Seiberg-Witten prepotential from Liouville classical conformal block

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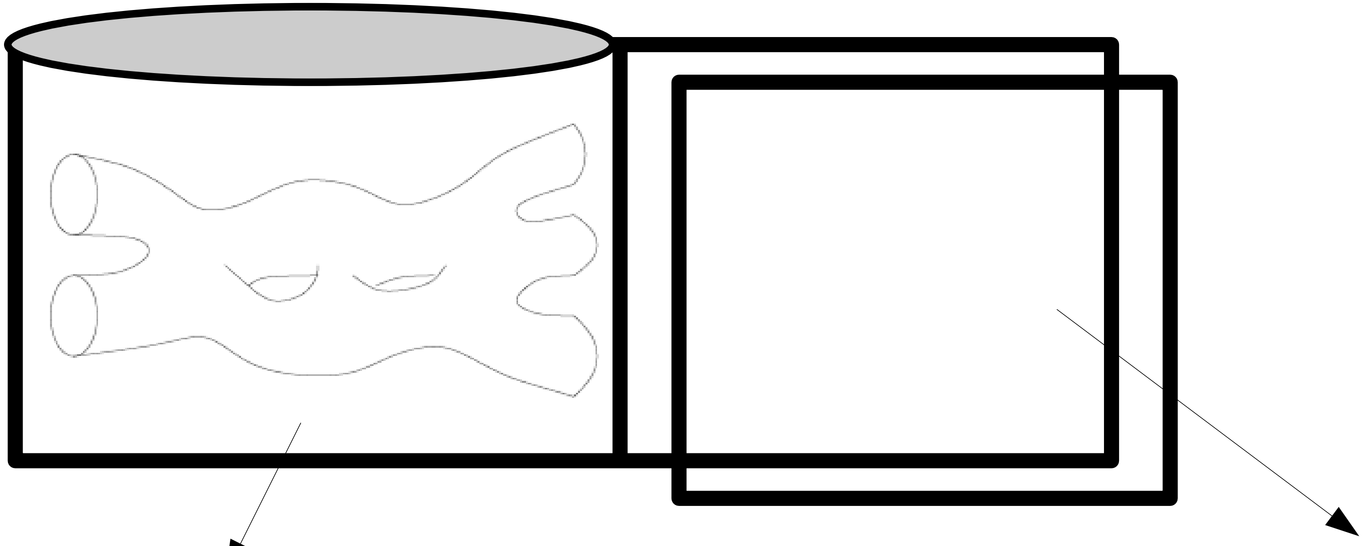
Based on 1008.4332

JHEP 1010 107 (2010)

Review: AGT conjecture



LFT on Riemann surface C **4D N=2 SU(2) SCFT (Coulomb phase)**

Information of C ⟺ **properties of 4D QFT**

dictionary

Ex

Complex moduli of C **UV gauge coupling**

.... and so on

AGT dictionary

Given any C, one has certain SCFT !!

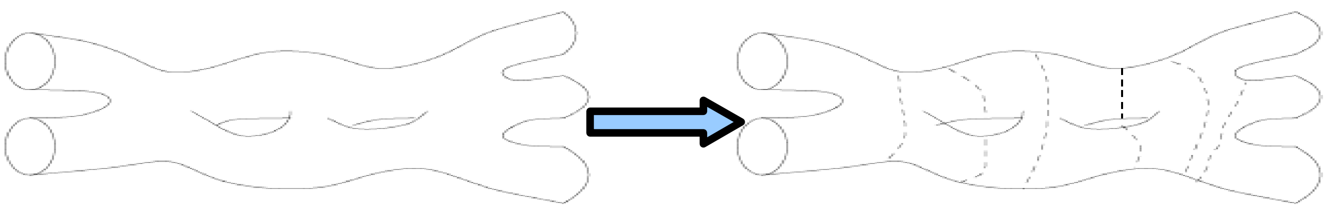
(I) # of gauge coupling (or group) = χ=(3g-3+n)

(II) flavor mass = weight of insertion

(III) Coulomb parameter = internal momentum

(IV) SW curve = double cover of C

Ex



χ=(3g-3+n)=8, # of Coulomb parameters is eight!!

Ways of bootstrap are many...

Main idea is …

Re-express C owing to Poincare, Koebe and Klein
(Uniformization theorem, about 100 years ago)

