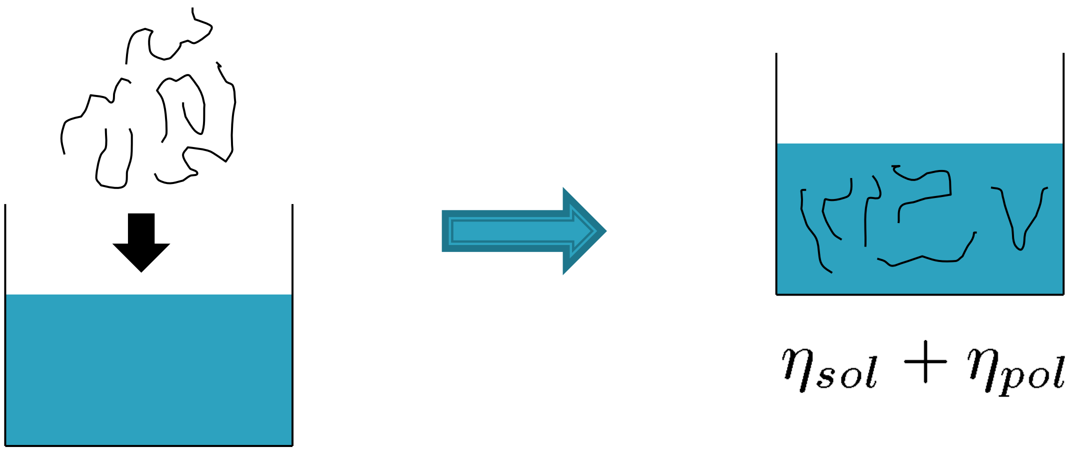
Membrane paradigm from the viewpoint of a fundamental string



Can we reproduce the viscosity of the fictitious membrane from a highly excited string?

What is the viscosity of the string?

In polymer physics,



η_{sol}

$\eta_{sol} + \eta_{pol}$

This is due to the fact that the stress tensor of the polymer itself is added to the stress tensor of the solvent.

Contents

2. Open string in highly excited states

Review of bosonic open string

Worldsheet action in flat background spacetime

$$S_0 = -\frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{-\gamma}\gamma^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X_\mu \qquad \begin{matrix}(\alpha = 0,1) \\ (\mu,\nu = 0,1,\dots d)\end{matrix}$$

Choosing the unit gauge,

$$\gamma_{\alpha\beta} = \eta_{\alpha\beta} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} ,$$

the action becomes

$$\boxed{S_0 = \frac{1}{4\pi\alpha'}\int d^2\sigma (\dot{X}^\mu)^2 - (X'^\mu)^2}$$

where $\dot{X}^\mu \equiv \frac{\partial X^\mu}{\partial \tau}, \quad X'^\mu \equiv \frac{\partial X^\mu}{\partial \sigma}$.

Mode expansion of X^μ for open string

$$X^\mu(\tau,\sigma) = \underbrace{\bar{x}^\mu + 2\alpha' p^\mu \tau}_{X^\mu_{cm}(\tau)} + i\sqrt{2\alpha'} \sum_{\substack{n\neq 0\\ n\in \mathbf{Z}}} \frac{\alpha_n^\mu}{n} e^{-in\tau}\cos(n\sigma),$$

where $$[\alpha_m^\mu, \alpha_n^\nu] = m\delta_{m+n,0}\eta^{\mu\nu}$$.

We choose the light-cone gauge,

$$X^+ = \bar{x}^+ + 2\alpha' p^+ \tau$$

$$X^\pm \equiv \frac{1}{\sqrt{2}}(X^0 \pm X^d)$$

Mass shell condition

$$M^2 = -p^\mu p_\mu \simeq \frac{N}{\alpha'}$$

where

$$N = \sum_{n=1}^{\infty} :\alpha^i_{-n}\alpha_{ni}: \qquad (\ i = 1, \cdots, d-1\ )$$

Observables in highly excited string states

Damour, Veneziano (1999)

In terms of the usual harmonic oscillators,

$$a_n^i = \frac{1}{\sqrt{n}}\alpha_n^i, \quad a_n^{i\dagger} = \frac{1}{\sqrt{n}}\alpha_{-n}^i \qquad (n > 0)$$

$$[a_m^i, a_n^{j\dagger}] = \delta^{ij}\delta_{mn}$$

The level of the open string becomes

$$N[N_n^i] = \sum_{n=1}^{\infty}\sum_{i=1}^{d-1} n N_n^i, \qquad N_n^i = a_n^{i\dagger} a_n^i$$

