Quantum Gravity

— discrete vs. continuum —

Yuki Sato[*][†]

Department of Physics, Graduate School of Science, Nagoya University
Furo-cho, Chikusa-ku, Nagoya 464-8602, Japan

Abstract

Over a long period of time, there exists the notorious or deserved-to-attack problem in quantizing the $(3+1)$-dimensional gravity, uncontrollable ultraviolet divergences. Against this open problem, a brand-new breakthrough has been proposed by P. Horava, which we call the Lifshitz-type gravity because it is closely related to the so-called Lifshitz point in the condensed matter physics. The most amazing thing is its power-counting renormalizability in $3+1$ dimensions. What is paid in return is the requirement of the strong anisotropy between space and time in high energies. However this theory is expected to flow naturally to the relativistic general relativity in low energies, which is not confirmed (up to January 2010).

In this thesis, we review the Lifshitz-type gravity, and try to uncover some aspects embedded in this profound theory. Further we would like to discuss the possibility that this Lifshitz-type gravity in $3+1$ dimensions may be the continuum limit of the $(3+1)$-dimensional causal dynamical triangulation of the Lorentzian space-time manifold, which has been pointed out by P. Horava. To carry it out, we model on the discrete-vs.-continuum structure in the 2-dimensional quantum gravity. In this line of thought, we review the 2-dimensional dynamical triangulation and its matrix model dual, and we would like to see how its continuum limit becomes the bosonic Liouville field theory with $c = 0$. Then we discuss the $(1+1)$-dimensional causal dynamical triangulation which poses the causality on the dynamical triangulation, and discuss its continuum limit. Taking advantage of the intriguing structure of the 2-dimensional quantum gravity, we investigate the relation between the Lifshitz-type gravity in $3+1$ dimensions and the causal dynamical triangulation in $3+1$ dimensions via the spectral dimension.

[*]Email: ysato(at)th.phys.nagoya-u.ac.jp

[†]This review article is based on author's master thesis submitted to Nagoya University in January 2010, and revised in November 2018.

Omnibus ex nihil ducendis sufficit unum. — J. A. Wheeler

Contents

1 Introduction

Every theory in physics has its applicable range. Extending its range stands for constructing the more fundamental theory. In the case of gravity, the corresponding fundamental theory is Einstein's general relativity. General relativity can predict the gravitational dynamics at the astronomical scale such as Mercury's precession rate of 43 seconds of arc per century. Up to the present date, there is no discrepancy between general relativity and experiments. Although general relativity is very strict, it is supposed to be broken down at "short" range. In the present accelerator experiments, the shortest scale which we can probe is about 10^{-17}cm, so that the "short" range means the range shorter than 10^{-17}cm. Once general relativity goes over its applicable range, we need to find out the more fundamental theory of gravity which covers the shorter range. That is *quantum gravity*. The word "quantum" is named after the theory which can describe the microscopic range, that is quantum mechanics. Is general relativity really affected by the quantum fluctuations at "short" range? We can answer this question by remembering Einstein's equation:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 8\pi G_N T_{\mu\nu},$$

where G_N is Newton's gravitational coupling constant. The left-hand side is the beautiful purely geometric part. On the other hand, the right-hand side is the matter part. The matter part must be quantized by quantum mechanics. Thus the metric in the left-hand side must be affected by the quantum fluctuations via the right-hand side of the equation. This is the reason why we need to construct quantum gravity.

However, when we try to construct quantum gravity there exists the difficult problem, uncontrollable ultraviolet divergences. The high momentum (ultraviolet) region is equivalent to the short region because of Heisenberg's uncertainty relation. Thus this problem means that the quantum fluctuation can be too large to handle in the short region. When we consider the small fluctuation of the metric from the flat background, this problem occurs. We call such a small fluctuation *the graviton* denoted by $h_{\mu\nu}$. If we expand the Einstein-Hilbert action,

$$S_{EH} = \frac{1}{16\pi G_N}\int d^4x\sqrt{-g}(R-2\Lambda),$$

with respect to the graviton $h_{\mu\nu}$, the uncontrollable or in other words nonrenormalizable ultraviolet divergences emerge. This fact can be also understood via the negative mass dimension of G_N, that is $[G_N] = -2$.

To overcome this situation it is natural that by considering the Einstein-Hilbert action as the low-energy effective action we insert the higher-derivative terms such as R^2, $R_{\mu\nu}R^{\mu\nu}$ into the Einstein-Hilbert action. Further expressing the graviton in terms of the momentum p, we can derive the quantum fluctuation of the gravitational field as

$$\langle |h_{\mu\nu}(p)|^2\rangle = \frac{1}{ap^2 + b(p^2)^2 + c(p^2)^3 + \dots} = \frac{\alpha}{p^2} + \frac{\beta}{p^2+m^2} + \frac{\gamma}{p^2+\tilde{m}^2} + \dots,$$

At first sight, this seems to cure the ultraviolet divergences because in the high momentum region the quantum fluctuation converges more quickly than p^{-2}. However, the other problem shows up. Taking a look at the second equality, we notice that letting the fluctuations converge more quickly than p^{-2}, at least one of coefficients (α, β, γ, ...) must be negative. In fact the coefficient expresses the transition probability to some state with the mass (m, $\tilde{m}$, $\dots$), and the negative coefficient means the negative probability [1]. This is not good news.

In the present, physicists are struggling to find out the true consistent quantum gravity both theoretically and phenomenologically. The most well-known candidate is *superstring theory.* This sophisticated theory avoids ultraviolet divergences by considering the fundamental element is the 1-dimensional string. In this case, the loop in the Feynmann diagram does not crumple, that is no ultraviolet divergence. Another candidate for quantum gravity is *loop quantum gravity.* This theory is based on the idea that the perturbative method is not suitable for the gravity so that quantum gravity must be formulated nonperturbatively. Thus loop quantum gravity is constructed in terms of not the path-integral formalism but the Hamiltonian formalism. Contrary to that, there exists the approach by using the path-integral method, and in this line the most prosperous one is the so-called *causal dynamical triangulation* which we focus on in this thesis. We can list lots of other possibilities. The existence of various models implies the difficulty of testing the quantum effects of gravity by expremental. Unfortunately, more or less every-existing theory has problems. It seems that getting the true quantum theory of gravity we need to overcome some barrier by some acrobatic point of view.

In the situation like above, the intriguing model has been proposed by P. Horava [2, 3]. In this thesis, we call it the *Lifshitz-type gravity* because it is closely related to the Lifshitz point in the condensed matter physics. The Lifshitz type gravity requires the different scaling between space and time at high-energy region, which makes the theory being power-counting renormalizable. Anisotropic scaling means the violation of the Lorentz symmetry which is the symmetry of the background space-time. Can this model be a consistent quantum gravity? We can say yes in the sense that if the space-time is an emergent object from some mechanism at high-energy scale, then the symmetry of the space-time, the Lorentz symmetry, is not fundamental but effective. The validity of this statement must be discussed seriously. In this thesis we would like to follow up the possibility that the Lorentz symmetry is the low-energy effective symmetry.

Furthermore, P. Horava also has proposed the possibility that the Lifshitz-type gravity may be the continuum limit of the causal dynamical triangulation [4]. The causal dynamical triangulation is the lattice-regularized gravity constructed by the Lorentzian path-integral. If we can verify this possibility, it must be meaningful for both theories. Further we may understand something about quantum theory of gravity. Even the tiny information is tremendously important if it is true. Our goal in this thesis is approaching this possibility as possible. Our tactics toward this purpose is modeling on the discrete-vs.-continuum structure of the 2-diemnsional pure quantum gravity.

This thesis consists of five parts:

- **Lifshitz-type gravity** (Detailed balance condition)
- 2 **dimensions** (String susceptibility)
- 4 **dimensions** (Spectral dimension)
- **Concluding remarks**
- **Appendices**

[1] This argument is based on [1].

We listed the title of each part, and add the keyword in the bracket. The first three are the main parts of this thesis. The each part has several sections.

—**Lifshitz-type gravity**—

In section 2, we introduce the basic idea of a Lifshitz point and its field theory. Through this section, one can understand the space-time anisotropy in the Lifshitz-type theory. In section 3, which is the key section in this part, we focus on the concept, the detailed balance condition. This is the guiding principle when we construct the action of Lifshitz-type theories. In section 4, we explain the Lifshitz-type gravity models, and then we try to find out some intriguing aspects of these models. This section is the one of main sections in this thesis. Section 5 is the discussion of this part.

—**2 dimensions**—

In section 6, based on the concept that the Liouville field theory with $c = 0$ is equivalent to the Euclidean 2-dimensional pure gravity, at first we see how the Liouville action emerges as a consequence of the Weyl anomaly. Then we construct the renormalized Liouville action, and using it we calculate the critical exponent, the string susceptibility. This is the indicative quantity when we discuss about the equivalence between the lattice-regularized theory and the continuum theory. In section 7, we review the Euclidean dynamical triangulation in detail. The goal of this section is understanding the concepts in the dynamical triangulation of the space-time and the matrix model dual. The important point here is the string susceptibility derived by the dynamical triangulation (or the matrix model). In section 8, we introduce the causal dynamical triangulation, and focus on how to impose the causality on the dynamical triangulation. In section 9, we generalize the causal dynamical triangulation by allowing the configurations which violates the causality “mildly”. Section 10 is the discussion of this part.

—**4 dimensions**—

In section 11, we extend the causal dynamical triangulation in $1 + 1$ dimensions to in $3 + 1$ dimensions. The important point here is the building block in the regularization, that is the 4-simplex. In section 12, we discuss about the one of main points in this thesis, the spectral dimension. We compare the spectral dimensions derived by the causal dynamical triangulation and the Lifshitz-type gravity. Section 13 is the discussion of this part.

—**Concluding remarks**—

In this part, we discuss the possibility that the continuum limit of the causal dynamical triangulation is the Lifshitz-type gravity, and comment other possibilities.

—**Appendices**—

In appendix A, we review the fundamental ideas of the ADM decomposition of general relativity. This helps when one reads the section of the Lifshitz-type gravity. In appendix B, we introduce the derivation of the so-called *Regge action*. This action becomes the basic concept in the lattice gravity. In appendix C, we explain about the matrix model. In this appendix, one can understand the large-N limit, the 'tHooft coupling, and the Vandermonde determinant. We calculate the free energy in the spherical limit.

Part I

Lifshitz-type gravity

2 Theories at Lifshitz point

First of all, we explain about the Lifshitz point [5] and its property based on the Landau theory in the condensed matter physics[2]. Then we will define the precise meaning of placing the theory at a Lifshitz point, which is the central idea of this section. As a consequence, the theory at a Lifshitz point is Lorentz violating one in general, which also can be seen at the end of this section.

2.1 Lifshitz point

Before discussions, we would like to review some terminologies and concepts in phase transitions. To begin, we define the phase as the region of a thermodynamic system where its physical properties are uniform, and call its variation *the phase transition*. More precisely, the phase transition is some drastic change of the medium in the system. The phase transition occurs at the characteristic point called *the critical point*, and the temperature at this point is called *the critical temperature*, T_c. For instance, the water has three phases, the solid, the liquid, and the vapor. Of course the medium is H_2O. In this case, the evaporation is one of the phase transition, and its critical temperature is about 100^0C. Often but not always the phase transition occurs among phases with different symmetry. If we focus on one symmetry and consider two phases with and without the symmetry, then we call the phase with the symmetry *the ordered phase* and do *the disordered phase* for others. As for the liquid/solid transition in the water, the liquid phase corresponds to the disordered phase, and on the other hand the solid phase does the ordered phase. In such a case, the symmetry is the translational symmetry.

In the following, we concentrate only on this class of phase transitions. Now we would like to know how we can distinguish different phases. It can be done by introducing the order parameter. We call the thermodynamic average of some operator an *order parameter* ϕ if it can distinguish the ordered phase from the disordered phase. Again for the water, the density is the order parameter. By using the order parameter, we can also classify the two types of phase transitions. The transition is called *the second order* if the order parameter rises continuously from zero in the variation of the temperature. Meanwhile the transition is called *the first order* if the order parameter jumps discontinuously to nonzero. Now let us step into the Landau theory. The basic concept of the Landau theory is expanding the free energy in terms of the order parameter in the sense that the order parameter is small, and according to the required symmetry of the phase we vary the coefficients of the order parameter. Nature is known to choose the lowest value of the free energy so that if we can know the value of the order parameter which gives the lowest free energy in each phase, then we can identify the different phases. This type of the free energy is called *Landau phenomenological free energy* because it can be determined phenomenologically.

The general form of this free energy (density) can be written as follows:

$$F = a\phi^2 + b\phi^4 + c\phi^6 + \cdots + \alpha(\nabla\phi)^2 + \beta(\nabla^2\phi)^2 + \cdots , \tag{2.1}$$

[2] Discussion is based on [6].

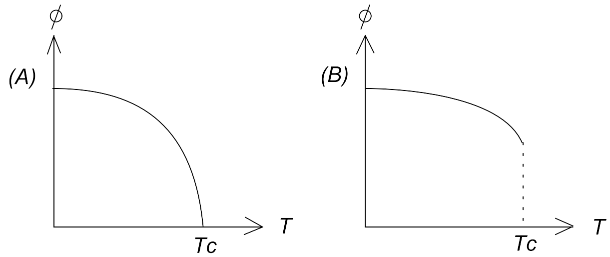


Figure 1: (A): The second order phase transition, (B): The first order phase transition

where ϕ is the order parameter and $\nabla\phi$ is its derivative. All coefficients are the function of the temperature. As for the second order phase transition, corresponding free energy can be written up to irrelevant terms such as

$$F \sim a\phi^2 + b\phi^4, \tag{2.2}$$

where b is greater than zero. In this case, the coefficient a can classify the phases. If we take ϕ as the magnetization, the region $a < 0$ shows the spontaneous magnetization whose free energy is the well know wine-bottle type. $a = 0$ region stands for the critical point (criticality). Explaining

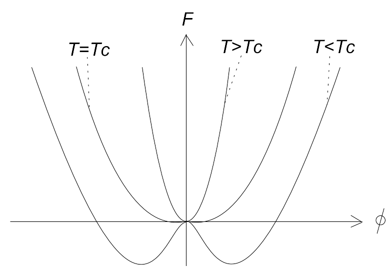


Figure 2: The spontenious magnetization

by words alone, in the first order case, the coefficient b plays a role of a in the second order case. We would like to introduce the Lifshitz point as the second example [5]. The Lifshitz point is the tricritical (end) point as to the vector-like (scalar-like) order parameter. The critical end point means that a line of the second order phase transition terminates by the first order line. In the Lifshitz type tricritical endpoint, the corresponding free energy is written by

$$F \sim a\phi^2 + b\phi^4 + \alpha(\nabla\phi)^2 + \beta(\nabla^2\phi)^2. \tag{2.3}$$

where $b > 0$ and $\beta > 0$. This type of the tricritical endpoint is obviously different from the standard tricritical endpoint. At the Lifshitz point, coefficients, a and α, become zero so that the free energy at the Lifshitz point is

$$F_{LS} \sim b\phi^4 + \beta(\nabla^2\phi)^2. \tag{2.4}$$

In high energies, the leading term of the above is $\beta(\nabla^2\phi)^2$, and if we construct the action containing this term as the potential, then we will call such a manipulation *placing the theory at a Lifshitz* *point* as a definition. For example, such a theory is written by the following action [6, 7]:

$$S = -\frac{1}{2}\int d^D x dt \Bigg[\Big(\partial_t \phi(x,t)\Big)^2 - \Big(\frac{1}{2}\partial_i \partial_i \phi(x,t)\Big)^2 \Bigg],$$

where the space-time dimension is $D+1$. The scalar obeying the above action is often called the Lifshitz scalar.