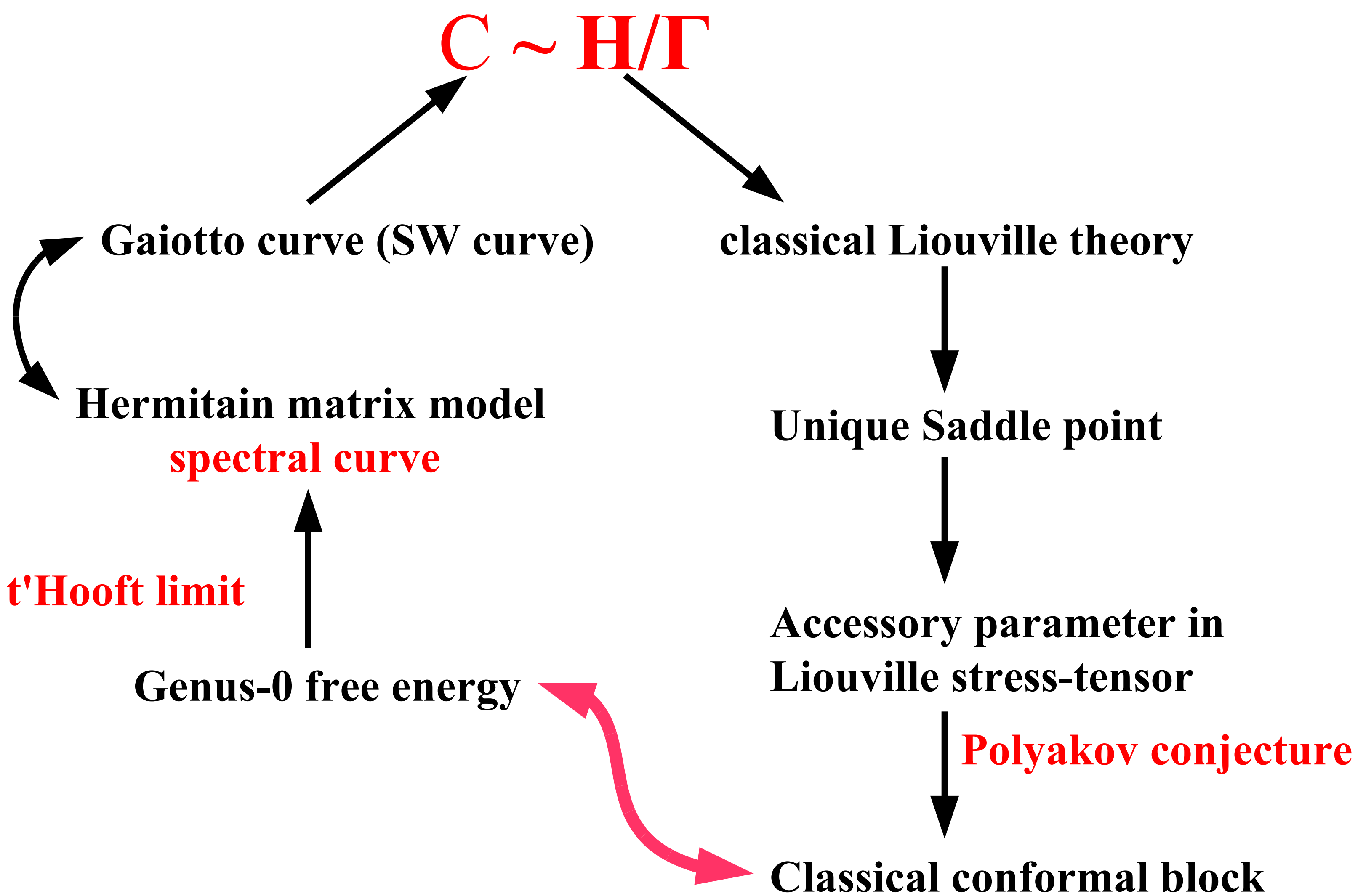
- Uniform punctured Riemann surface by upper half-plane **H** (universal cover)
- Endow C ~ **H/Γ** with hyperbolic metric w/ discrete Fuchsian group **Γ**

C ~ **H/Γ**

Gaiotto curve (SW curve)

Liouville equation
(classical Liouville theory)

Classical version of AGT conjecture !!


C ~ H/Γ
Gaiotto curve (SW curve)
classical Liouville theory
Hermitain matrix model
spectral curve
Unique Saddle point
t'Hooft limit
Accessory parameter in
Liouville stress-tensor
Genus-0 free energy
Polyakov conjecture
Classical conformal block

Plan of talk

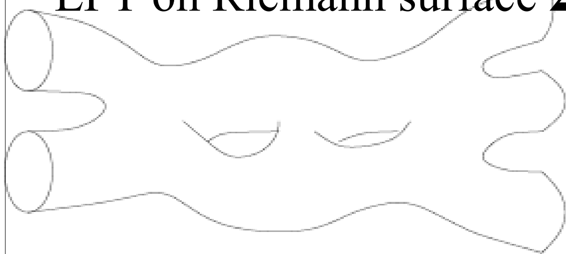
- Introduction—very brief review of AGT relation (2pp)
- What is Uniformization (均一化) (6pp)
- Big picture (2pp)
- Example I, II & III (4pp)

(extract SW prepotential from classical conformal block)

- Summary (1pp)