Number operator

Consider the following "canonical partition function",

$$\begin{aligned} Z(\beta) &= tr(e^{-\beta N}) \\ &= \sum_{\{N_n^i\}} \langle \{N_n^i\}| \exp(-\beta N) | \{N_n^i\}\rangle \\ &= \sum_{\{N_n^i\}} \exp(-\beta N[N_n^i]) \sim \exp\left(\frac{(d-1)\pi^2}{6\beta}\right) \end{aligned}$$

$$(N_n^i = 0, 1, 2, \cdots)$$

Since the density matrix is defined by

$$\rho = \frac{\exp(-\beta N)}{Z} \quad ,$$

an expectation value of an observable is evaluated by

$$\langle \mathcal{O} \rangle_\beta = tr(\rho \mathcal{O})$$

($\mathcal{O}$: Observable)

The mean value of the level and the fluctuation

$$\begin{cases} \bar{N} = \langle N \rangle_\beta = \dfrac{(d-1)\pi^2}{6\beta^2} \\ \Delta N = \sqrt{\langle (N-\bar{N})^2 \rangle_\beta} = \sqrt{\dfrac{(d-1)\pi^2}{3\beta^3}} \end{cases} \Rightarrow \frac{\Delta N}{\bar{N}} \sim \beta^{1/2}$$

If $\beta \ll 1$, we can obtain observables in highly excited string states.

Mass of the string

$$M^2 \simeq \frac{\bar{N}}{\alpha'} = \frac{(d-1)\pi^2}{6\beta^2\alpha'}.$$

Entropy of the string

$$\boxed{S = -tr(\rho \ln \rho) = 2\pi\sqrt{\frac{(d-1)\bar{N}}{6}}.}$$

This is consistent with the Cardy formula,

$$S = 2\pi\sqrt{\frac{cN}{6}},$$ (c : central charge)

with $c = d - 1.$

3. Shear viscosity of a highly excited string

Stress tensor of the open string

Source term

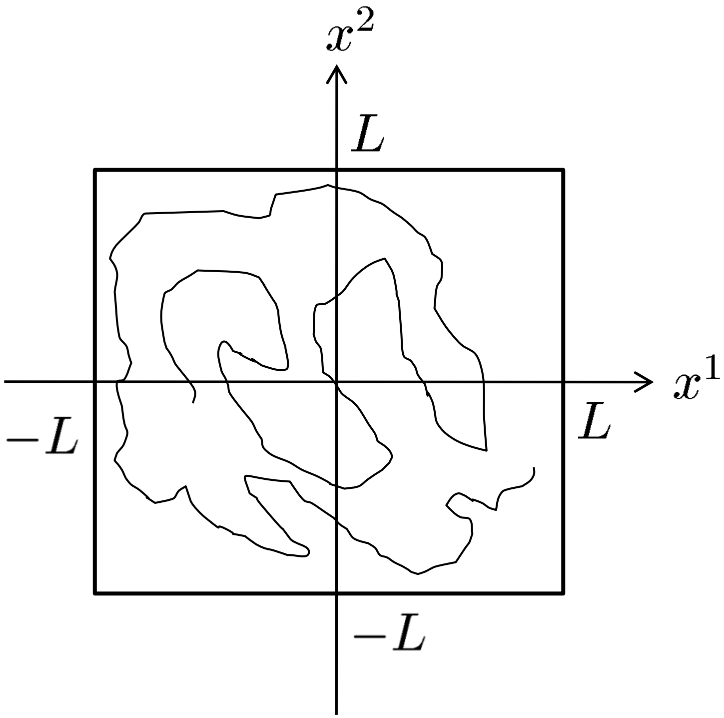
$$S_1 = \frac{1}{4\pi\alpha'} \int d^2\sigma (\dot{X}^\mu \dot{X}^\nu - X'^\mu X'^\nu) h_{\mu\nu}(X)$$
$$= \frac{1}{4\pi\alpha'} \int d^{d+1}x \int d^2\sigma (\dot{X}^\mu \dot{X}^\nu - X'^\mu X'^\nu) \delta^{d+1}(x - X) h_{\mu\nu}(x)$$
$$= \frac{1}{2} \int d^{d+1}x \, T^{\mu\nu}(x) h_{\mu\nu}(x)$$

After Insertion of the light-cone gauge and integration over τ ,

$$T^{\mu\nu}(x^+, x^-, x^i) = \frac{1}{4\pi\alpha'^2 p^+} \int_0^\pi d\sigma (\dot{X}^\mu \dot{X}^\nu - X'^\mu X'^\nu) \delta(x^- - X^-) \delta^{d-1}(x^i - X^i),$$

with $$\tau = \frac{x^+}{2\alpha' p^+}.$$

Since the stress tensor trivially vanishes outside the string sizes, we restrict the ranges of the spatial coordinates as follows:



$$-L_- \leq x^- - X_{cm}^- \leq L_-,$$
$$-L \leq x^i \leq L,$$

where we have chosen $\bar{x}^\mu = p^i = 0$.