2.2 Space-time anisotropy

In this subsection, we would like to see that the theory at the Lifshitz point is anisotropic between space and time. We can carry it out easily by considering about the conformal dimension of the Lifshitz scalar. If we pick up the kinetic part of the Lifshitz scalar action:

$$S = -\frac{1}{2}\int d^D x dt (\partial_t \phi(x,t))^2,$$

and assuming the scalings of x^i and t like:

$$x^i \to b x^i \quad t \to b^Z t, \tag{2.5}$$

where b is some constant, and we call the constant Z the dynamical critical exponent, and then we can calculate the conformal dimension, or mass dimension of ϕ by its kinetic term:

$$0 = [dt] + [d^D x] + 2[\partial_t] + 2[\phi] = -Z - D + 2Z + 2[\phi].$$

Thus we get

$$[\phi] = \frac{D-Z}{2}. \tag{2.6}$$

Futhermore, if we add the potential term

$$S = -\frac{1}{2}\int d^D x dt \Bigg[\Big(\partial_t \phi(x,t)\Big)^2 - \Big(\frac{1}{2}\partial_i \partial_i \phi(x,t)\Big)^2 \Bigg], \tag{2.7}$$

then we also calculate the conformal dimension of the adove action like this:

$$0 = [dt] + [d^D x] + 4[\partial_i] + 2[\phi] = -Z - D + 4 + 2\left(\frac{D-Z}{2}\right).$$

Thus we determine the dynamical critical exponent Z:

$$Z = 2. \tag{2.8}$$

This expresses the strong anisotropy between space and time. We can also understand that this anisotropy is originated from the Lifshitz type potential term as we mentioned before. It is also possible that we construct the more general Lifshitz type action by using the following potential:

$$((\nabla)^Z \phi)^2. \tag{2.9}$$

In this thesis, the action having the above potential is called the *Z Lifshitz-type action.* Of course, Z=1 means Lorentzian theory. It is meaningful to mention about something intriguing at this point, namely we can get the general Z Lifshitz-type action by using the different way which is called *detailed balance condition*. Such a condition is closely related to the stochastic quantization, and this lets us understand that theories at a Lifshitz point are strongly disturbed by some "quantum" fluctuations as we will see in the later sections.

As the conclusion of this subsection, let us consider about the mechanism of the Lorentz symmetry breaking. One can remember the free energy at the Lifshitz point is

$$F_{LS} \sim b\phi^4 + \beta(\nabla^2\phi)^2,$$

where $b > 0$ and $\beta > 0$ as before. If the Lorentz symmetry breaking were possible, it would turn into reality only at high energy, maybe at Planck scale. On the other hand, what about the low energy region? We can consider about this question from various points of view. One of them is the assumption that the system is getting away from a Lifshitz point as coming to the IR. This can be realized by restoring the terms vanishing at a Lifshitz point. Namely

$$F \sim a\phi^2 + b\phi^4 + \alpha(\nabla\phi)^2 + \beta(\nabla^2\phi)^2.$$

If we assume all the coefficients except for β are large enough to neglect the Lorentz-violating terms, then it seems to recover the Lorentz symmetry in the low energy region in the sense that the dynamical critical exponent is not two but one. But we do not have any tool to answer how and when coefficients are away from zero.

3 Detailed balance condition

What we would like to handle in this section is the detailed balance condition, which is closely related to the stochastic quantization[3]. At first, just by using the detailed balance condition without explaining its origin we try to relate the partition function in D flat dimensions with the density of the ground state wave functional in $D+1$ dimensions. In the free scalar case, one can notice that the $(D+1)$-dimensional action is precisely the Z=2 Lifshitz-type action. This procedure helps us to uncover the secrets of theories at a Lifshitz point. Then we would like to get all messy components together with the help of the stochastic quantization.

3.1 Lifshitz scalar

We take the D -flat-dimensional Euclidean free scalar action as our setting:

$$W = \frac{1}{2}\int d^D\mathbf{x}(\partial_i\phi\partial_i\phi), \tag{3.1}$$

where $\mathbf{x} = (x_1, x_2, \cdots, x_D)$. The partition function of the above is

$$\mathcal{Z} = \int \mathcal{D}\phi(\mathbf{x})\exp(-W[\phi(\mathbf{x})]). \tag{3.2}$$

Now we assume that the configuration space in some $(D+1)$-dimensional theory coincides with the above all $\phi(\mathbf{x})$. Or in other words, by the above statement, we define the $(D+1)$-dimensional

[3]This section is based on [2, 8, 9, 10]

theory. The wavefunctional of $(D+1)$-dimensional theory is denoted by $\Psi[\phi(\mathbf{x},t)]$. In this situation, we would like to assume that the ground state wave functional $\Psi_0[\phi(\mathbf{x},t)]$ reproduces the weight of the partition function at some fixed time, that is

$$\exp(-W[\phi(\mathbf{x})]) = \Psi_0^*[\phi(\mathbf{x},t)]\Psi_0[\phi(\mathbf{x},t)]. \tag{3.3}$$

What is the $(D+1)$-dimensional theory which satisfies the above relation? We can answer this question by taking the $(D+1)$-dimensional action as

$$S = \frac{1}{2}\int dtd^D\mathbf{x}\left[(\dot{\phi})^2 - \frac{1}{4}(\Delta\phi)^2\right], \tag{3.4}$$

where $\Delta = \partial_i\partial_i$. This action is the Lifshitz-type action as we have seen. In addition, this action must satisfy the important condition

$$\left(\frac{1}{4}\Delta\phi\right)^2 = \left(\frac{1}{2}\frac{\delta W}{\delta\phi(\mathbf{x})}\right)^2\Bigg|_{\phi(\mathbf{x})=\phi(\mathbf{x},\mathbf{t})}. \tag{3.5}$$

This condition is called *detailed balance condition* which causes the strong anisotropy between space and time. The rest of our work is that its ground state wave functional really reproduces the partition function in D dimensions. To make our work complete, we construct the $(D+1)$-dimensional Hamiltonian as:

$$H = \frac{1}{2}\int d^D\mathbf{x}\left[P^2 + \frac{1}{4}(\Delta\phi)^2\right], \tag{3.6}$$

where we take the representation of P as

$$P(\mathbf{x},t) = -i\frac{\delta}{\delta\phi(\mathbf{x},t)}, \tag{3.7}$$

in an agreement with the equal-time canonical comutation relation:

$$[\phi(\mathbf{x},t), P(\mathbf{y},t)] = i\delta^{(D)}(\mathbf{x}-\mathbf{y}). \tag{3.8}$$

We take the somewhat mandatry procedure, that is that we set the operators like:

$$\mathcal{Q}(\mathbf{x},t) = iP(\mathbf{x},t) - \frac{1}{2}\Delta\phi(\mathbf{x},t), \tag{3.9}$$

and its complex conjugate

$$\bar{\mathcal{Q}}(\mathbf{x},t) = -iP(\mathbf{x},t) - \frac{1}{2}\Delta\phi(\mathbf{x},t). \tag{3.10}$$

Then calculating the anticommutator of these leads

$$\begin{aligned}\{\mathcal{Q}(\mathbf{x},t), \bar{\mathcal{Q}}(\mathbf{x},t)\} &= 4\left[\frac{1}{2}\left(P^2 + \frac{1}{4}(\Delta\phi)^2\right)\right] \\ &= 4\mathcal{H},\end{aligned}$$

where $\mathcal{H}$ denotes the Hamiltonian density up to the UV divergent zero-point energy. Thus the Hamiltonian is

$$H = \frac{1}{4}\int d^D\mathbf{x}\{\mathcal{Q}(\mathbf{x},t), \bar{\mathcal{Q}}(\mathbf{x},t)\}. \tag{3.11}$$

We introduce the ground state wavefunctional $\Psi_0[\phi(\mathbf{x},t)]$ in the following sense:

$$\mathcal{H}\Psi_0[\phi(\mathbf{x},t)] = \{\mathcal{Q}(\mathbf{x},t), \bar{\mathcal{Q}}(\mathbf{x},t)\}\Psi_0[\phi(\mathbf{x},t)] = 0. \tag{3.12}$$

We can choose $\mathcal{Q}\Psi_0[\phi] = 0$ or $\bar{\mathcal{Q}}\Psi_0[\phi] = 0$ as the answer. If we choose the former one, then we can derive the ground state as follows:

$$\begin{aligned}\mathcal{Q}(\mathbf{x},t)\Psi_0[\phi(\mathbf{x},t)] &= \Big(\frac{\delta}{\delta\phi(\mathbf{x},t)} - \frac{1}{2}\Delta\phi(\mathbf{x},t)\Big)\Psi_0[\phi(\mathbf{x},t)] \\ &= 0. \end{aligned} \tag{3.13}$$

Thus

$$\begin{aligned}\frac{\delta\Psi_0[\phi(\mathbf{x},t)]}{\delta\phi(\mathbf{x},t)} &= \frac{1}{2}\Delta\phi(\mathbf{x},t)\Psi_0[\phi(\mathbf{x},t)] \\ &= \pm\frac{1}{2}\frac{\delta W}{\delta\phi(\mathbf{x})}\bigg|_{\phi(\mathbf{x})=\phi(\mathbf{x},\mathbf{t})}\Psi_0[\phi(\mathbf{x},t)]. \end{aligned} \tag{3.14}$$

In the last line, we have just used the detailed balance condition. The ambiguity of the sign is originated with the choice: $\mathcal{Q}\Psi_0[\phi] = 0$ or $\bar{\mathcal{Q}}\Psi_0[\phi] = 0$. If we choose the minus sign, then we get

$$\Psi_0[\phi(\mathbf{x},t)] = \exp\Big(-\frac{1}{4}\int d^D\mathbf{x}\partial_i\phi\partial_i\phi\Big), \tag{3.15}$$

up to some constant. This is exactly what we want, that is that this ground state wave functional can reproduce the weight of the partition function in D dimensions. The most important thing in this subsection is that the Lifshitz-type action can be constructed by the detailed balance condition, which relates the D-dimensional Euclidean theory with the $(D+1)$-dimensional Lifshitz-type theory. In the following subsection, we will see more precisely this profound relation.

3.2 Stochastic quantization

In this subsection, we would like to see the detailed balance condition from the different point of view, that is the stochastic quantization. The interesting concept of this theory helps us to understand what the origin of the detailed dalance condition is, and gives us a reason why the detailed balance condition is related to the strong anisotropy between space and time to some extent.

The stochastic quantization was proposed by Parisi and Wu in 1981 [**?**]. The key idea of this theory is to take the Euclidean quantum field theory as the equilibrium limit of a statistical system coupled to a thermal reservoir. Namely, the dynamical motion of the Euclidean fields are disturbed by some random force originated from the heat reservoir. In the equilibrium limit, the stochastic averages over these fields are identified with the Euclidean vacuum expectation values. Namely, what we get in this process is equivalent to the one derived from the path-integral procedure. In the stochastic quantization, there are two types of formulations: the Langevin formulation and the Fokker-Plank formulation. In the following, we will concentrate on the former one, the Langevin formulation.

3.2.1 Langevin formulation

First of all though, we have to introduce three technical terms:

- Stochastic variable
- Stochastic process
- Markov process

First, the stochastic variable $\mathcal{X}$ is defined by its possible values x and the probability distribution over these values is written by $P(x)$, which often expresses the random fluctuation. The second one, the stochastic process is the process that depends on the stochastic variables and the time which is often called *fictitious time*. The third one, the Markov process is the stochastic process that the probability for the system to some state at some time depends not on the whole history but only on the most recent time.

One of interesting examples of the Markov process is the Brownian motion, which can be described by the Langevin equation:

$$m\frac{d}{dt}\vec{v}(t) = -c\vec{v}(t) + \vec{\eta}(t), \tag{3.16}$$

which expresses the particle of the mass m moves in a liquid with the friction coefficient c disturbed by the stochastic vector $\vec{\eta}$ originated from the effect of the collision with other molecules. η is just the stochastic variable which is called *white Gaussian noise* because of the assumption that η has the Gaussian distribution.

Next, let us try to apply the above Langevin equation to the Euclidean field theory. The procedure is as follows. First, we prepare the field $\phi(x)$ described by the classical D-dimensional Euclidean action W, and supplement them with an extra time t, the fictitious time, like:

$$\phi(x) \to \phi(x,t). \tag{3.17}$$

Then we assume that the field $\phi(x,t)$ obeys the Langevin equation:

$$\frac{\partial\phi(x,t)}{\partial t} = -\left(\frac{1}{2}\frac{\delta W}{\delta\phi}\right)\Bigg|_{\phi(x)=\phi(x,t)} + \eta(x,t). \tag{3.18}$$

The first term in the right hand side is just the seed of the detailed balance condition. The correlations of the white Gaussian noise η are given by

$$\begin{aligned} \langle\eta(x,t)\rangle_\eta &= 0, \\ \langle\eta(x_1,t_1)\eta(x_2,t_2)\rangle_\eta &= 2\delta(t_1-t_2)\delta^{(D)}(x_1-x_2). \end{aligned} \tag{3.19}$$

In general, the correlations of the odd numbers of η vanish while those of the even numbers do not. Given some initial value t_0, we can get the η-dependent solution $\phi_\eta(x,t)$. We notice that the $\phi_\eta(x,t)$ is also the stochastic variable. Then we can write down its correlation functions as follows:

$$\langle\phi_\eta(x_1,t_1)\cdots\phi_\eta(x_m,t_m)\rangle_\eta = \frac{1}{\int\mathcal{D}\eta\exp(-\frac{1}{2}\int d^Dxdt\ \eta^2(x,t))}$$

$$\times \int \mathcal{D}\eta \phi_\eta(x_1,t_1)\cdots\phi_\eta(x_m,t_m)\exp\left(-\frac{1}{2}\int d^Dx dt\ \eta^2(x,t)\right). \quad (3.20)$$

The main idea of the stochastic quantization is

$$\langle\phi(x_1)\cdots\phi(x_m)\rangle = \lim_{t\to\infty}\langle\phi_\eta(x_1,t)\cdots\phi_\eta(x_m,t)\rangle_\eta. \quad (3.21)$$

The left hand side is the correlation function of the D-dimensional Euclidean field theory. Applying this procedure to the Euclidean scalar field theory gives us the clear understanding of the relation between the stochastic quantization and the detailed balance condition so that in the following subsection we will discuss about the scalar field case.