Introduction—very brief review of AGT relation

LFT on Riemann surface Σ



6D N=(0,2) A-type/Σ

partial twist

4D N=2 SU(2) SCFT

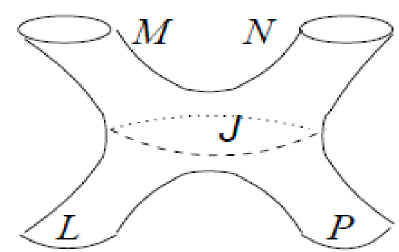
Complex moduli of Σ ⟷ UV gauge coupling

(for two-sphere, puncture needed)

Ex

χ=(3g-3+n)=1 for (g,n)=(0,4) or (g,n)=(1,1)

n=4 w/ (0,1,λ,∞), four-punctured P^1 ~ torus

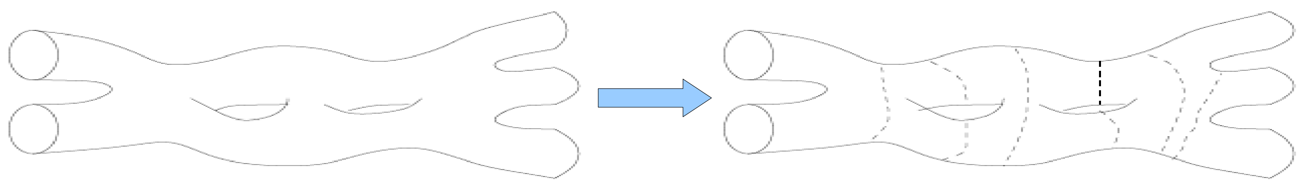


UV gauge coupling λ ⟷ IR gauge coupling τ (lambda function)

AGT dictionary

(I) # of gauge coupling (or group) = χ=(3g-3+n)
(II) flavor mass = weight of insertion
(III) Coulomb parameter = internal momentum
(IV) SW curve = Σ

Ex



χ=(3g-3+n)=8, # of Coulomb parameters is eight!!
Ways of bootstrap are many...

What is Uniformization (均一化)

- Uniform punctured Riemann surface by upper half-plane (universal cover)
- Endow $\Sigma \sim \mathbf{H}/\Gamma$ with hyperbolic metric

→ conformal factor φ

→ negative constant curvature → Liouville equation

→ saddle point (e.o.m.) of Liouville field theory (LFT)

→ classical regime of LFT

Dawn(曙) of the bridge in between <u>LFT</u> & <u>SW curve</u>!!

Without relying on AGT conjecture!!

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From conformal factor φ we can have

→ stress-tensor T of LFT : $T_L(z) = Q\partial_z^2\phi - (\partial_z\phi)^2$ / $T(z) \equiv \frac{1}{2}\partial_z^2\varphi_{cl} - \frac{1}{4}(\partial_z\varphi_{cl})^2$

→ 2nd order ODE of Fuchsian type $\left(\partial_z^2 + T(z)\right)\Psi = 0$ on Σ

→ determine T by either **monodromy** on Σ or **Polyakov conjecture (Ward identity)**

Ployakov conjecture: $$c_i = -\frac{\partial\mathcal{S}[\varphi_{cl}(\delta_i, z_i)]}{\partial z_i}, \qquad i \neq (0, 1, \infty).$$

$$c_2(x) = -\frac{\partial\mathcal{S}_{cl}[\xi_i; x]}{\partial x} = \left(\frac{\partial}{\partial x} f_\delta \left[\begin{smallmatrix}\delta_3 & \delta_2 \\ \delta_4 & \delta_1\end{smallmatrix}\right](x)\right)_{p=p_s(x)}$$

Basically, it is Ward identity of T (next page)

Make possible a direct connection to classical conformal block f

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Ward identity of T

$$\langle T(z) X_\alpha \rangle = \sum_{i=1}^{n} \left(\frac{\Delta_{\alpha_i}}{(z-z_i)^2} + \frac{\partial_{z_i}}{z-z_i} \right) \langle X_\alpha \rangle \qquad X_\alpha = V_{\alpha_1}(z_1) \ldots V_{\alpha_n}(z_n)$$

$$T = \sum_{i=1}^{n-1} \left(\frac{\Delta_{\alpha_i}}{(z-z_i)^2} + \frac{c_i}{z-z_i} \right)$$

For four-punctured sphere

$$c_2(x) = -\frac{\partial \mathcal{S}_{cl}[\xi_i; x]}{\partial x} = \left(\frac{\partial}{\partial x} f_\delta \left[\begin{array}{cc} \delta_3 & \delta_2 \\ \delta_4 & \delta_1 \end{array} \right](x) \right)_{p=p_s(x)}$$

Accessory parameter is determined by Polyakov conjecture,
proved by mathematicians [Leon Takhtajan et al]