For free open string,

$$L \sim l_s \sqrt{l_s M},$$
$$L_- \sim \frac{(l_s M)^{3/2}}{p^+}.$$

To obtain the viscosity, we just have to consider the long wave length limit.

→ Zero modes for spatial directions

Fourier expansions of the delta functions

$$\delta(x^- - X^-) = \frac{1}{2L_-}\sum_{n\in\mathbf{Z}} \exp\left(i\frac{\pi n}{L_-}(x^- - X^-)\right)$$

$$\delta(x^i - X^i) = \frac{1}{2L}\sum_{n\in\mathbf{Z}} \exp\left(i\frac{\pi n}{L}(x^i - X^i)\right)$$

Zero mode of the stress tensor

$$\bar{T}^{\mu\nu}(x^+) = \frac{1}{4\pi\alpha'^2 p^+ V_d}\int_0^\pi d\sigma(\dot{X}^\mu \dot{X}^\nu - X'^\mu X'^\nu),$$

where $V_d = 2^d L_- L^{d-1}.$

Kubo's formula for shear viscosity

$$\eta = \lim_{\omega\to 0} \frac{Im[G^R_{12,12}(\omega)]}{\omega},$$

$$G^R_{ij,kl}(\omega) \equiv iV_d \int dx^+ e^{i\omega x^+} \theta(x^+) \langle : [\bar{T}_{ij}(x^+), \bar{T}_{kl}(0)] : \rangle_\beta$$

(We have assumed that nonvanishing components of the metric perturbation are h_{ij} and they only depend on x^+ .)

Using

$$\bar{T}_{ij}(x^+) = \frac{1}{2\alpha' p^+ V_d} \sum_{\substack{n\neq 0 \\ n\in \mathbf{Z}}} \alpha^i_n \alpha^j_n e^{-2in\tau} \quad , \quad \langle : a^{i\dagger}_n a^j_m : \rangle_\beta = \frac{\delta_{nm}\delta^{ij}}{e^{\beta n} - 1},$$

We obtain

$$\eta = \sqrt{\frac{6}{d-1} \frac{M l_s}{2V_d}}.$$

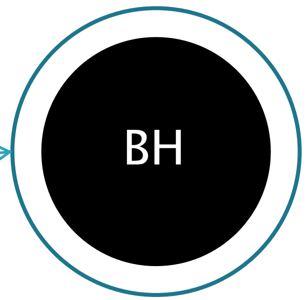
$$\frac{\eta}{s} = \frac{3}{2\pi(d-1)} = \frac{3}{2\pi c}$$

4. Shear viscosity of a string on a stretched horizon and black hole membrane paradigm

Difference between fictitious membrane and highly excited string

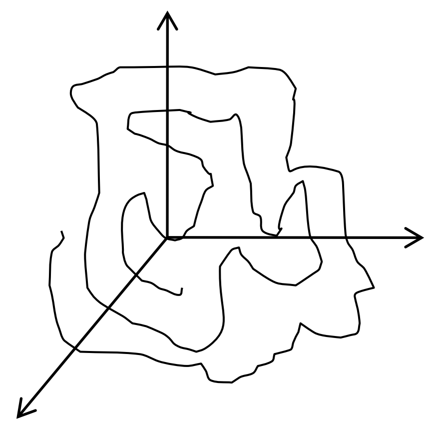
Membrane paradigm

$(d-1)$ dimensional surface



No radial thickness.

Highly excited string



Distributed in d spatial dimensions

	Membrane paradigm	Highly excited string
Mass dimension of stress tensor	d	$d+1$
Mass dimension of shear viscosity	$d-1$	d

We consider the longitudinally reduced stress tensor of the string.