3.2.2 Application to scalar field

The purpose of this subsection is to see that in the context of the stochastic quantization the Lifshitz-type theory is strongly disturbed by random forces, and in the equilibrium limit this becomes a lower dimensional Euclidean theory at the level of correlation functions. Consider the action of the free scalar field in D flat dimensions with the Euclidean signature,

$$W[\phi] = \frac{1}{2}\int d^Dx\big[\partial_i\phi(x)\partial_i\phi(x)\big] \quad (3.22)$$

where i runs from 1 to D. Then what we need to do is supplementing the extra time t:

$$\phi(x) \to \phi(x,t).$$

We require this new field $\phi(x,t)$ evolves in the fictious time direction, in other words, obeys the Langevin equation:

$$\begin{aligned}\partial_t\phi(x,t) &= -\left(\frac{1}{2}\frac{\delta W}{\delta\phi}\right)\bigg|_{\phi(x)=\phi(x,t)} + \eta(x,t)\\ &= \frac{1}{2}\partial_i\partial_i\phi(x,t) + \eta(x,t).\end{aligned} \quad (3.23)$$

The scalar field which satisfies the above equation is η-dependent-stochastic variable $\phi_\eta(x,t)$. In this line of arguments, we can describe the correlation function:

$$\langle\phi_\eta(x_1,t_1)\cdots\phi_\eta(x_m,t_m)\rangle_\eta = \frac{1}{\mathcal{Z}}\int \mathcal{D}\eta\phi_\eta(x_1,t_1)\cdots\phi_\eta(x_m,t_m)\exp\left[-\frac{1}{2}\int d^Dx dt\ \eta^2(x,t)\right], \quad (3.24)$$

where the partition function $\mathcal{Z}$ is

$$\mathcal{Z} = \int \mathcal{D}\eta \exp\left(-\frac{1}{2}\int d^Dx dt\ \eta^2(x,t)\right). \quad (3.25)$$

This means that ϕ is disturbed by the white Gaussian noise η. In the equilibrium limit, the Langevin equation is

$$0 = \partial_t\phi(x,t) = \frac{1}{2}\partial_i\partial_i\phi(x,t) + \eta(x,t). \quad (3.26)$$

Since $\eta(x,t) = 0$ in the equilibrium limit, the equation of motion of the Euclidean D-dimensional theory (the free scalar) holds:

$$\frac{\delta W}{\delta\phi} = -\partial_i\partial_i\phi(x,t) = 0. \quad (3.27)$$

Next, we consider the equal time correlator satisfying the following relation:

$$\lim_{t\to\infty}\langle\phi_\eta(x_1,t)\cdots\phi_\eta(x_m,t)\rangle_\eta = \langle\phi(x_1)\cdots\phi(x_m)\rangle, \tag{3.28}$$

which is the central idea of the stochastic quantization. For simplicity, we will just consider the partition function (3.25):

$$\mathcal{Z} = \int \mathcal{D}\eta \exp\left[-\frac{1}{2}d^Dxdt\ \eta^2(x,t)\right].$$

If we change the variable from η to ϕ, then we get

$$\begin{aligned}\mathcal{Z} &= \int \mathcal{D}\phi\ \det\left[\frac{\delta\eta}{\delta\phi}\right]\Bigg|_{\eta=\partial_t\phi(x,t)-\frac{1}{2}\partial_i\partial_i\phi(x,t)} \exp\Bigg(-\frac{1}{2}\int d^Dxdt\ \left[\partial_t\phi(x,t)-\left(\frac{1}{2}\partial_i\partial_i\phi(x,t)\right)^2\right]\Bigg)\\ &\equiv \int \mathcal{D}\phi\ \det\left[\frac{\delta\eta}{\delta\phi}\right]\Bigg|_{\eta=\partial_t\phi(x,t)-\frac{1}{2}\partial_i\partial_i\phi(x,t)} \exp(-S).\end{aligned} \tag{3.29}$$

where the Jacobian is just the constant so that we can neglect it, and S is the $(D+1)$-dimensional action. We can read off the explicit form of S:

$$\begin{aligned}S &= -\frac{1}{2}\int d^Dxdt\ \left(\partial_t\phi(x,t)-\frac{1}{2}\partial_i\partial_i\phi(x,t)\right)^2\\ &= -\frac{1}{2}\int d^Dxdt\left[(\partial_t\phi(x,t))^2+\left(\frac{1}{2}\partial_i\partial_i\phi(x,t)\right)^2\right] \equiv S_L,\end{aligned} \tag{3.30}$$

where we have used the fact that the cross-term, $(\partial_t\phi)\partial_i\partial_i\phi$, becomes the total derivative so that we have neglected it. We notice that the last form of the $(D+1)$-dimensinal action obeys just Z=2 Lifshitz-type action! One can check this exactly by rotating the fictitious time coordinate into Lorentzian one because the action derived by the stochastic quantization is just the $(D+1)$-dimensional Euclidean action. Summing up all the calculations, we get

$$\langle\phi(x_1)\cdots\phi(x_m)\rangle = \lim_{t\to\infty}\frac{1}{\mathcal{Z}}\int \mathcal{D}\phi\exp(-S_L)\phi_\eta(x_1,t)\cdots\phi_\eta(x_m,t), \tag{3.31}$$

where we redefine the partition function as

$$\mathcal{Z} = \int \mathcal{D}\phi\exp(-S_L). \tag{3.32}$$

Consequently, we can understand that the random force causes the strong anisotropy between space and (fictitious) time. Namely, the stochastic field $\phi_\eta(x,t)$ is disturbed by some "quantum" effects, and in the end, it reaches the equilibrium. In this situation, what the stochastic field obeys is the detailed balance condition. As we have seen before, the point where the field governed by this condition is the Lifshitz point.

Alternatively, we can put the fermion into the $(D+1)$-dimensional action S_L. It is achieved by using the Gaussian integration. Remembering the determinat emerged by changing the variable, and introducing two types of fermions, we get

$$\det\left[\frac{\delta\eta}{\delta\phi}\right]\Bigg|_{\eta=\partial_t\phi(x,t)-\frac{1}{2}\partial_i\partial_i\phi(x,t)} = \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\exp\Big(-\int dtd^Dx\ \bar{\psi}(x,t)(\partial_t-\frac{1}{2}\partial_i\partial_i)\psi(x,t)\Big). \tag{3.33}$$

In this case, we get the action below:

$$S_L = \int d^D x dt \left[-\frac{1}{2}(\partial_t \phi(x,t))^2 - \frac{1}{8}(\partial_i \partial_i \phi(x,t))^2 + \bar{\psi}(x,t) \left(\partial_t - \frac{1}{2}\partial_i \partial_i \right) \psi(x,t) \right]. \tag{3.34}$$

Although adding fermions into the action is very interesting to argue, we do not treat it in this thesis.

4 Gravity at Lifshitz point

In this section, we place the gravity at a Lifshitz point [3][4]. To do that, it is good for us to remember the ADM-decomposed action in the general relativity:

$$S_{EH} = \frac{2}{\kappa^2} \int dt d^3 x \Big[N\sqrt{q}(\mathrm{tr}K^2 - (\mathrm{tr}K)^2 + R^{(3)}) + (\text{surface terms}) \Big],$$

where κ is the Einstein's gravitational constant. In the following discussions, we obey the conventions appeared in Appendix A. Just for simplicity, we would like to rewrite the above action by using the *De Witt metric* given by

$$\mathcal{G}^{ijkl} \equiv \frac{1}{2}(q^{ik}q^{jl} + q^{il}q^{jk}) - q^{ij}q^{kl}. \tag{4.1}$$

We can write down the action as follows:

$$S_{EH} = \frac{2}{\kappa^2} \int dt d^3 x \Big[N\sqrt{q}(K_{ij}\mathcal{G}^{ijkl}K_{kl} + R^{(3)}) + (\text{surface terms}) \Big]. \tag{4.2}$$

If we focus on the only compact topologies, we can omit the surface terms, and hereafter we will consider only on such compact topologies in this thesis. We call the first term in the action *kinetic term*, and *potentail term* for the second one because the first one has time derivatives and the second one does not. We are ready to place the gravity at a Lifshitz point. As in the case of ADM decomposition in the general relativity, we fix the space-time topology to the one which is diffeomorphic to S (space) $\times R$ (time). We try to construct the D-dimensional Lifshits-type gravity action. The procedure is leaving the kinetic term as in the general relativity, and then replacing the potential term with the one which satisfies the detailed balance condition by using the lower dimensional action, that is

$$S_{HL} = \int dt d^D x N\sqrt{q} \left[\frac{2}{\kappa^2} K_{ij}\mathcal{G}^{ijkl}K_{kl} - \frac{\kappa^2}{8} \left(\frac{\delta W}{\sqrt{q}\delta q_{ij}} \right) \mathcal{G}_{ijkl} \left(\frac{\delta W}{\sqrt{q}\delta q_{kl}} \right) \right], \tag{4.3}$$

where W is some $(D-1)$-dimensional Euclidean action. In fact, the kinetic term is not exactly the same as in the general relativity, namely the coupling constant λ is put into the De Witt metric such as

$$\mathcal{G}^{ijkl} \equiv \frac{1}{2}(q^{ik}q^{jl} + q^{il}q^{jk}) - \lambda q^{ij}q^{kl}. \tag{4.4}$$

The position of λ is arbitrary. Its inverse is defined by

$$\mathcal{G}_{ijkl} \equiv \frac{1}{2}(q_{ik}q_{jl} + q_{il}q_{jk}) - \frac{\lambda}{D\lambda - 1} q_{ij}q_{kl}, \tag{4.5}$$

[4]This section is based on [3].

in the following sense:

$$\mathcal{G}^{ijmn}\mathcal{G}_{mnkl} = \frac{1}{2}(\delta^k{}_i\delta^l{}_j + \delta^l{}_i\delta^k{}_j). \tag{4.6}$$

Our desire in this section is taking the Lifshitz type gravity as a candidate for the quantum gravity. In the following subsections, we would like to see the interesting properties of the Lifshitz type gravity theory.

4.1 $Z = 2$ Lifshitz-type gravity in D dimensions

To begin with, we concentrate on the $Z = 2$ Lifshitz-type gravity that is expected to have the 4th order derivative terms at most. If we remember the part of the potential term, $(\frac{\delta W}{\sqrt{q}\delta q_{ij}})\mathcal{G}_{ijkl}(\frac{\delta W}{\sqrt{q}\delta q_{kl}})$, we notice that the action W that has 2nd order derivative terms is the candidate. Remembering the interesting notion of the stochastic quantization, that is that the theory disturbed by random forces is related to the lower dimensional theory in the equilibrium, we can guess that the Lifshitz-type gravity is related to the Euclidean gravity in the equilibrium. This is very attractive and reasonable to some extent although it is merely an assumption. Thus we take W as D-dimensional Euclidean Einstein-Hilbert action:

$$W = \frac{1}{{\kappa_W}^2}\int d^Dx\sqrt{q}R^{(D)}. \tag{4.7}$$

where ${\kappa_W}^2$ is some coupling constant. The Ricci scalar really has the 2nd order derivative terms as we desire. We can calculate the functional derivative of W as to the spacial metric:

$$\frac{\delta W}{\delta q_{ij}} = -\frac{1}{{\kappa_W}^2}(R^{(D)ij} - \frac{1}{2}R^{(D)}q^{ij})\sqrt{q}. \tag{4.8}$$

The quadratic term of this can be written by

$$\begin{aligned}
\left(\frac{\delta W}{\sqrt{q}\delta q_{ij}}\right)\mathcal{G}_{ijkl}\left(\frac{\delta W}{\sqrt{q}\delta q_{kl}}\right) &= \left[\frac{1}{{\kappa_W}^2}\left(R^{(D)ij} - \frac{1}{2}R^{(D)}q^{ij}\right)\right]\mathcal{G}_{ijkl}\left[\frac{1}{{\kappa_W}^2}\left(R^{(D)ij} - \frac{1}{2}R^{(D)}q^{kl}\right)\right] \\
&= \frac{1}{{\kappa_W}^4}R^{(D)ij}R^{(D)}{}_{ij} \\
&\quad + \frac{1}{{\kappa_W}^4}\frac{R^{(D)2}}{4}\left[\frac{\lambda}{D\lambda - 1}(-4 - D^2 + 4D) + (D-4)\right] \\
&= \frac{1}{{\kappa_W}^4}R^{(D)ij}R^{(D)}{}_{ij} + \frac{1}{{\kappa_W}^4}\left[\frac{1 - \lambda - D/4}{D\lambda - 1}\right]R^{(D)2} \\
&\equiv \frac{1}{{\kappa_W}^4}\left[R^{(D)ij}R^{(D)}{}_{ij} + aR^{(D)2}\right].
\end{aligned} \tag{4.9}$$

In the later discussion, we omit the label, (D). The total action is

$$S_{HL} = \int dtL_{HL} = \int dtd^DxN\sqrt{q}\left[\frac{2}{\kappa^2}K_{ij}\mathcal{G}^{ijkl}K_{kl} - \frac{\kappa^2}{8{\kappa_W}^4}\left(R^{ij}R_{ij} + aR^2\right)\right]. \tag{4.10}$$

Next we try to find the symmetry of this theory. In the case of the general relativity, the corresponding symmetry is the four dimensional diffeomorphism. However the Lifshitz-type action has only the spacial diffeomorphism at first sight. In fact, we can find the symmetry in D dimensions which extends the spacial diffeomorphism and narrows $(D+1)$-dimensional diffeomorphism. That is the *foliation-preserving deffeomorphism.*

4.1.1 Foliation-preserving diffeomorphism

In the general relativity, the speed of light is a constant. Contraly to that, the Lifshitz-type theory has no support for it to be constant, that is that the speed of light may excess or diminish the value determined by the relativity. Thus it is interesting to restore the speed of light in the space-time metric in the general relativity and consider the diffeomorphism transformation of it where the light speed is close to infinity. The finite terms in this limit have the possibility to be the symmetry of the Lifshitz-type theory. At first, we restore the light speed c in the metric. If we relate the time t with the zero component of x^μ

$$x^0 = ct, \tag{4.11}$$

then we get

$$\begin{aligned} g(\partial_0, \partial_0) &= g(c^{-1}\partial_t, c^{-1}\partial_t) \\ &= g(c^{-1}N(cn) + c^{-1}N^m\partial_m, c^{-1}N(cn) + c^{-1}N^l\partial_l) \\ &= -N^2 + \frac{1}{c^2}N^mN_m, \end{aligned}$$

and

$$\begin{aligned} g(\partial_0, \partial_i) &= g(c^{-1}\partial_t, \partial_i) \\ &= g(c^{-1}N(cn) + c^{-1}N^m\partial_m, \partial_i) \\ &= \frac{1}{c}N_i. \end{aligned}$$

Thus the space-time metric is

$$g_{\mu\nu} = \begin{pmatrix} -N^2 + \frac{1}{c^2}N^mN_m & \frac{1}{c}N_i \\ \frac{1}{c}N_i & q_{ij}. \end{pmatrix} \tag{4.12}$$

Next, let us consider the space-time diffeomorphism

$$t \to t + cf(t,x) + \mathcal{O}(1/c), \tag{4.13}$$

$$x^i \to x^i + \zeta(t,x) + \mathcal{O}(1/c^2), \tag{4.14}$$

where f and ζ are the generators along the time and the spacial directions respectively. Then the Lie derivative of the space-time metric can be written as follows:

$$\mathcal{L}_{f,\zeta}g_{\mu\nu} = g_{\mu t}\partial_\nu f + g_{\mu i}\partial_\nu\zeta^i + g_{\nu t}\partial_\mu f + g_{\nu i}\partial_\mu\zeta^i + \partial_t g_{\mu\nu}f + \partial_i g_{\mu\nu}\zeta^i. \tag{4.15}$$