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$$c_2(x) = -\frac{\partial \mathcal{S}_{cl}[\xi_i; x]}{\partial x} = \left(\frac{\partial}{\partial x} f_\delta \left[\begin{array}{cc} \delta_3 & \delta_2 \\ \delta_4 & \delta_1 \end{array}\right](x)\right)_{p=p_s(x)}$$

classical geometry Σ ⟺ **classical conformal block**

Reason: factorization of LFT action as ***b goes to 0***

$$\mathcal{S}_{cl}[\xi_1, \cdots, \xi_4; x] = \mathcal{S}^{(3)}(\delta_4, \delta_3, \delta_s) + \mathcal{S}^{(3)}(\delta_s, \delta_2, \delta_1) - f_{\delta_s, \delta_i}(x) - f_{\delta_s, \delta_i}(\bar{x}).$$

Ultimately, we find ***f***=SW prepotential (instanton part)

Reason: analogous to Hermitian matrix model at ***large-N***

?

$$c_2(x) = \frac{\partial}{\partial x} \hbar^{-2} \mathcal{F}_0$$

$$\langle T_M(z)\rangle \to \mathcal{W}'(z)^2 + f(z)$$

$$Z = \frac{1}{\mathrm{vol}U(N)} \int_{N\times N} dM \exp\left(\frac{1}{\hbar} \mathrm{Tr}\, \mathcal{W}(M)\right) = \exp\left(\sum_{g\geq 0} \hbar^{2g-2} \mathcal{F}_g\right)$$