Longitudinally reduced stress tensor

The stress tensor of the string can be written as

$$T^{ij}(x^+, x^-, x^i) = \frac{1}{2L_-}\sum_{n\in\mathbf{Z}} \tilde{T}^{ij}(x^+, n, x^i) e^{i\frac{\pi n}{L_-}x^-} \quad .$$

We define the longitudinally reduced stress tensor

$$\boxed{T_r^{ij}(x^+, x^i) \equiv \tilde{T}^{ij}(x^+, n=0, x^i)}$$

$$\Rightarrow \quad T_r^{ij}(x^+, x^i) = \frac{1}{4\pi\alpha'^2 p^+}\int_0^\pi d\sigma(\dot{X}^i\dot{X}^j - X'^i X'^j)\delta^{d-1}(x^k - X^k).$$

(Mass dimension = d)

The zero mode of $T_r^{ij}(x^+, x^i)$ for the transverse directions

$$\begin{aligned}\bar{T}_r^{ij}(x^+) &= \frac{1}{4\pi\alpha'^2 p^+(2L)^{d-1}}\int_0^\pi d\sigma(\dot{X}^i\dot{X}^j - X'^i X'^j)\\ &= 2L_-\bar{T}^{ij}(x^+),\end{aligned}$$

Shear viscosity of the longitudinally reduced string

Since $$\bar{T}_r^{ij} = \eta_r \frac{\partial h_{ij}}{\partial x^+},$$

the shear viscosity of the longitudinally reduced string is

$$\eta_r = \sqrt{\frac{6}{d-1}\frac{Ml_s}{2V_{d-1}}},$$

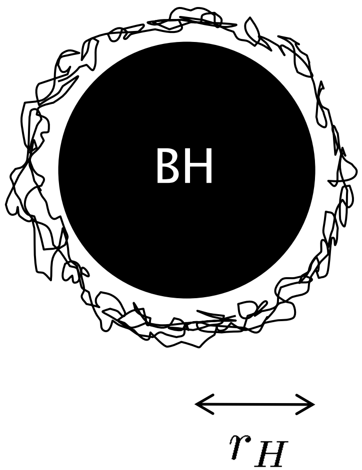
where $V_{d-1} \equiv (2L)^{d-1}$ is the volume of the transverse size of the string.

$\frac{\eta}{s}$ does not change if the string is longitudinally reduced

because this quantity is dimensionless.

$$\frac{\eta}{s} = \frac{3}{2\pi(d-1)} = \frac{3}{2\pi c}$$

Shear viscosity of the longitudinally reduced string on the stretched horizon



Shear viscosity of the longitudinally reduced string in the flat background

$$\eta_r = \sqrt{\frac{6}{d-1}\frac{M l_s}{2V_{d-1}}}$$

On the stretched horizon, we have to replace

$$M \to E_{sh} \sim \frac{r_H^{d-1}}{G l_s} \quad , \quad V_{d-1} \to r_H^{d-1}$$

$$\eta_r^{sh} \sim \frac{E_{sh} l_s}{r_H^{d-1}} \sim \frac{1}{G}.$$

This is consistent with the membrane paradigm

$$\eta_{BH} = \frac{1}{16\pi G}$$

<u>Consistency with string-black hole correspondence</u>

If $g_s^2 \sim \frac{1}{l_s M}$, a highly excited string becomes a black hole

with $r_H \sim l_s$.

At the critical string coupling, the shear viscosity of the string will be

$$\eta_r \sim \frac{l_s M}{l_s^{d-1}}$$

On the other hand, the shear viscosity in the membrane paradigm becomes

$$\eta_{BH} \sim \frac{1}{g_s^2 l_s^{d-1}} \sim \frac{l_s M}{l_s^{d-1}}$$

Consistent!

About the ratio of the shear viscosity to entropy density

In our estimate, $\frac{\eta}{s}$ does not change even if we put the string on the stretched horizon.

$$\frac{\eta}{s} = \frac{3}{2\pi(d-1)} = \frac{3}{2\pi c}$$

On the other hand,

$$\frac{\eta_{BH}}{s_{BH}} = \frac{1}{4\pi}.$$

If $c = 6$, $\frac{\eta}{s}$ of the string matches with that of the membrane paradigm.

5. Summary and comments

- We have obtained the shear viscosity and $\frac{\eta}{s}$ of the highly excited string by using the Kubo's formula.

- We have estimated the shear viscosity and $\frac{\eta}{s}$ of the string on the stretched horizon of the black hole.

- The results are consistent with the black hole membrane paradigm.

- We have not considered the self-interactions of the highly excited string. This will lead to the g_s corrections to shear viscosity.

- It is important to investigate whether the correct numerical coefficient of the shear viscosity in the membrane paradigm can be derived from superstring theory.

- We have not discussed the bulk viscosity because we could not reproduce the negative bulk viscosity of the membrane paradigm from the highly excited string on the stretched horizon.

- It is interesting to find transport coefficients of a highly excited string when source fields are given by other fields instead of metric.