We try to write down the components explicitly:

$$\begin{aligned} \mathcal{L}_{f,\zeta}g_{ij} &= \delta q_{ij} \\ &= (cf + \mathcal{O}(1/c))(1/c\partial_t) + (\zeta^l + \mathcal{O}(1/c^2))\partial_l g_{ij} + g_{i0}\partial_j(cf + \mathcal{O}(1/c)) \\ &\quad + g_{il}\partial_j(\zeta^l + \mathcal{O}(1/c^2)) + g_{jo\partial_i}(cf + \mathcal{O}(1/c)) + g_{il}\partial_i(\zeta^l + \mathcal{O}(1/c^2)), \end{aligned} \tag{4.16}$$

$$\mathcal{L}_{f,\zeta}g_{0i} = \delta(1/cN_i)$$

$$
\begin{aligned}
&= (cf + \mathcal{O}(1/c))(1/c\partial_t)(1/c\partial_t) + (\zeta^l + \mathcal{O}(1/c^2))\partial_l(1/cN_i) \\
&+ (-N^2 + 1/c^2 N^m N_m)\partial_i(cf + \mathcal{O}(1/c)) + (1/cN_l)\partial_i(\zeta^l + \mathcal{O}(1/c^2)) \\
&+ N_i/c(1/c\partial_t)(cf + \mathcal{O}(1/c)) + g_{il}(1/c\partial_t)(\zeta^l + \mathcal{O}(1/c^2)),
\end{aligned}
\tag{4.17}
$$

$$
\begin{aligned}
\mathcal{L}_{f,\zeta} g_{00} &= \delta(-N^2 + 1/c^2 N^m N_m) \\
&= (cf + \mathcal{O}(1/c))(1/c\partial_t)(-N^2 + 1/c^2 N^m N_m) \\
&+ (-N^2 + 1/c^2 N^m N_m)(1/c\partial_t)(cf + \mathcal{O}(1/c)) \\
&+ (1/cN_i)(1/c\partial_t)(\zeta^l + \mathcal{O}(1/c^2)) \\
&+ (-N^2 + 1/c^2 N^m N_m)(1/c\partial_t)(cf + \mathcal{O}(1/c)) \\
&+ N_i/c(1/c\partial_t)(\zeta^i + \mathcal{O}(1/c^2)) + (\zeta^i + \mathcal{O}(1/c^2))\partial_i(-N^2 + 1/c^2 N^m N_m).
\end{aligned}
\tag{4.18}
$$

If the generator along the time direction depends only on the time

$$
t \to t + cf(t) + \mathcal{O}(1/c), \tag{4.19}
$$

then these transformations converge in $c \to \infty$. In this situation, by summing up all results we can define the restricted diffeomorphism which is finite in $c \to \infty$:

$$
\delta q_{ij} \equiv \lim_{c\to\infty} \delta_{f,\zeta} q_{ij} = f\partial_t q_{ij} + \zeta^l \partial_l q_{ij} + q_{il}\partial_j \zeta^l + q_{jl}\partial_i \zeta^l, \tag{4.20}
$$

$$
\delta N_i \equiv \lim_{c\to\infty} \delta_{f,\zeta} N_i = f\partial_t N_i + \zeta^l \partial_l N_i + N_l \partial_i \zeta^l + N_i \partial_t f + q_{il}\partial_t \zeta^l, \tag{4.21}
$$

$$
\delta N \equiv \lim_{c\to\infty} \delta_{f,\zeta} N = f\partial_t N + \zeta^l \partial_l N + N\partial_t f. \tag{4.22}
$$

These restricted diffeomorphisms are called *foliation-preserving diffeomorphisms*. We can show that the Lifshitz-type gravity action is invariant under the foliation-preserving diffeomorphism up to total derivatives. Additionally, we pose the *projectability condition* on the lapse function, that is that we take the lapse function that depends only on the time, $N = N(t)$. This is because we would like to avoid lots of problems which are caused in the case of $N = N(t, x)$. The projectability condition is equivalent to taking the gauge condition $\partial_i N = 0$ in some (unknown) theory that is invariant under the diffeomorphism. Thus the projectability condition is merely the gauge fixing condition so that it is reasonable for us to impose this condition on the Lifshitz-type gravity.

4.1.2 Hamiltonian formalism

The Hamiltonian formalism breaks the covariance of space and time explicitly so that we can assume that the Lifshitz-type gravity with anisotropic scaling between space and time is suitable for the Hamiltonian formalism. Under this thought, we try to investigate the Hamiltonian and the structure of constraints which is very different from that of the Einstein's general relativity. What we need to do first is constructing the conjugate momentum of the spacial metric.

$$
p^{ij} \equiv \frac{\delta L_{HL}}{\delta(\partial_t q_{ij})} = \frac{2\sqrt{q}}{\kappa^2} G^{ijkl} K_{kl}. \tag{4.23}
$$

The momenta conjugate to $N(t)$ and $N^i(t,x)$ are zero because the time derivative as to these variables do not exist in the Lagrangian. Thus these momenta are the primary constraints:

$$p_N \equiv \frac{\delta L_{HL}}{\delta(\partial_t N)} = 0, \qquad\qquad p_{\vec{N}}{}^i \equiv \frac{\delta L_{HL}}{\delta(\partial_t \vec{N})} = 0. \tag{4.24}$$

Then what we need to do is imposing the consistency condition of above constraints as to the time evolution. To do that, we define the Hamiltonian density $\mathcal{H}_{HL}$ by the Legendre transformation of the Lagrangian density $\mathcal{L}_{HL}$:

$$\begin{aligned}
\mathcal{H}_{HL} &\equiv p^{ij}\partial_t q_{kl} - \mathcal{L}_{HL} \\
&= 2Np^{ij}K_{ij} + 2p^{ij}\nabla_i N_j - Np^{ij}K_{ij} + \frac{\sqrt{q}N\kappa^2}{8{\kappa_W}^4}\left(R^{ij} - \frac{1}{2}Rq^{ij}\right)\mathcal{G}_{ijkl}\left(R^{kl} - \frac{1}{2}q^{kl}\right) \\
&= N\left[\frac{\kappa^2}{2\sqrt{q}}p^{ij}\mathcal{G}_{ijkl}p^{kl} + \frac{\sqrt{q}}{8{\kappa_W}^4}\left(R^{ij} - \frac{1}{2}Rq^{ij}\right)\mathcal{G}_{ijkl}\left(R^{kl} - \frac{1}{2}q^{kl}\right)\right] \\
&\quad + N_j\left[-2\nabla_i p^{ij}\right] + \text{(a total derivative)} \\
&\equiv NC + N_l C^l + \text{(a total derivative)}.
\end{aligned} \tag{4.25}$$

In the third equality, we used the fact that the momentum p_{ij} had the determinant $\sqrt{q}$, that is that we got the total derivative term, and we also used the relation $K_{ij} = \frac{\kappa^2}{2\sqrt{q}}p^{kl}\mathcal{G}_{klij}$. We set the Poisson brackets as follows:

$$\{q_{ij}(x), p^{kl}(y)\}_{PB} = \frac{1}{2}(\delta_i^k\delta_j^l + \delta_j^k\delta_i^l)\delta^{(D)}(x-y), \tag{4.26}$$

$$\{N, p_N\}_{PB} = 1, \tag{4.27}$$

$$\{N_i(x), p_N^j(y)\}_{PB} = \delta_i^j\delta^{(D)}(x-y). \tag{4.28}$$

Now let us impose the consistency conditions of the primary constraints as to the time evolution:

$$\frac{d}{dt}p_N = \{p_N, \int d^D x\mathcal{H}_{HL}\}_{PB} = \int d^D x C \approx 0, \tag{4.29}$$

and

$$\frac{d}{dt}p_{\vec{N}}{}^i(x) = \{p_{\vec{N}}{}^i(x), \int d^D x\mathcal{H}_{HL}\}_{PB} = -C^i \approx 0. \tag{4.30}$$

We notice the interesting thing that the constraint C is non-local, which is caused by the projectability condition, $N = N(t)$. The constraint C^i takes the same form in the general relativity so that we can get the constraint algebras by using the analogy in the general relativity:

$$\left\{\int d^D x C(x), \int d^D y C(y)\right\}_{PB} = 0, \tag{4.31}$$

$$\left\{\int d^D x C^i N_i(x), \int d^D y C(y)\right\}_{PB} = 0, \tag{4.32}$$

$$\left\{\int d^D x C^i N_i(x), \int d^D y C^j M_j(y)\right\}_{PB} = \int d^D x(N^i\partial_i M^j - M^i\partial_i N^j)C_j(x). \tag{4.33}$$

Thus the secondary constraints form the closed Lie algebras. Additionally, we can understand that these constraints are consistent with the time evolution like

$$\frac{d}{dt}\int d^D x C(x) = \left\{\int d^D x C(x), H_{HL}\right\}_{PB} \approx 0, \tag{4.34}$$

$$\frac{d}{dt}\int d^D x C^i N_i(x) = \left\{\int d^D x C^i N_i(x), H_{HL}\right\}_{PB} \approx 0. \tag{4.35}$$

The above equations mean that there is no extra constraint. Thus we get the all constraints in the Z=2 Lifshitz-type gravity.

4.1.3 Linear approximation

We try to understand the physical degrees of freedom in the Z=2 Lifshitz-type gravity. It can be assumed that there exists extra physical degrees of freedom compared to that of the general relativity because the symmetry in the Lifshitz-type gravity is the foliation-preserving diffeomorphism which is smaller than in the general relativity. For preparation to check the degrees of freedom, we expand the theory around the Minkowski bacground with perturbations like:

$$q_{ij} = \delta_{ij} + \kappa_W h_{ij}, \quad N = 1, \quad N_i = 0, \tag{4.36}$$

where the perturbation h_{ij} is small. As a convention, we define the ratio of two coupling constants, κ and κ_W as

$$\gamma \equiv \frac{\kappa}{\kappa_W}. \tag{4.37}$$

In this situation, if we take the linear terms as to h_{ij}, then the kinetic term in the action is

$$S_{HL(K)} = \frac{1}{2\gamma^2}\int d^D x \Big[(\partial_t h_{ij})^2 - \lambda(\partial_t h)^2\Big], \tag{4.38}$$

where $h = h_{ii}$. Under the linear approximation, we use the equality, =, up to the higher order terms of h_{ij}. We need to fix the residual guage, that is the time-independent spacial diffeomorphism. At some fixed time surface, we impose the following constraint:

$$\partial_i h_{ij} - \lambda \partial_j h = 0. \tag{4.39}$$

This gauge choice must be consistent with the time evolution so that we also require:

$$\partial_i \partial_t h_{ij} - \lambda \partial_j \partial_t h = 0, \tag{4.40}$$

which can be given by the $N_i's$ equation of motion of the full Lifshitz-type gravity action with $N_i = 0$. We can check this easily:

$$\begin{aligned}
\frac{\delta S_{HL}}{\delta N_l(y)} &= \frac{1}{2}\int dt d^D x \frac{1}{\gamma^2}\Big[2\left\{-\partial_i(\delta^l_j \delta^{(D+1)}(x-y)) - \partial_j(\delta^l_i \delta^{(D+1)}(x-y))\right\} \\
&\quad \times (\partial_t h_{ij} - \partial_i N_j - \partial_j N_i) - 2\lambda \partial_l(\partial h - 2\partial_i N^i)\Big] \\
&= \frac{2}{\gamma^2}\Big[\partial_i(\partial_t h_{il} - \partial_i N_l - \partial_l N_i) - \lambda \partial_l(\partial_t h - 2\partial_i N^i)\Big] = 0.
\end{aligned}$$

For $N_i = 0$ gauge, we get

$$\partial_i \partial_t h_{ij} - \lambda \partial_j \partial_t h = 0.$$

In $\lambda = 1$ case, we cannot impose the gauge fixing condition, $\partial_i h_{ij} - \lambda \partial_j h = 0$, because the linearized Ricci scalar can be written as follows.

$$-\kappa_W \partial_i (\partial_j h_{ij} - \partial_i h) = R. \tag{4.41}$$

It is impossible to take $R = 0$ by the gauge transformation. We can check the above equation as follows:

$$\begin{aligned} R_{ij} &= R^k_{ikj} \\ &= \partial_i \Gamma^k_{kj} - \partial_k \Gamma^k_{ij} + \Gamma^k_{im}\Gamma^m_{kj} - \Gamma^k_{km}\Gamma^m_{ij} \\ &= \frac{\kappa_W}{2}(\partial_i \partial_j h - \partial_i \partial_k h_{kj} - \partial_j \partial_k h_{ki} + \partial_k \partial_k h_{ij}), \end{aligned} \tag{4.42}$$

where Γ^i_{jk} is the spacial Christoffel symbol. In the last equality, we used

$$\Gamma^i_{jk} = \frac{\kappa_W}{2} \delta_{kl}(\partial_i h_{lj} + \partial_j h_{li} - \partial_l h_{ij}).$$

Thus we get the relation (4.41). We introduce the new variable as:

$$H_{ij} = h_{ij} - \lambda \delta_{ij} h. \tag{4.43}$$

By using H_{ij}, we can rewrite the residual gauge fixing condition as follows.

$$\partial_i H_{ij} = 0. \tag{4.44}$$

This condition is consistent with the time evolution as we mentioned. Thus we can understand that H_{ij} is transverse. Next, let us decompose H_{ij} into the traceless mode $\tilde{H}_{ij}$ and the trace mode H:

$$H_{ij} = \tilde{H}_{ij} + \frac{1}{D-1}\left(\delta_{ij} - \frac{\partial_i \partial_j}{\partial^2}\right) H. \tag{4.45}$$

Under this decomposition, we get

$$\partial_i \tilde{H}_{ij} = 0, \quad h = \frac{H}{1 - \lambda D}, \tag{4.46}$$

where $H = H_{ii}$. What we can understand from the above expressions is that $\tilde{H}_{ij}$ is the transverse traceless mode so that the physical degrees of freedom of $\tilde{H}_{ij}$ is

$$\mathrm{DoF}[\tilde{H}_{ij}] = {}_D C_2 - 1 = \frac{(D+1)(D-2)}{2}. \tag{4.47}$$

This is exactly the same with the degrees of freedom of the graviton in the general relativity. In addition to that, there exists the spinless particle H in the Lifshitz-type gravity originated from the restricted deffeomorphism. By combining above results, we can write down the graviton h_{ij}:

$$h_{ij} = \tilde{H}_{ij} + \frac{1}{D-1}\left(\delta_{ij} - \frac{\partial_i \partial_j}{\partial^2}\right) H + \lambda \frac{H}{1 - \lambda D} \delta_{ij}$$

$$= \tilde{H}_{ij} - \frac{1}{D-1}\frac{\partial_i\partial_j}{\partial^2}H + \frac{1-\lambda}{(D-1)(1-\lambda D)}\delta_{ij}H. \tag{4.48}$$

In the limit, $\lambda \to 1$, which can be the case of the general relativity, the scalar graviton H takes the form of the pure gauge. Of course, we can not take $\lambda = 0$ in this gauge as we mentioned. Next let us rewrite the kinetic term in terms of $\tilde{H}_{ij}$ and H:

$$\begin{aligned} S_{HL(K)} &= \frac{1}{2\gamma^2}\int dtd^Dx\Big[(\partial_t H_{ij} + \lambda\delta_{ij}\partial_t h)^2 - \lambda(\partial_t h)^2\Big] \\ &= \frac{1}{2\gamma^2}\int dtd^Dx\Big[(\partial_t\tilde{H}_{ij})^2 + \frac{(\partial_t H)^2}{D-1} + 2\lambda(\partial_t H)(\partial_t h) + (D\lambda - 1)\lambda(\partial_t h)^2\Big] \\ &= \frac{1}{2\gamma^2}\int dtd^Dx\Big[(\partial_t\tilde{H}_{ij})^2 + \frac{\lambda-1}{(D-1)(D\lambda-1)}(\partial_t H)^2\Big]. \end{aligned} \tag{4.49}$$

We can notice that the coefficient of $(\partial_t H)^2$ has the possibility to be negative. The negative sign in front of the kinetic term means the negative energy, and the particle with negative energy is the ghost. It is worthwhile to investigate the physical region of the scalar graviton in terms of the parameter region of λ. Assuming $D > 1$, H is physical in the regions, $\lambda > 1$ and $\lambda < 1/D$. On the other hand, in the region, $1/D < \lambda < 1$, H is the ghost (See Figure 3).