Gaiotto curve ⟺ **Nekrasov partition function**

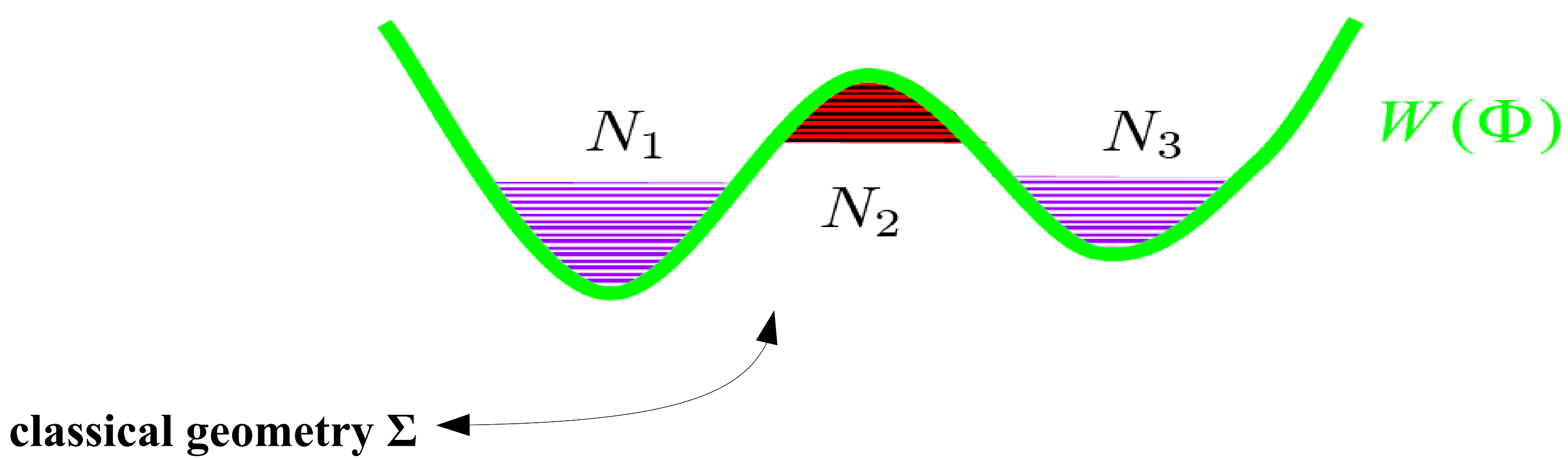
Degression to Hermitian Matrix Model

$$Z = \int_{N\times N} d\Phi \; e^{\mathrm{Tr}\, W(\Phi)/\hbar}$$

't Hooft limit

$$N \to \infty,\ \hbar \to 0,\ \hbar N = t\ constant$$

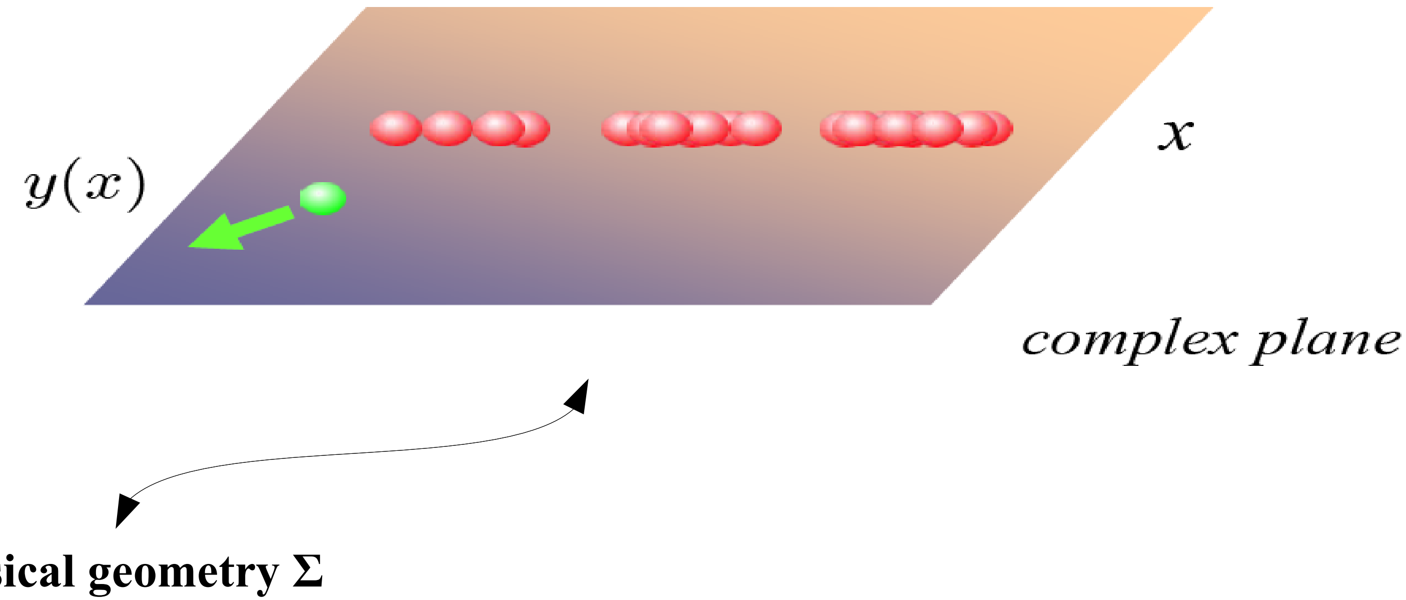
Filling fractions $t_i = \hbar N_i$

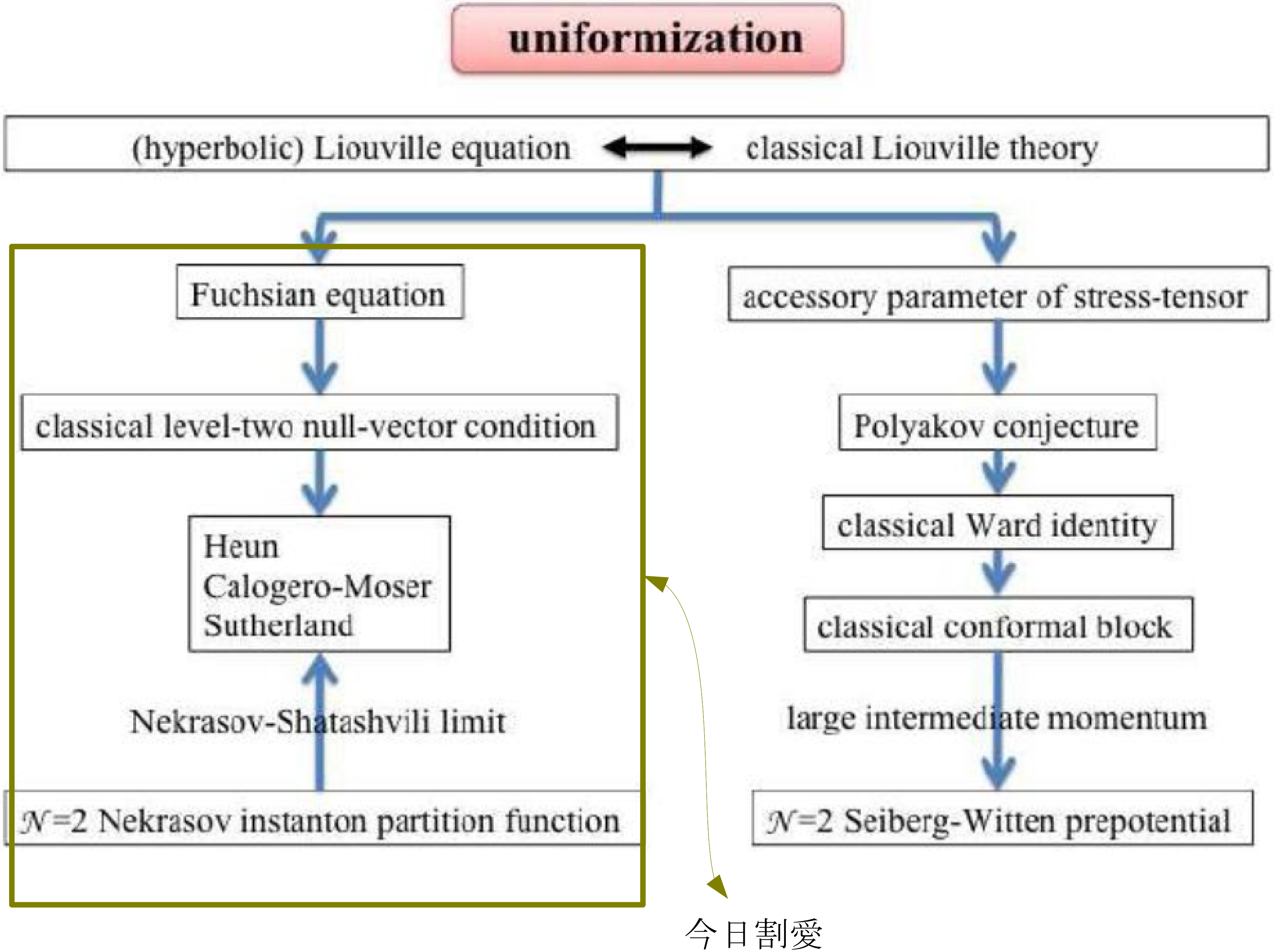


$$Z = \int d^N\lambda \prod_{i<j} (\lambda_i - \lambda_j)^2 \exp \sum_i W(\lambda_i)/\hbar$$

Effective force/resolvent

$$y(x) = \frac{dS_{eff}}{dx} = W'(x) - 2\hbar \mathrm{Tr} \frac{1}{x - \Phi}$$