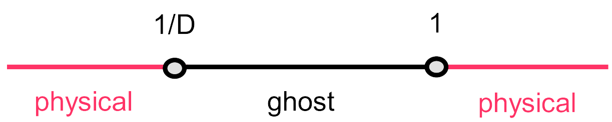


Figure 3: The range of λ

Next we would like to know the dispersion relations. To get them, we introduce the potential part in the linearized Lifshitz-type gravity without showing explicit calculation:

$$S_{HL(P)} = \frac{\gamma^2}{32}\int dtd^Dx\Big[\partial^2\tilde{H}_{ij}\partial^2\tilde{H}_{ij} + \frac{(D-2)^2(\lambda-1)^3}{(D-1)(D\lambda-1)^3}(\partial^2 H)^2\Big]. \tag{4.50}$$

We can write down the action of the transverse traceless mode $\tilde{H}_{ij}$

$$S_{HL(\tilde{H}_{ij})} = \frac{1}{2\gamma^2}\int dtd^Dx\Big[(\partial_t\tilde{H}_{ij})^2 - \frac{\gamma^4}{16}\partial^2\tilde{H}_{ij}\partial^2\tilde{H}_{ij}\Big], \tag{4.51}$$

and its equation of motion is

$$\Big[(\partial_t)^2 + \frac{\gamma^4}{16}(\partial^2)^2\Big]\tilde{H}_{ij} = 0. \tag{4.52}$$

We introduce the spacial momentum k and the the angular velocity ω, and take $\tilde{H}_{ij}$ as the Fourier-transformed one:

$$\tilde{H}_{ij}(t,x) = \int \frac{d\omega d^Dk}{(2\pi)^{D+1}}e^{i(kx-\omega t)}\tilde{H}'_{ij}(\omega,k). \tag{4.53}$$

If we put them into the equation of motion, then we get

$$\omega^2 = \frac{\gamma^4}{16}(k^2)^2. \tag{4.54}$$

This is the dispersion relation for the transverse traceless graviton. Similar to the above procedure, we can get the dispersion relation for the scalar graviton, that is

$$\omega^2 = \frac{\Gamma^4}{16}(k^2)^2, \tag{4.55}$$

where

$$\Gamma^4 = \frac{(D-2)^2(\lambda-1)^2}{(D\lambda-1)^2}\gamma^4. \tag{4.56}$$

4.1.4 Relevant deformation

We would like to construct the theory which is nonrelativistic in the ultraviolet region and would-be relativistic in the infrared region. Without breaking the detailed balance condition, we can carry it out by adding the cosmological constant to the D-dimensional-Euclidean action W that is lifted by the detailed balance condition. The explicit form of W is

$$W = \frac{1}{{\kappa_W}^2}\int dx^D\sqrt{q}(R-2\Lambda_W), \tag{4.57}$$

where Λ_W is the cosmological constant in D dimension. Let us calculate a part of the potential term as follows.

$$\begin{aligned}
\left(\frac{\delta W}{\sqrt{q}\delta q_{ij}}\right)\mathcal{G}_{ijkl}\left(\frac{\delta W}{\sqrt{q}\delta q_{kl}}\right) &= \left[\frac{1}{{\kappa_W}^2}\left(R^{ij}-\frac{1}{2}Rq^{ij}\right)+\Lambda_W q^{ij}\right] \\
&\quad \times \mathcal{G}_{ijkl}\left[\frac{1}{{\kappa_W}^2}\left(R^{ij}-\frac{1}{2}Rq^{kl}+\Lambda_W q^{kl}\right)\right] \\
&= \frac{1}{{\kappa_W}^4}\Bigg[R^{ij}R_{ij}+aR^2 \\
&\quad + \Lambda_W\left(-2\frac{R}{D\lambda-1}+\frac{DR}{D\lambda-1}\right)+{\Lambda_W}^2\left(-\frac{D}{D\lambda-1}\right)\Bigg] \\
&\equiv \frac{1}{{\kappa_W}^4}\left[R^{ij}R_{ij}+aR^2-M^2R+\frac{D(1-D\lambda)}{(D-2)^2}M^4\right],
\end{aligned} \tag{4.58}$$

where M is some scale defined by $M^2 \equiv \frac{D-2}{1-D\lambda}\Lambda_W$ which is a scale of this theory, and a is same as before. The total Z=2 Lifshitz-type action is

$$S_{HL} = \int dtd^DxN\sqrt{q}\left[\frac{2}{\kappa^2}K_{ij}\mathcal{G}^{ijkl}K_{kl}-\frac{\kappa^2}{8{\kappa_W}^4}\left(R^{ij}R_{ij}-aR^2-M^2R+\frac{D(1-D\lambda)}{(D-2)^2}M^4\right)\right]. \tag{4.59}$$

At a glance, the action above seems to be the Einstein-Hilbert action in the low energy limit as we desired. However if we try to identify the above as the Einstein-Hilbert action, we are forced to vary the God-given units. We can see such a deformation of constants by restoring the speed of light c and the Newton constant G_D in the Einstein-Hilbert action with the cosmological constant Λ:

$$S_{EH} = \frac{1}{16G_N}\int cdtd^DxN\sqrt{q}\left[\frac{1}{c^2}K_{ij}\mathcal{G}^{ijkl}K_{kl}+R-2\Lambda\right].$$

By comparing the above one and the Z=2 Lifshitz-type gravity action with the relevant deformation, we can get the three effective constants:

$$c = \frac{\gamma^2}{4}M, \tag{4.60}$$

$$G_N = \frac{\kappa_W^2}{8\pi M}, \tag{4.61}$$

$$\Lambda = \frac{1}{2}\frac{D}{D-2}\Lambda_W. \tag{4.62}$$

Putting aside the consideration of the above deformed constants, let us count the mass dimension of those couplings. In the Z=2 Lifshitz-type gravity, there exists three coupling constants, κ, κ_W, and λ. To get the mass dimension of the three, we need to check the mass dimension of fields:

$$[q_{ij}] = 0, \quad [N_i] = 1, \quad [N] = 0. \tag{4.63}$$

This is because the mass dimension of the $D+1$-dimensional metric is zero. By using the above and the fact that the action must be dimensionless, we can get the mass dimension of couplings:

$$[\kappa] = \frac{2-D}{2}, \quad [\kappa_W] = \frac{2-D}{2}, \quad [\lambda] = 0. \tag{4.64}$$

In $D = 2$, all coupling constants become marginal so that the Z=2 Lifshitz-type gravity in $2+1$ dimensions is renormalizable. However, unfortunately this theory cannot be a candidate of the $(2+1)$-dimensional Einstein gravity in the infrared region. One can understand this by noticing that a scale of this theory M is zero at $D = 2$ so that there exists no relevant potential term. If this theory had the general-relativity-like terms, we do not know whether or not λ naturally flows to 1 at the low energy fixed point for now. But the Z=2 Lifshitz-type gravity with Lifshitz scalers in $2+1$ dimensions has very interesting features in the context of quantizing membranes[2], which we do not consider in this thesis. In dimensions greater than two, coupling constants have negative mass dimensions, that is that it is hard to consider the $Z = 2$-type theory as a candidate of the quantum gravity.

The discussion above is based on the relevant deformation with preserving the detailed balance. If we break this condition "softly", we can get the non-vanishing Einstein gravity like terms, which preserves the scaling dimension of the time. We use the word, softly, in the sense that the theory breaking the detailed balance softly does not satisfy the detailed balance at the infrared region but satisfies at the ultraviolet region. Such a theory can be written by

$$\begin{aligned} S_{HL} = &\int dt d^D x N\sqrt{q}\left[\frac{2}{\kappa^2}K_{ij}\mathcal{G}^{ijkl}K_{kl} - \frac{\kappa^2}{8{\kappa_W}^4}(R^{ij}R_{ij} - aR^2)\right] \\ &+ \int dt d^D x N\sqrt{q}(-M^{D+2} + \mu^D R), \end{aligned} \tag{4.65}$$

where M and μ are couplings with unit mass dimension. The former part of the action is the one derived by the detailed balance condition of the D dimensional Einstein-Hilbert action. Additionally, if we take

$$\mu^D \equiv \frac{\kappa^2}{8\kappa_W^4}(\tilde{\mu})^2, \quad M^{D+2} \equiv \frac{\kappa^2}{8\kappa_W^4}(\tilde{M})^4, \tag{4.66}$$

then the action can be rewritten by

$$S_{HL} = \int dt d^D x N\sqrt{q}\left[\frac{2}{\kappa^2}K_{ij}\mathcal{G}^{ijkl}K_{kl} - \frac{\kappa^2}{8{\kappa_W}^4}(R^{ij}R_{ij} - aR^2 - \tilde{\mu}^2 R + \tilde{M}^4)\right]. \tag{4.67}$$

The speed of light, the Newton constant and the cosmological constant are

$$c = \frac{\gamma^2}{4}\tilde{\mu}, \quad G_N = \frac{\kappa_W^2}{8\pi\tilde{\mu}}, \quad \Lambda = \frac{\tilde{M}^4}{2\tilde{\mu}^2}. \tag{4.68}$$

The mass dimension of the coupling constants are $[\kappa] = \frac{2-D}{2}$, $[\kappa_W] = \frac{2-D}{2}$ and $[\lambda] = 0$ so that in $D = 2$ we can get the renormalizable theory.

We can consider the other possibility, breaking the detailed balance. It actually runs off the concept of this thesis so that we do not follow this possibility any longer.

4.2 $Z = 3$ Lifshitz-type gravity in $3 + 1$ dimensions

As we have seen above, the $(2+1)$-dimensional $Z = 2$ Lifshitz-type gravity with the softly broken detailed balance condition is a candidate to the quantum gravity in the sense that it is power-counting renormalizable. What we would like to seek next is a $(3+1)$-dimensional candidate. To derive such a theory, it is natural to consider about the critical exponent Z greater than two because the higher spacial derivative terms may lead to the good convergence of the graviton propergator. Thus we will construct the theory with the next higher value of Z, that is 3. Now let us place the gravity at a$Z = 3$ Lifshitz point, which can be done by lifting the 3-dimensional Euclidean action:

$$W = \frac{1}{\omega^2}\int d^3x\epsilon^{ijk}\left[\Gamma^m_{il}\partial_j\Gamma^l_{km} + \frac{2}{3}\Gamma^n_{il}\Gamma^l_{jm}\Gamma^m_{kn}\right] \tag{4.69}$$

where ω^2 is the dimensionless positive coupling constant, and Γ^i_{jk} is the Christoffel symbol compatible with $\nabla_i q_{jk} = 0$. This action is known as the gravitational Chern-Simons action. This has the third order derivative terms so that we can construct the Z=3 Lifshitz-type gravity by lifting up W. Varying W in terms of the Christoffel symbol, we get

$$\begin{aligned}\delta W &= \frac{1}{\omega^2}\int d^3x\epsilon^{ijk}2\delta\Gamma^n_{il}\left(\partial_j\Gamma^l_{kn} + \Gamma^l_{jm}\Gamma^m_{kn}\right)\\ &= \frac{1}{\omega^2}\int d^3x\epsilon^{ijk}\delta\Gamma^n_{il}R^l_{jkn},\end{aligned}$$

If we consider the variation $\delta\Gamma$ in terms of the variation of the metric δq, then we get

$$\delta W = \frac{1}{\omega^2}\int d^3x\epsilon^{ijk}R^l_{jkn}\left[\frac{1}{2}q^{nm}(\nabla_i\delta q_{ml} + \nabla_l\delta q_{mi} - \nabla_m\delta q_{il})\right].$$

Before evaluating the above variation, we need to know an interesting property of the Riemann tensor in 3 dimensions. The Riemann tensor can be decomposed into its symmetric Ricci trace $R_{ij} = R^k_{ikj}$ and its traceless Weyl tensor C^l_{ikj}. In 3 dimensions, the Weyl tensor vanishes identically because the Riemann tensor and the Ricci tensor have the same degrees of freedom, 6, and these two are equivalent like below:

$$R_{ijkl} = q_{ik}\tilde{R}_{jl} + q_{jl}\tilde{R}_{ik} - q_{il}\tilde{R}_{jk} - q_{jk}\tilde{R}_{il}, \tag{4.70}$$

where

$$\tilde{R}_{ij} \equiv R_{ij} - \frac{1}{4}q_{ij}R. \tag{4.71}$$

By using this property, we can evaluate the variation of W:

$$\begin{aligned}
\delta W &= \frac{1}{\omega^2}\int d^3x\epsilon^{ijk}(\delta^l_k\tilde{R}_{jn} + q_{jn}\tilde{R}^l_k - \delta^l_n\tilde{R}_{jk} - q_{jk}\tilde{R}^l_n)\left[\frac{1}{2}q^{nm}(\nabla_i\delta q_{ml} + \nabla_l\delta q_{mi} - \nabla_m\delta q_{il})\right] \\
&= \frac{1}{\omega^2}\int d^3x\epsilon^{ijk}(\tilde{R}^m_j\nabla_k\delta q_{mi}) \\
&= \frac{1}{\omega^2}\int d^3x\nabla_k(\epsilon^{ijk}\tilde{R}^m_j\delta q_{mi}) - \frac{1}{\omega^2}\int d^3x\epsilon^{ijk}(\nabla_k\tilde{R}^m_j)\delta q_{mi} \\
&= \frac{1}{\omega^2}\int d^3x\epsilon^{ijk}(\nabla_j\tilde{R}^m_k)\delta q_{mi}.
\end{aligned}$$

Thus we get

$$\frac{1}{\sqrt{q}}\frac{\delta W}{\delta q_{im}} = \frac{1}{\omega^2}\left(\frac{1}{\sqrt{q}}\epsilon^{ijk}\nabla_j\tilde{R}^m_k\right) \equiv \frac{1}{\omega^2}C^{im}. \tag{4.72}$$

C^{im} is called *Cotton-York tensor*. This tensor has the pretty interesting features. It is symmetric and traceless:

$$C^{ij} = C^{ji}, \quad q_{ij}C^{ij} = 0, \tag{4.73}$$

and it is covariantly conserved:

$$\nabla_i C^{ij} = 0. \tag{4.74}$$

Now we can write down the Z=3 Lifshitz-type gravity action explicitly:

$$\begin{aligned}
S_{HL} &= \int dtd^3xN\sqrt{q}\left[\frac{2}{\kappa^2}K_{ij}\mathcal{G}^{ijkl}K_{kl} - \frac{\kappa^2}{2}\left(\frac{\delta W}{\sqrt{q}\delta q_{ij}}\right)\mathcal{G}_{ijkl}\left(\frac{\delta W}{\sqrt{q}\delta q_{kl}}\right)\right]. \\
&= \int dtd^3xN\sqrt{q}\left[\frac{2}{\kappa^2}K_{ij}\mathcal{G}^{ijkl}K_{kl} - \frac{\kappa^2}{2\omega^4}C_{ij}C^{ij}\right].
\end{aligned}$$

We can write $C_{ij}C^{ij}$ more explicitly:

$$\begin{aligned}
C_{ij}C^{ij} &= \left[\epsilon^{ikl}\nabla_k\left(R^j_l - \frac{1}{4}R\delta^j_l\right)\right]\left[\epsilon_{imn}\nabla^m\left(R^n_j - \frac{1}{4}R\delta^n_j\right)\right] \\
&= (\delta^k_m\delta^l_n - \delta^k_n\delta^l_m)\left[\nabla_k\left(R^j_l - \frac{1}{4}R\delta^j_l\right)\nabla^m\left(R^n_j - \frac{1}{4}R\delta^n_j\right)\right] \\
&= -\frac{1}{8}(\nabla_m R)(\nabla^m R) + (\nabla_i R_{jk})(\nabla^i R^{jk}) - (\nabla_n R_{mj})(\nabla^m R^{jn}) \\
&\quad + \frac{1}{2}\nabla^m R\nabla_n(R^n_m - \frac{1}{2}\delta^n_m R) \\
&= -\frac{1}{8}(\nabla_m R)(\nabla^m R) + (\nabla_i R_{jk})(\nabla^i R^{jk}) - (\nabla_n R_{mj})(\nabla^m R^{jn}).
\end{aligned}$$

In the last line, we used the fact that $\nabla_n(R^n_m - \frac{1}{2}\delta^n_m R) = 0$. Thus the total action is

$$\begin{aligned}
S_{HL} = \int dtd^3xN\sqrt{q}\Bigg[&\frac{2}{\kappa^2}K_{ij}\mathcal{G}^{ijkl}K_{kl} \\
&- \frac{\kappa^2}{2\omega^4}\left(-\frac{1}{8}\nabla_m R\nabla^m R + \nabla_i R_{jk}\nabla^i R^{jk} - \nabla_n R_{mj}\nabla^m R^{jn}\right)\Bigg].
\end{aligned} \tag{4.75}$$

One can notice that the potential terms have the sixth order spacial derivatives, which means that this action is exactly at a Z=3 type Lifshitz point. The symmetry of the above action is again the foliation-preserving diffeomorphism.

4.2.1 Linear approximation

We can investigate the linear approximation of the $Z = 3$ Lifshitz-type theory by almost the same way as in $Z = 2$. To begin with, expanding the field around the Minkowski background with perturbations

$$q_{ij} = \delta_{ij} + \omega h_{ij}, \quad N = 1, \quad N_i = 0. \tag{4.76}$$

If we take the same gauge fixing conditions as in the case of the Z=2, then we get the kinetic term of the Z=3 Lifshitz-type gravity as:

$$S_{HL(K)} = \frac{1}{2\gamma^2}\int dtd^3x\left[(\partial_t\tilde{H}_{ij})^2 + \frac{\lambda-1}{2(3\lambda-1)}(\partial_t H)^2\right], \tag{4.77}$$

where $\gamma = \frac{\kappa}{\omega}$. This is exactly the same form as the $Z = 2$ case because the form of the kinetic term is universal. Thus we can do the same argument in the $Z = 2$ case about the physical degrees of freedom of the graviton so that we do not follow the same discussion in this thesis. The different thing is the potential term, which implies the different dispersion relation. Without showing the explicit calculation, we shall write down the potential term in the linear approximation:

$$S_{HL(P)} = -\frac{\gamma^2}{8}\int dtd^3x\left[\tilde{H}_{ij}(\partial^2)^3\tilde{H}_{ij}\right]. \tag{4.78}$$

The scalar graviton vanishes in the above potential part because of the conformal properties of the Cotton-York tensor. The action of $\tilde{H}_{ij}$ is as follows:

$$S_{HL(\tilde{H}_{ij})} = \int dtd^3x\left[\frac{1}{2\gamma^2}(\partial_t\tilde{H}_{ij})^2 + \frac{\gamma^2}{8}\tilde{H}_{ij}(\partial^2)^3\tilde{H}_{ij}\right]. \tag{4.79}$$

The equation of motion is

$$\left[\frac{1}{\gamma^2}(\partial_t)^2 - \frac{\gamma^2}{4}(\partial^2)^3\right]\tilde{H}_{ij} = 0. \tag{4.80}$$

Putting the Fourier transformed field $\tilde{H}_{ij}(\omega, k)$ into the equation of motion as before, we get the dispersion relation:

$$\omega^2 = \frac{\gamma^4}{4}(k^2)^3. \tag{4.81}$$

4.2.2 Relevant deformation

As in the case of $Z = 2$, we would like to construct the theory that is expected to flow to the Einstein gravity at low energy region and be a candidate to the quantum gravity in the sense of ultraviolet complete theory. At first, we try to construct such an action with preserving the detailed balance condition. It is natural to add the Einstein-Hilbert action with the cosmological constant to the Chern-Simons action:

$$W = \frac{1}{\omega^2}\int \omega_3(\Gamma) + \mu\int d^3x\sqrt{q}(R - 2\Lambda_W), \tag{4.82}$$

where

$$\omega_3(\Gamma) \equiv d^3x\epsilon^{ijk}\left(\Gamma^m_{il}\partial_j\Gamma^l_{km} + \frac{2}{3}\Gamma^n_{il}\Gamma^l_{jm}\Gamma^m_{kn}\right). \tag{4.83}$$

μ is the coupling constant with the unit mass dimension, and Λ_W is the cosmological constant whose mass dimension is 2. The potential term can be calculated like:

$$
\begin{aligned}
&-\frac{\kappa^2}{2}\left(\frac{\delta W}{\sqrt{q}\delta q_{ij}}\right)\mathcal{G}_{ijkl}\left(\frac{\delta W}{\sqrt{q}\delta q_{kl}}\right) \\
&= -\frac{\kappa^2}{2}\left[\left\{\frac{1}{\omega^2}C^{ji} - \frac{\mu}{2}\left(R^{ij} - \frac{1}{2}q^{ij}R + \Lambda_W q^{ij}\right)\right\}\right. \\
&\quad \left.\times\, \mathcal{G}_{ijkl}\left\{\frac{1}{\omega^2}C^{kl} - \frac{\mu}{2}\left(R^{kl} - \frac{1}{2}q^{kl}R + \Lambda_W q^{kl}\right)\right\}\right] \\
&= -\frac{\kappa^2}{2\omega^4}C^{ij}C_{ij} - \frac{\kappa^2\mu^2}{8}\left(R^{ij}R_{ij} + \frac{\lambda + 1/4}{1-3\lambda}R^2 + \frac{1}{3\lambda-1}\Lambda_W R - \frac{3}{3\lambda-1}\Lambda_W^2\right) \\
&\quad + \frac{\kappa^2\mu}{2\omega^2}(R^{ij} - \frac{1}{2}q^{ij}R + \Lambda_W q^{ij})\mathcal{G}_{ijkl}C^{kl}.
\end{aligned}
$$

The last term can be evaluated by using the antisymmetric property of ϵ^{ijk}, that is

$$
\begin{aligned}
\frac{\kappa^2\mu}{2\omega^2}\left(R^{ij} - \frac{1}{2}q^{ij}R + \Lambda_W q^{ij}\right)\mathcal{G}_{ijkl}C^{kl} &= \frac{\kappa^2\mu}{2\omega^2}\epsilon^{imn}\left(\nabla_m R^j_n - \frac{1}{4}\nabla_m R\delta^j_n\right) \\
&\quad \times\left(R_{ij} - \frac{1}{2}q_{ij}R + \Lambda_W q_{ij}\right) \\
&= \frac{\kappa^2\mu}{2\omega^2}\epsilon^{imn}\nabla_m R^j_n R_{ij}.
\end{aligned}
$$

Summing up all the results, we get the Z=3 Lifshitz-type gravity action with relevant deformation:

$$
\begin{aligned}
S_{HL} = \int dtd^3xN\sqrt{q}\Bigg[&\frac{2}{\kappa^2}K_{ij}\mathcal{G}^{ijkl}K_{kl} - \frac{\kappa^2}{2\omega^4}C^{ij}C_{ij} \\
&- \frac{\kappa^2\mu^2}{8}\left(R^{ij}R_{ij} + \frac{\lambda+1/4}{1-3\lambda}R^2 + \frac{1}{3\lambda-1}\Lambda_W R - \frac{3}{3\lambda-1}\Lambda_W^2\right) + \frac{\kappa^2\mu}{2\omega^2}\epsilon^{imn}\nabla_m R^j_n R_{ij}\Bigg].
\end{aligned}
\tag{4.84}
$$

To match the terms relevant in the low energy with those of the general relativity, we need to redefine three constants as before. First, the effective Newton constant is derived by the comparison among kinetic terms, that is

$$G_N = \frac{\kappa^2}{32\pi c}. \tag{4.85}$$

If we compare among R terms, we get the effective speed of light:

$$c = \frac{\kappa^2\mu}{4}\sqrt{\frac{\Lambda_W}{1-3\lambda}}. \tag{4.86}$$

From the no derivative terms, we get the effective cosmological constant:

$$\Lambda = \frac{3}{2}\Lambda_W. \tag{4.87}$$

If the coupling λ flows to 1 at low energy fixed point, then this action seems to be the Einstein-Hilbert action in the infrared region. What we need to do next is counting the dimensions of the couplings. From the dimensional analysis as to the kinetic term, we can calculate the mass dimension of κ:

$$[\kappa] = 0. \tag{4.88}$$

Thus this theory is power-counting renormalizable so that the Z=3 Lifshitz-type gravity with relevant deformation is a candidate to the quantum gravity.

5 Discussions

Let us sum up key ideas in the Lifshitz-type theory. There are three prominent features, those are

- **Detailed balance condition**
  $\rightarrow$ Explain the origin of the anisotropy
- **Foliation-preserving diffeomorphism** (only for gravity)
  $\rightarrow$ Extra scalar DoF
- **Projectability condition** (only for gravity)
  $\rightarrow$ Closed constraint algebra

In the following, we coment on the above three. To begin, we box up ideas of the **detailed balance condition**. As we have seen, the detailed balance condition is some kind of duality in the sense that it connects the D-dimensional theory and the $(D+1)$-dimensional theory. Namely, imposing the detailed balance condition, the $(D+1)$-dimensional theory with anisotropic scaling can be naturally derived by the stochastic quantization of the D-dimensional theory. Thus with the help of the detailed balance condition, we can understand that the origin of the anisotropy between space and time is some "quantum" fluctuations induced by the white Gaussian noise. Although one can construct the Lifshit-type theory without the detailed balance condition, in that case we can say nothing about the origin of the anisotropy. Noting that this anisotropy is quite important especially for the Lifshitz-type gravity. This is because the most fruitful outcome of putting the gravity at a Lifshitz point is the power-counting renormalizability, which is originated with the anisotropic scaling labeled by the critical exponent Z. In the present stage, we can not make a decision on the importance of the detailed balance condition in the context of the Lifshitz-type gravity, though. The next one is the **foliation-preserving diffeomorphism**. This is the symmetry of the Lifshitz-type gravity. Compared to the general relativity, the total diffeomorphism is clearly restricted in the Lifshitz-type gravity, and this fact causes an extra degree of freedom, the scalar graviton. The last one is the **projectability condition**. Unless we impose this condition on the Lifshitz-type gravity, the Poisson bracket of constraints does not form a closed algebra and further it creates the extra second-class constraints. In such a situation, iterating the calculus until we derive the closed algebra, all the derived second-class constraints become over-constraints, 0 physical degree of freedom (See [11] and for the simplified model [12]). Thus for the consistent Lifshitz-type gravity we must pose the projectability condition on it. Then what is the physical meaning of the projectability condition? In fact we can consider it as the gauge fixing condition as we have mentioned before. Namely speaking, the Lifshitz-type gravity is supposed to be the theory with the fixed gauge as $N(x,t) = N(t)$ against some bigger theory.

The Lifshitz-type theory has tremendously interesting concepts. However considering it as the true quantum gravity we need to check lots of facts such as

- Does the coupling constant λ really flows to 1 in the low-energy region? $\rightarrow$ closely related to the unitarity of the scalar graviton.
- Does the speed of light and some other fundamental "constants" truly shift those values?
- Does the "power-counting renormalizability" valid in the theory with the anisotropic scaling?

There exists some more jobs to be done. At this stage, the Lifshtz-type gravity is still the developing model.

Part II

2 dimensions

6 Liouville field theory

In this section, we will discuss about the Liouville field theory somewhat briefly[5]. Our goal is the derivation of the string susceptibility in the context of the Liouville field theory. Right from the start, we would like to show how the Liouville action can emerge in the bosonic string theory. What we would like to make a point here is that the dimension of the target space in the string theory can be taken as the number of matters in the 2-dimensional gravity. In this sense, the so-called *string susceptibility* has been calculated in the conformal gauge by Distler and Kawai [13] (independently by David [14]), which we will follow in this section.

6.1 Emergence of Liouville action

At first, we introduce the Euclidean bosonic string action in d dimensions:

$$S_X(X;g) = \frac{1}{8\pi}\int d^2\xi\sqrt{g}g^{ab}\partial_a X^I\partial_b X^I, \tag{6.1}$$

where the small letter runs 1 to 2, and the capital one does 1 to d. Here X^I are the coordinates of the bosonic string as usual. This theory is the conformal field theory so that the action is invariant under the world-sheet diffeomorphism and the Weyl transformation. The corresponding partition function is

$$Z = \frac{1}{V(G)}\int[\mathcal{D}g][\mathcal{D}_g X]e^{-S_X(X;g)}, \tag{6.2}$$

where $V(G)$ is the volume of the symmetry group. The integral measure $[\mathcal{D}_g X]$ is defined by the Gaussian integration of its fluctuation:

$$\int[\mathcal{D}_g\delta X]e^{-\|\delta X\|_g^2} \equiv 1, \tag{6.3}$$

where

$$\|\delta X\|_g^2 = \int d^2\xi\sqrt{g}\delta X^I\delta X^I. \tag{6.4}$$

Similarly the integration measure $[\mathcal{D}g]$ can be defined by using the norm:

$$\|\delta g\|_g^2 = \int d^2\xi\sqrt{g}(g^{ac}g^{bd} + u g^{ab}g^{cd})\delta g_{ab}\delta g_{cd}, \tag{6.5}$$

where u is a non-negative constant. The action is invariant under the diffeomorphism and the Weyl scaling as we have mentioned. Thus if we change the metric g by using the Weyl rescaling,

$$g_{ab} \to \tilde{g}_{ab} = e^{\phi}g_{ab}, \tag{6.6}$$

where ϕ is some field with an argument ξ, then the action (6.1) is invariant. However, the integration measure $[\mathcal{D}_g X]$ is not invariat under the Weyl rescaling, which is known as *the Weyl anomaly*. To

[5] This section is based on [13, 14, 15, 16, 17].

see how it occurs, we use the well-known trick, and its procedure is as follows. At first, we define the function, W:

$$W = \int [\mathcal{D}_g X] e^{-S_X(X;g)}. \tag{6.7}$$

Then we implement the Weyl rescaling:

$$\begin{aligned} W \to \tilde{W} &= \int [\mathcal{D}_{\tilde{g}} X] e^{-S_X(X;\tilde{g})} \\ &= \int [\mathcal{D}_{\tilde{g}} X] e^{-S_X(X;g)} \\ &\equiv e^{-F(\phi;g)} \int [\mathcal{D}_g X] e^{-S_X(X;g)} \\ &= e^{-F} W. \end{aligned} \tag{6.8}$$

For preparation we introduce the expectation value of the energy-momentum tensor and something important around that:

$$\langle T_{ab} \rangle_{\tilde{g}} \equiv -\frac{4\pi}{\sqrt{\tilde{g}}} \frac{\delta}{\delta \tilde{g}^{ab}} F. \tag{6.9}$$