uniformization
(hyperbolic) Liouville equation
classical Liouville theory
Fuchsian equation
classical level-two null-vector condition
Heun
Calogero-Moser
Sutherland
Nekrasov-Shatashvili limit
$\mathcal{N}$=2 Nekrasov instanton partition function
accessory parameter of stress-tensor
Polyakov conjecture
classical Ward identity
classical conformal block
large intermediate momentum
$\mathcal{N}$=2 Seiberg-Witten prepotential
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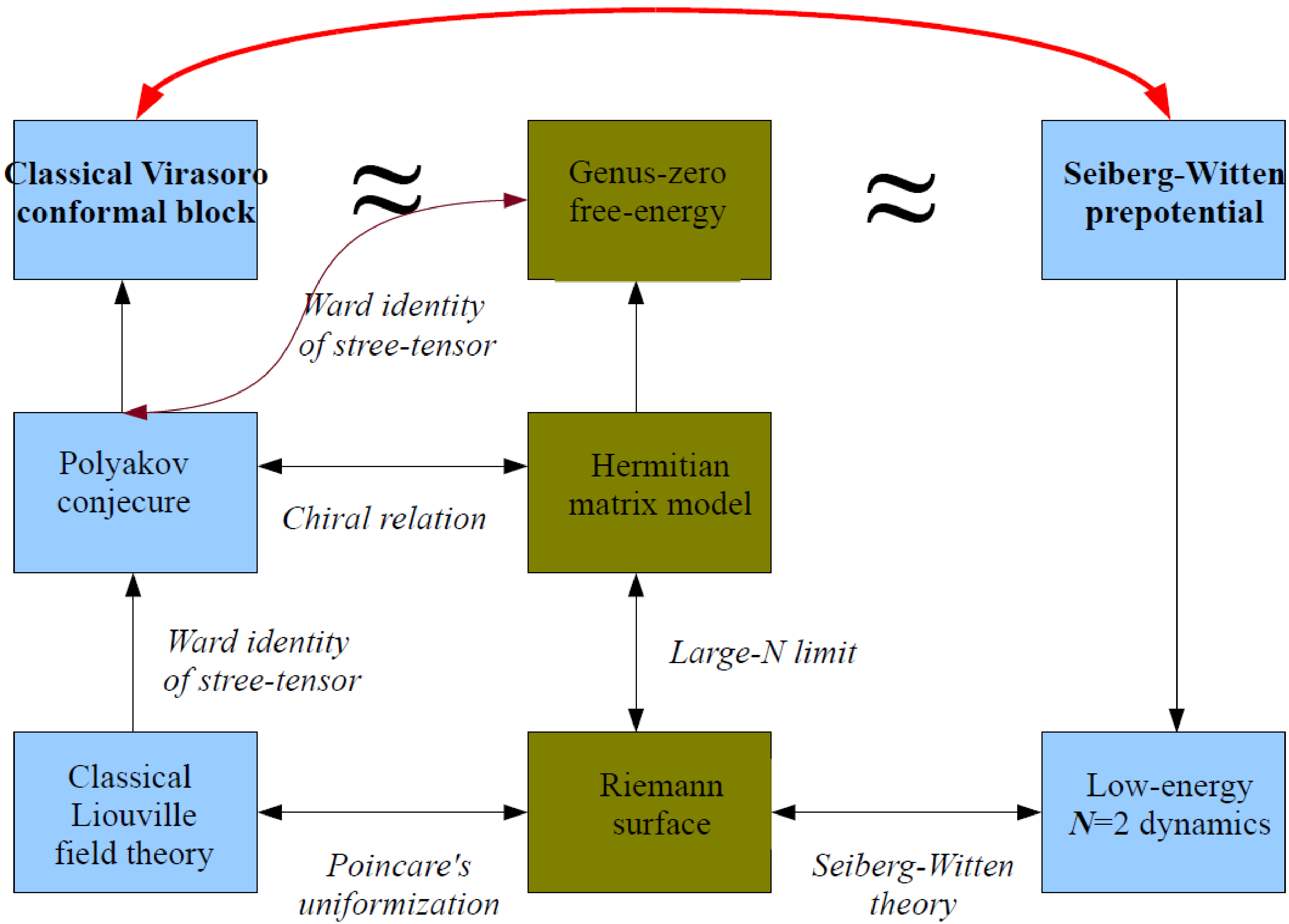
Why classical Virasoro conformal block reproduces Seiberg-Witten prepotential ?

Unexpected or not ...



Example I, II & III

(extract SW prepotential from classical conformal block)

(I) All $m_i = 0$

(II) $m_1 = m_2 = \frac{\xi}{4}$ **and** $m_3 = m_4 = 0$

(III) Arbitrary four flavor masses

bare flavor masses and weights assigned to punctures are related by

$$\begin{cases} \xi_1 = m_1 + m_2 + \frac{1}{2}, & \xi_2 = -m_1 + m_2 + \frac{1}{2}, \\ \xi_3 = m_3 + m_4 + \frac{1}{2}, & \xi_4 = -m_3 + m_4 + \frac{1}{2}, \end{cases}$$

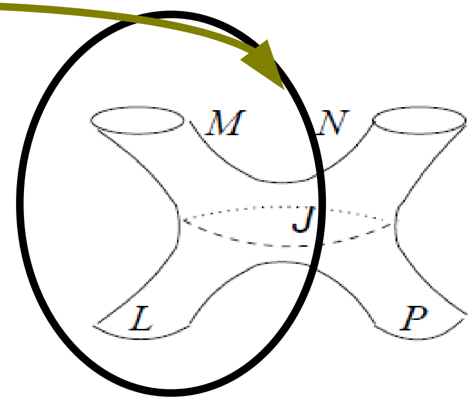
Example (I)

$$f_{\delta_s,\delta_i=\frac{1}{4}}(x)$$
$$= (p_s^2 - \frac{1}{4}) \log x + (p_s^2 + \frac{1}{4})\frac{x}{2} + (\frac{13p_s^2}{16} + \frac{9}{32} + \frac{1}{256p_s^2 + 256})\frac{x^2}{4} + \cdots$$

$$\rightarrow a^2 \left(\log x + \frac{x}{2} + \frac{13x^2}{64} + \frac{23x^3}{192} + \cdots \right)$$

$$a \equiv \oint dz \sqrt{T(z)} = \frac{\ell(\gamma_{12})}{4\pi b}$$

$$x \equiv \exp(2\pi i \tau_{UV})$$



Perfect agreement, up to a perturbative piece -log16

Example **(II)** $m_1 = m_2 = \frac{\xi}{4}$ **and** $m_3 = m_4 = 0$

$$f_{\frac{1}{4}+p^2}\left[\begin{array}{cc} \frac{1}{4} & \frac{1-\xi^2}{4} \\ \frac{1}{4} & \frac{1}{4} \end{array}\right](x) = (p^2 - \frac{1-\xi^2}{4}) \log x + (\frac{1-\xi^2}{8} + \frac{p^2}{2})x$$
$$+\left(\frac{9(1-\xi^2)}{128} + \frac{13p^2}{64} + \frac{(1-\xi^2)^2}{1024(1+p^2)}\right)x^2 + \mathcal{O}(x^3)$$

$$\mathcal{F}^{SW}_{inst} = (a^2 - m^2)\log x + (a^2+m^2)\frac{x}{2} + \left(13a^2 + 18m^2 + \frac{m^4}{a^2} + \mathcal{O}(a^{-4})\right)\frac{x^2}{64} + \mathcal{O}(x^3)$$

carrying out $(a^2, m^2) \to (p^2, -\frac{\xi^2}{4})$. up to a perturbative piece

Example (III) Arbitrary four flavor masses

$$f_{\delta,\delta_i}(x) = (\delta - \delta_1 - \delta_2)\log x + \frac{(\delta+\delta_1-\delta_2)(\delta+\delta_3-\delta_4)}{2\delta}x + \mathcal{O}(x^2)$$

$$\to (a^2 + 2m_1^2 + 2m_2^2)\log x + \frac{a^4 - 4a^2(m_1m_2 + m_3m_4) + 16m_1m_2m_3m_4}{2a^2}x + \cdots$$

$$\longrightarrow \quad (a^2 - m^2)\log x + \frac{(a^4 + 2a^2m^2 + m^4)}{2a^2}x + \cdots$$

Especially,

for $m_1 = m_2 = m_3 = m_4 = \frac{i}{2}m$ which agrees to (3.48) of [24] up to the $U(1)$ part