The only possible form of the trace part of the energy-momentum tensor is

$$\langle T^a_a \rangle_{\tilde{g}} = -\frac{d}{12} \tilde{R}(\xi) + a, \tag{6.10}$$

where $\tilde{R}(\xi)$ is the Ricci scalar defined by $\tilde{g}$, d is the dimension of the target space and a is some constant. This is known as *the trace anomaly relation*. The last one we need is the Weyl-transformed form of the Einstein-Hilbert term:

$$\tilde{R}\sqrt{\tilde{g}} = \sqrt{g}(R - \nabla^2 \phi). \tag{6.11}$$

Let us consider the variation of F under the fluctuation of ϕ:

$$-\delta F = \frac{\delta \tilde{W}}{\tilde{W}}. \tag{6.12}$$

The left hand side is

$$\begin{aligned} -\delta F &= -\int d^2\xi \left(\frac{\delta F}{\delta \tilde{g}^{ab}} \right) \delta \tilde{g}^{ab} \\ &= \frac{1}{4\pi} \int d^2\xi \sqrt{\tilde{g}} \delta \tilde{g}^{ab} \langle T_{ab} \rangle_{\tilde{g}} \\ &= -\frac{1}{4\pi} \int d^2\xi \sqrt{\tilde{g}} \langle T^a_a \rangle_{\tilde{g}} \delta\phi \quad (\because \tilde{g}^{ab} = e^{-\phi} g^{ab}) \\ &= -\frac{1}{4\pi} \int d^2\xi \sqrt{\tilde{g}} \left(-\frac{d}{12} \tilde{R} + a \right) \delta\phi \\ &= -\frac{1}{4\pi} \int d^2\xi \left[-\frac{d}{12} \sqrt{g}(R - \nabla^2\phi)\delta\phi + a\sqrt{g} e^{\phi} \delta\phi \right] \\ &= \delta \left[\frac{d}{48\pi} \int d^2\xi \sqrt{g} \left(\frac{1}{2} g^{ab} \partial_a \phi \partial_b \phi + \phi R - \frac{12a}{d} e^{\phi} \right) \right]. \end{aligned}$$

Thus we get

$$F = -\frac{d}{48\pi}\int d^2\xi\sqrt{g}\left(\frac{1}{2}g^{ab}\partial_a\phi\partial_b\phi + \phi R - \frac{12a}{d}e^{\phi}\right) \equiv -\frac{d}{48\pi}S_L. \tag{6.13}$$

The reason we denote F as S_L daringly is that the above action is called *Liouville action*. Letting the discussion go, we parametrize the metric g by a diffeomorphism f and a Weyl scaling ω:

$$f^*g = e^{\omega}\hat{g}, \tag{6.14}$$

where f^* is an action of a diffeomorphism, and $\hat{g}$ is the fiducial metric parametrized by the moduli. This gauge is the so-called *conformal gauge*. We introduce the ghost fields, b and c, as the Jacobian $J(b,c)$ when we trade the integration over the metric g_{ab} for the integration over the infinitesimal generator of a diffeomorphism ϵ_a and a Weyl scaling ω:

$$[\mathcal{D}g] = [\mathcal{D}_g\omega][\mathcal{D}_g\epsilon]J(b,c) = [\mathcal{D}_g\omega][\mathcal{D}_g\epsilon]([\mathcal{D}_g b][\mathcal{D}_g c]e^{-S_{gh}(b,c;g)}), \tag{6.15}$$

where S_{gh} is the ghost action. As in the case of the matter field X, the ghost action is invariant under a Weyl rescaling, $g \to \tilde{g} = e^{\phi}g$. However, the measure does not, that is

$$[\mathcal{D}_{\tilde{g}}b][\mathcal{D}_{\tilde{g}}c]e^{-S_{gh}(b,c;\tilde{g})} = e^{-\frac{26}{48\pi}S_L}[\mathcal{D}_g b][\mathcal{D}_g c]e^{-S_{gh}(b,c;g)}. \tag{6.16}$$

As a consequence, we find:

$$[\mathcal{D}\tilde{g}][\mathcal{D}_{\tilde{g}}X] = e^{\frac{d-26}{48\pi}S_L}[\mathcal{D}_g X][\mathcal{D}_g b][\mathcal{D}_g c][\mathcal{D}_{\tilde{g}}\omega][\mathcal{D}_{\tilde{g}}\epsilon]e^{-S_{gh}(b,c;g)}. \tag{6.17}$$

This is pretty interesting. We can divide the diffeomorphism group corresponding to $[\mathcal{D}_{\tilde{g}}\epsilon]$ because no term depends on ϵ. If the dimension of the target space is 26, which is the critical string, then no term shows up under Weyl rescalings so that we can also devide the Weyl symmetry group corresponding to $[\mathcal{D}_{\tilde{g}}\omega]$. However if d is not equivalent to 26, then we can not divide the Weyl symmetry group. Namely, we must integrate over ω. Thus when the dimension of the target space is not 26 (,which is the so-called *noncritical string*), the partition function is

$$Z = \int [\mathcal{D}_g\omega][\mathcal{D}_g b][\mathcal{D}_g c][\mathcal{D}_g X]e^{-S_{gh}(b,c;g)-S_X(X;g)}. \tag{6.18}$$

At first sight, it seems that in $d \neq 26$ there exists no Weyl symmetry at the level of the quantum mechanics. But in fact, the Weyl anomaly is cancelled by the miraculous way. We will see this fact in the next subsection. As a conclusion of this subsection, we add some comments. In the above discussions we have neglected the integration of the moduli parameter τ. But this treatment does not hurt the discussion because the measure of τ is independent of ω.

6.2 Renormalized Liouville action

In this subsection, we focus on the noncritical string ($d \neq 26$) with the conformal gauge ($g = e^{\omega}\hat{g}$). For the later discussion, let us introduce an extra term breaking the Weyl symmetry explicitly, that is the bulk cosmological constant term. The partition function is

$$Z = \int [\mathcal{D}_g\omega][\mathcal{D}_g b][\mathcal{D}_g c][\mathcal{D}_g X]\exp\left[-\left(S_X(X;g) + S_{gh}(b,c;g) + \frac{\mu}{2\pi}\int d^2\xi\sqrt{g}\right)\right], \tag{6.19}$$

where μ is the cosmological constant, and again we have omitted the integration over the moduli. The biggest problem is the measure $[\mathcal{D}_g\omega]$. To see this, it is good to write down the norm for $\delta\omega$:

$$\|\delta\omega\|_g^2 = \int d^2\xi\sqrt{g}(\delta\omega)^2 = \int d^2\xi\sqrt{\hat{g}}e^{\omega}(\delta\omega)^2. \tag{6.20}$$

The measure defined above is very hard to handle because the dependence of ω is complex, not Gaussian. To save this situation, we introduce the new field Φ given by

$$\|\delta\Phi\|_{\hat{g}}^2 = \int d^2\xi\sqrt{\hat{g}}(\delta\Phi)^2. \tag{6.21}$$

This form is good to use. Additionally we change the all g_{ab} in the measure into $\hat{g}_{ab}$, and pick up the Jacobian:

$$[\mathcal{D}_g\omega][\mathcal{D}_g b][\mathcal{D}_g c][\mathcal{D}_g X] = [\mathcal{D}_{\hat{g}}\Phi][\mathcal{D}_{\hat{g}}b][\mathcal{D}_{\hat{g}}c][\mathcal{D}_{\hat{g}}X]J(\Phi,\hat{g}). \tag{6.22}$$

We assume that the Jacobian may be written by the action with the properties, the locality and the general coordinate invariance:

$$S(\Phi;\hat{g}) = \frac{1}{8\pi}\int d^2\xi\sqrt{\hat{g}}(\hat{g}^{ab}\partial_a\Phi\partial_b\Phi - Q\hat{R}\Phi + 4\lambda e^{\alpha\Phi}), \tag{6.23}$$

where Q, λ, and α are some constants. This action is the same form as the Liouville action up to coefficients. Now we relate the two metrics:

$$g_{ab} = e^{\alpha\Phi}\hat{g}_{ab}. \tag{6.24}$$

Under this setting, what we would like to do is determining the coefficients, Q, λ, and α satisfying that the total central charge is zero! First, we adjust λ to cancel μ. Then we can notice that the original theory is invariant under the shift:

$$\hat{g}_{ab} \to e^{\sigma}\hat{g}_{ab} \quad , \quad \Phi \to \Phi - \sigma/\alpha. \tag{6.25}$$

This shift must be taken simultaneously because such a combination can let g_{ab} be invariant. In the following, we use a kind of magic. This simultaneous shift leads to the equality:

$$[\mathcal{D}_{e^{\sigma}\hat{g}}\Phi][\mathcal{D}_{e^{\sigma}\hat{g}}b][\mathcal{D}_{e^{\sigma}\hat{g}}c][\mathcal{D}_{e^{\sigma}\hat{g}}X]e^{-S(\Phi-\sigma/\alpha;e^{\sigma}\hat{g})} = [\mathcal{D}_{\hat{g}}\Phi][\mathcal{D}_{\hat{g}}b][\mathcal{D}_{\hat{g}}c][\mathcal{D}_{\hat{g}}X]e^{-S(\Phi;\hat{g})}. \tag{6.26}$$

The variable of integration $\Phi - \sigma/\alpha$ is a dummy one so that we can rename it Φ^*, and furthermore we can call it Φ. This means very surprising result that the total conformal anomaly is vanished! In other words, the total central charge is vanished. We already know the central charge as to the matter + the ghost, that is

$$c_{(X+gh)} = d - 26. \tag{6.27}$$

Thus the central charge of Φ-system $c_{(\Phi)}$ has the following relation:

$$0 = c_{(X+gh)} + c_{(\Phi)} = d - 26 + c_{(\Phi)}, \tag{6.28}$$

We can also calculate $c_{(\Phi)}$ using the OPE among the two energy-momentum tensors. For deriving it, we try to calculate the energy-momentum tensor with respect to the action:

$$S(\Phi;\hat{g}) = \frac{1}{8\pi}\int d^2\xi\sqrt{\hat{g}}(\hat{g}^{ab}\partial_a\Phi\partial_b\Phi - Q\hat{R}\Phi). \tag{6.29}$$

We use the convenient expressions:

$$\hat{G}^{ab} \equiv \hat{R}^{ab} - \frac{1}{2}\hat{g}^{ab}\hat{R}, \tag{6.30}$$

and

$$\hat{w}_a \equiv \hat{g}^{cd}\hat{\nabla}_a \delta\hat{g}_{cd} - \hat{\nabla}^b \delta\hat{g}_{ab}. \tag{6.31}$$

The variation of $S(\Phi;\hat{g})$ with respect to $\hat{g}_{ab}$ is as follows:

$$\begin{aligned}
\delta S(\Phi;\hat{g}) =& \frac{1}{8\pi}\int d^2\xi\sqrt{\hat{g}}\delta\hat{g}_{cd}\left(-\partial^c\Phi\partial^d\Phi + \frac{1}{2}\hat{g}^{cd}\partial_a\Phi\partial_a\Phi\right) \\
&+ \frac{1}{8\pi}\int d^2\xi\sqrt{\hat{g}}\delta\hat{g}_{cd}Q(\hat{G}^{cd}) + \frac{1}{8\pi}\int d^2\xi\sqrt{\hat{g}}Q(\hat{\nabla}^a\hat{w}_a) \\
=& \frac{1}{8\pi}\int d^2\xi\sqrt{\hat{g}}\delta\hat{g}_{cd}\left[-\partial^c\Phi\partial^d\Phi + \frac{1}{2}\hat{g}^{cd}\partial_a\Phi\partial_a\Phi + Q\hat{G}^{cd}\Phi + Q(\hat{g}^{cd}\partial^2\Phi - \partial^c\partial^d\Phi)\right].
\end{aligned} \tag{6.32}$$

This leads the classical the energy-momentum tensor like:

$$\begin{aligned}
T^{cd}(\xi) \equiv& \frac{4\pi}{\sqrt{\hat{g}}}\frac{\delta}{\delta\hat{g}_{cd}}S \\
=& \frac{1}{2}\left(-\partial^c\Phi\partial^d\Phi + \frac{1}{2}\hat{g}^{cd}\partial_a\Phi\partial_a\Phi + Q\hat{G}^{cd}\Phi + Q(\hat{g}^{cd}\partial^2\Phi - \partial^c\partial^d\Phi)\right).
\end{aligned} \tag{6.33}$$

To compute the OPE, we introduce the complex coordinates:

$$z = \xi^1 + i\xi^2 \quad , \quad \bar{z} = \xi^1 - i\xi^2. \tag{6.34}$$

Adopting this coordinate, we can get the holomorphic part of the energy-momentum tensor:

$$T_{zz}(z) = -\left(\frac{1}{2}\partial\Phi\partial\Phi + Q\partial\partial\Phi\right), \tag{6.35}$$

where $\partial = \partial_z$. For simplicity, we write $T_{zz}(z)$ as $T(z)$. Of course, we can also get the antiholomorphic part. Now we take the normal ordering and the OPE given by

$$:\mathcal{F}::\mathcal{J}:= \exp\left(-\frac{1}{2}\int d^2z_1 d^2z_2 \log|z_1 - z_2|^2 \frac{\delta}{\delta\Phi_L(z_1,\bar{z}_1)}\frac{\delta}{\delta\Phi_R(z_2,\bar{z}_2)}\right):\mathcal{F}\mathcal{J}:, \tag{6.36}$$

for some operators $\mathcal{F}$ and $\mathcal{J}$. $\frac{\delta}{\delta\Phi_L(z_1,\bar{z}_1)}$ acts on the left hand side of the operator, and $\frac{\delta}{\delta\Phi_R(z_2,\bar{z}_2)}$ does on the right. In this manner, we can calculate the OPE among energy-momentum tensors:

$$\begin{aligned}
:T(z_1)::T(z_2): \sim& \frac{1}{2}(1+3Q^2)\frac{1}{(z_1-z_2)^4} - \frac{Q}{(z_1-z_2)^3}:\partial\Phi(z_2): \\
&- \frac{Q}{(z_1-z_2)^2}:\partial^2\Phi(z_2):.
\end{aligned} \tag{6.37}$$

Thus we can read off the central charge $c_{(\Phi)}$:

$$c_{(\Phi)} = 1 + 3Q^2. \tag{6.38}$$

Comparing with the one derived above, we get

$$Q = \sqrt{(25-d)/3}. \tag{6.39}$$

It does not matter if we take $Q = \sqrt{(25-d)/3}$ or $Q = -\sqrt{(25-d)/3}$ because we can absorb it into the field redefinition. The last thing to be determined is α. To do this, we set the conformal weight of $e^{\alpha\Phi}$ is $(1,1)$. Let us derive the conformal weight of $e^{\alpha\Phi}$ in a different manner, and set it one. This can be done by evaluating the OPE:

$$:T(z_1)::e^{\alpha\Phi}(z_2):\sim -\frac{1}{2}\alpha(\alpha+Q)\frac{1}{(z_1-z_2)^2}:e^{\alpha\Phi}(z_2):. \tag{6.40}$$

Now we take the conformal weight as one, $1 = -\frac{1}{2}\alpha(\alpha+Q)$ which leads

$$\alpha^2 + Q\alpha + 2 = 0. \tag{6.41}$$

This can be easily calculated, and the solutions are

$$\alpha_\pm = -\frac{1}{2\sqrt{3}}\Big(\sqrt{25-d} \mp \sqrt{1-d}\Big). \tag{6.42}$$

We would like to understand α is real or pure imaginary because we think of $e^{\alpha\Phi}$ as the Weyl scaling. The allowed region under this restriction is

$$d \geq 25 \quad , \quad d \leq 1. \tag{6.43}$$

Especially for the range $d \geq 25$, Q and α both become pure imaginary, and to make α real we need to shift $\Phi \to i\Phi$. This redefinition causes the negative sign of the kinetic term which means the breaking down of the unitarity. We therefore restrict the dimension of the target-space as $d \leq 1$.