Eguchi & Maruyoshi JHEP 1007 (2010) 081
arXiv:1006.0828

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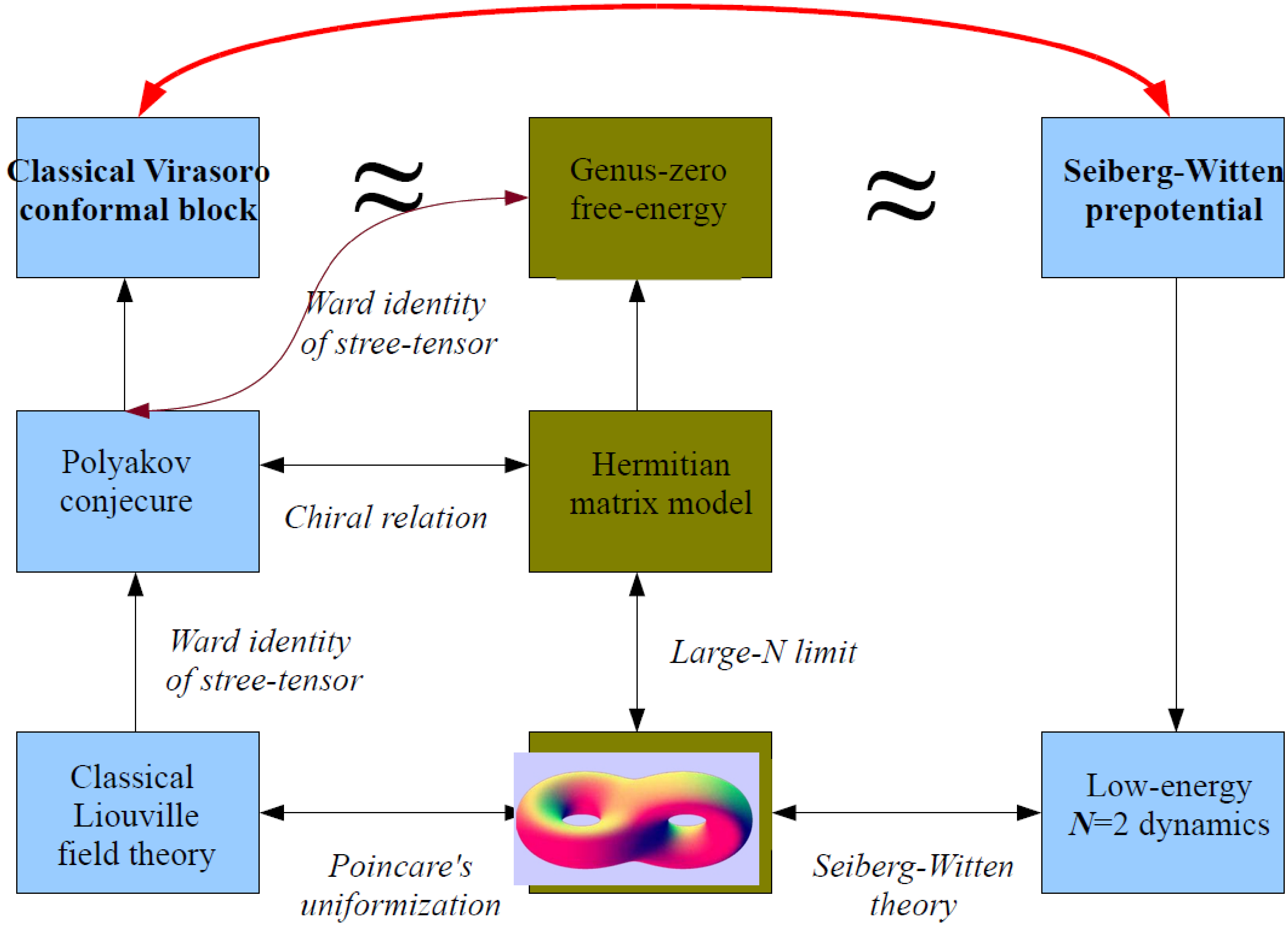
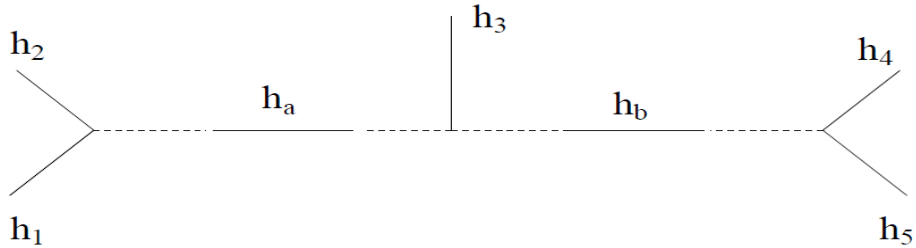
Unexpected or not ...



THE END



$$\mathcal{F}^{5pt}_{g=0} = 1 + \frac{(-h_1+h_2+h_a)(h_3+h_a-h_b)}{2h_a}q_1 + \frac{(-h_4+h_5+h_b)(h_3-h_a+h_b)}{2h_b}q_2 + \cdots$$