6.3 String susceptibility

In this subsection, we try to derive the string susceptibility in the context of the Liouville field theory. To do this, we restrict the Riemann surface to the one with the area A, and this prescription can be done by introducing the partial partition function:

$$Z(A) = \int [\mathcal{D}_{\hat{g}}\Phi][\mathcal{D}_{\hat{g}}X][\mathcal{D}_{\hat{g}}b][\mathcal{D}_{\hat{g}}c]e^{-S}\delta\Big(\int e^{\alpha\Phi}\sqrt{\hat{g}}d^2\xi - A\Big). \tag{6.44}$$

The string susceptibility γ_{str} is defined by

$$Z(A) \sim Z(1)A^{\gamma_{str}-3}. \tag{6.45}$$

In the following, we evaluate γ_{str} by the scaling discussion. Considering the shift of the field like:

$$\Phi \to \Phi + \rho/\alpha, \tag{6.46}$$

where ρ is some constant. The part being affected by this shift in the action is just one, and its change is

$$\begin{aligned} S &\to S - \frac{Q\rho}{8\pi\alpha}\Big(\int d^2\xi\sqrt{\hat{g}}\hat{R}\Big) \\ &= S - \frac{Q\rho}{2\alpha}(\chi) \end{aligned}$$

$$= S - \frac{\rho}{\alpha} Q(1-h), \tag{6.47}$$

where χ is the Euler character, and h is the number of genus. In the second line, we used the Gauss-Bonnet's theorem:

$$\int d^2\xi \sqrt{\hat{g}} \hat{R} = 4\pi\chi = 4\pi(2-2h). \tag{6.48}$$

The δ-function is also shifted:

$$\delta\left(\int e^{\alpha\Phi}\sqrt{\hat{g}} d^2\xi - A\right) \to e^{-\rho}\delta\left(\int e^{\alpha\Phi}\sqrt{\hat{g}} d^2\xi - e^{-\rho}A\right) \tag{6.49}$$

As for the measures, if we redefine $\Phi + \rho/\alpha \equiv \Phi$ as before, then the measures are invariant. Thus we get

$$Z(A) = (e^{-\rho})^{-[Q(1-h)/\alpha - 1]} Z(e^{-\rho}A) \tag{6.50}$$

Adjusting ρ to satisfy $e^{-\rho}A = 1$, $Z(A)$ can be written as

$$Z(A) = A^{[(Q(1-h)/\alpha + 2) - 3]} Z(1) \tag{6.51}$$

By comparing the definition, finally we can find the string susceptibility:

$$\begin{aligned} \gamma_{str} &= \frac{Q(1-h)}{\alpha} + 2 \\ &= (1-h)\left[d - 25 - \sqrt{(25-d)(1-d)}\right]/12 + 2 \end{aligned} \tag{6.52}$$

7 2-dimensional dynamical triangulation

Now we would like to see how the discretized 2-dimensional quantum gravity can be constructed, and furthermore we do uncover a variety of meaningful traits [6].

7.1 Continuum theory

To carry out our plan, at first we introduce the Euclidean continuum 2-dimensional gravitational action, and especially, we restrict our attention to the compact, connected and orientable surface with b boundaries. We denote such a manifold M. The action is

$$S = -\frac{1}{4\pi G_N}\left(\int_M d^2\xi\sqrt{g}R + \sum_{i=1}^{b}\oint_{\partial_i M} ds(\xi) 2k_g\right) + \Lambda\int_M d^2\xi\sqrt{g} + \sum_{i=1}^{b} Z_i \oint_{\partial_i M} ds(\xi). \tag{7.1}$$

There are three coupling constants in the above action, G_N, Λ, and Z_i. Those are the gravitational coupling constant, the cosmological constant, and the cosmological constant of the ith boundary, respectively. We denote R is the Ricci scalar based on the metric g, k_g is the geodesic curvature. The explicit form of the geodesic curvature is given by $k_g = (T^a \nabla_a T^b) n_b$ where T^a is the vector field along the closed boundaries, and n^b is the vector field being nomal to the geodesic. The last portion of the action is introduced because we fix the boundary lengths. Noting that boundary loops must be contractible. This is because unless the boundary loop is contractible, such a loop breaks up the genus of the surface (See Figure 4 and 5).

[6] This section is based on [18, 19, 20, 21, 22, 23, 24, 25, 26]. In particular, dynamical triangulations have been introduced in [28].

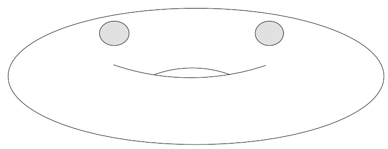


Figure 4: The surface with contractible boundary loops

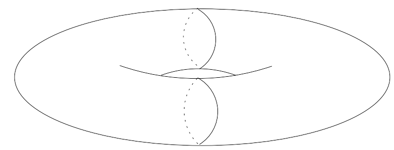


Figure 5: The surface with uncontractible boundary loops

What a great remarkable thing in 2-dimensional gravity is its topological feature. Although the curvature is locally flat, globally it is not because of the Gauss-Bonnet's theorem:

$$\int_M d^2\xi\sqrt{g}R + \sum_{i=1}^{b}\oint_{\partial_i M} ds(\xi)2k_g = 4\pi\chi, \tag{7.2}$$

where χ is the Euler character defined by the genus h and the number of boundaries b; $\chi = 2-2h-b$. The action can be rewritten in the following form:

$$S_\chi = -\frac{\chi}{G_N} + \Lambda V_g + \sum_{i=1}^{b} Z_i L_{i(g)}, \tag{7.3}$$

where

$$V_g \equiv \int_{M_h} d^2\xi\sqrt{g}, \tag{7.4}$$

and

$$L_{i(g)} \equiv \oint_{\partial_i M} ds(\xi). \tag{7.5}$$

$L_{i(g)}$ is the length of the i th boundary. The quantization of the 2-dimensional surface can be done by path-integral over all the metric fluctuations and all the genus contributions. Thus the partition function is

$$Z(G_N,\Lambda) = \sum_{h=0}^{\infty} e^{\frac{\chi}{G_N}} Z_\chi(\Lambda), \tag{7.6}$$

where

$$Z_\chi(\Lambda) = \int \frac{[Dg(\tau_\chi)]}{V(\mathit{diff})}\exp\Big[-\Lambda V_g\Big]. \tag{7.7}$$

τ_χ is a set of the moduli parameters of the surface with the Euler character χ, and $V(\mathit{diff})$ is the gauge volume with respect to the diffeomorphism. The generating functional for boundary loops is

$$W(G_N,\Lambda;Z_1,\cdots,Z_b) = \sum_\chi e^{\frac{\chi}{G_N}}\int\frac{[Dg(\tau_\chi)]}{V(\mathit{diff})}\exp\Big[-\Lambda V_g - \sum_{i=1}^{b} Z_i L_{i(g)}\Big]. \tag{7.8}$$

Of course the length of the boundary $L_{i(g)}$ is the dynamical variable, and there exists the diffeomorphism equivalence class with respect to $L_{i(g)}$. In short, we need to fix this gauge. To conduct this, it is good for us to introduce the so-called *Hartle-Hawking wave functional* [19]:

$$\tilde{W}(G_N,\Lambda;L_1,\cdots,L_b) = \sum_\chi e^{\frac{\chi}{G_N}}\int\frac{[Dg(\tau_\chi)]}{V(\mathit{diff}')}\exp\Big[-\Lambda V_g\Big]\prod_{i=1}^{b}\delta(L_i - L_{i(g)}), \tag{7.9}$$

where $V(diff')$ is the volume of the residual gauge. The Laplace transformation of this Hartle-Hawking wave functional leads

$$W(G_N,\Lambda;Z_1,\cdots,Z_b)=\int_0^\infty\prod_{i=1}^b d\mathrm{L}_i e^{-Z_iL_i}\tilde{W}(G_N,\Lambda;L_1,\cdots,L_b). \tag{7.10}$$

As a result, we can get the generating functional of the 2-dimensional gravity.

7.2 Discretized theory

Let us try to construct the discretized theory of Euclidean gravity in 2 dimensions. At first, we would like to approximate the 2-dimensional surface by equilateral piecewise linear triangles. A piecewise linear triangle is not a curved but a flat triangle. By such a treatment, we can construct the gravity theory without coordinates. Namely, the dynamical variable is the number of ways that we connect triangles, which is called *triangulation*. Such a theory is called *dynamical triangulation*. Contrary to that, if we fix the triangulation, then the length of the edge (or in other words, the link) of triangles becomes the dynamical variable. Such a formalism is called *Regge calculus* [18]. In this thesis, we focus not on the Regge calculus but on the dynamical triangulation.

Dynamical	Fixed	Classes of lattice gravities
Links	Triangulations	The Regge calculus
Triangulations	Links	The dynamical triangulation

Table 1: The classification of the lattice gravity

How can we get the triangulated gravitational action? A good way to know it is taking advantage of the continuum action, that is

$$S_\chi=-\frac{\chi}{G_N}+\Lambda V_g+\sum_{i=1}^b Z_iL_{i(g)}.$$

We define the triangulation as T. Let us try to investigate terms in the above action one by one. First, the curvature term, $-\frac{\chi}{G_N}$, already reduces to the topological quantity. But we would like to see how this quantity can be realized in the dynamical triangulation. To do that, we rewrite the Euler character χ. For preparation, we introduce the number of vertices V, the number of edges E, and the number of faces F. Additionally, we also introduce the number of triangles at ith vertex N_i. We can notice the two ralations:

$$2E=\sum_i N_i, \tag{7.11}$$

$$3F=2E. \tag{7.12}$$

Adding to that, we introduce the Euler's theorem $\chi=V-E+F$. Using these relations, we can rewrite the Euler character:

$$\chi=V-E+F$$

$$
\begin{aligned}
&= \sum_i \left(1 - \frac{E}{3}\right) \\
&= \sum_i \left(1 - \frac{N_i}{6}\right).
\end{aligned}
$$

This equation shows

$$-\frac{\chi}{G_N} = -\frac{1}{G_N}\sum_i\left(1-\frac{N_i}{6}\right). \tag{7.13}$$

We can understand that a curvature term is really a curvature. If $N_i = 6$, the curvature at the ith vertex is zero (Figure 6). Similarly, $N_i < 6$ is the positive curvature (Figure 8), and $N_i > 6$ is the negative curvature (Figure 7). Next, we need to find out the counter part of the volume element

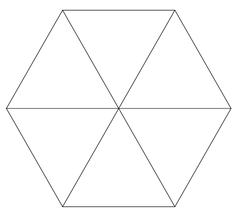


Figure 6: $N_i = 6$

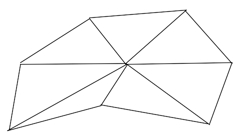


Figure 7: $N_i > 6$

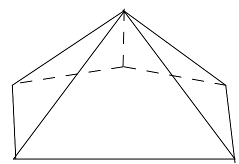


Figure 8: $N_i < 6$

V_g in the discretized context. It can be done easily by setting that each triangle has the unit area. In this situation, the number of triangles in T, $N_t \equiv \sum_i N_i$, becomes the total area of the surface. The last one is also easily found if we set the i^{th} length of the boundary to the number of links lined along the boundary which is denoted by l_i. Summing up all the results, we can write down the action of the dynamically triangulated surface:

$$
\begin{aligned}
S_T(G_N,\mu,\lambda_1,\cdots,\lambda_b) &= -\frac{1}{G_N}\sum_i\left(1-\frac{N_i}{6}\right)+\mu N_t+\sum_{i=1}^{b}\lambda_i l_i \\
&= -\frac{\chi}{G_N}+\mu N_t+\sum_{i=1}^{b}\lambda_i l_i,
\end{aligned}
\tag{7.14}
$$

where μ and λ_i are the bare cosmological constant of the bulk and the ith bare cosmological constant of the boundary component. In fact, we can always adjust the area and the length of edges per triangle by changing the value of couplings. In the following discussion, just for simplicity, we fix the topology of the surface as S^2 with b boundaries which we denote S_b^2. Under this situation, the Euler character is just the number, so that we neglect it and rewrite the action as

$$S_T(\mu,\lambda_1,\cdots,\lambda_b) = \mu N_t + \sum_{i=1}^{b}\lambda_i l_i. \tag{7.15}$$

Let us introduce the partition function as

$$Z(\mu) = \sum_{T\in\mathcal{T}}\frac{1}{C_T}e^{-S_T(\mu)} = \sum_{T\in\mathcal{T}}\frac{1}{C_T}e^{-\mu N_t} \equiv \sum_{k=0}^{\infty}e^{-\mu k}\left(\sum_{T\in\mathcal{T},N_t=k}\frac{1}{C_T}\right)$$

$$\equiv \sum_{k=0}^{\infty} e^{-\mu k} Z(k), \tag{7.16}$$

where C_T is the symmetry factor of T in $\mathcal{T}$ whose continuum analog is the volume of the isometry group. In the discretized theory, of course there exists no diffeomorphism invariance. However, the discrete symmetries which keep triangulation invariant exist, and its duplication (or order) is C_T. Additionally, we can change the length of the boundary so that we also have to fix its degrees of freedom. In the statistical-mechanical point of view, $Z(\mu)$ looks like the ground canonical partition function, and on the other hand, $Z(k)$ is the canonical partition function because k means the number of triangles so that μ seems to be the chemical potential for creating one triangle. We introduce the generating function by summing up all the triangulations:

$$w(\mu, \lambda_1, \cdots, \lambda_b) = \sum_{l_1, \cdots, l_b} \sum_{T \in \mathcal{T}(l_1, \cdots, l_b)} \frac{1}{C_T} e^{-S_T(\mu, \lambda_1, \cdots, \lambda_b)}. \tag{7.17}$$

We call this function *loop function* because this has the same form of the loop function in the matrix model as we will see. Of couse, in the above loop function, there exists the degrees of freedom changing the length of boundaries. Analogous to the continuum case, we can introduce the regularized Hartle-Hawking wave function to fix this gauge:

$$w(\mu, l_1, \cdots, l_b) = \sum_{T \in \mathcal{T}(l_1, \cdots, l_b)} \frac{1}{C_T} e^{-S_T(\mu)} \equiv \sum_k e^{-\mu k} w_{k, l_1, \cdots, l_b}, \tag{7.18}$$

where $w_{k, l_1, \cdots, l_b}$ is the number of triangulations in $\mathcal{T}(l_1, \cdots, l_b)$ with k triangles. Thus the Laplace transformation of the above leads

$$\begin{aligned} w(\mu, \lambda_1, \cdots, \lambda_b) &= \sum_{l_1, \cdots, l_b} w(\mu, l_1, \cdots, l_b) e^{-\sum_{i=1}^{b} \lambda_i l_i} \\ &= \sum_{k, l_1, \cdots, l_b} w_{k, l_1, \cdots, l_b} e^{-\mu k} e^{-\sum_{i=1}^{b} \lambda_i l_i} \\ &= \sum_{k, l_1, \cdots, l_b} w_{k, l_1, \cdots, l_b} g^k z_1^{-l_1} \ldots z_b^{-l_b}, \end{aligned} \tag{7.19}$$

where

$$g \equiv e^{-\mu}, \quad z_i \equiv e^{\lambda_i}. \tag{7.20}$$

If there is no ith boundary component, $l_i = 0$, then ith boundary shrinks to a point. Letting the shrinked boundary have the weight $\frac{1}{z_i}$, we add such factors into $w(\mu, \lambda_1, \cdots, \lambda_b)$, and we redefine it[7] by

$$w(g, z_1, \cdots, z_b) = \sum_{k, l_1, \cdots, l_b} w_{k, l_1, \cdots, l_b} g^k z_1^{-l_1 - 1} \ldots z_b^{-l_b - 1}. \tag{7.21}$$

We take this expression of the loop function as our starting point.

[7] The redefinition is done because we would like to construct the same loop function derived by the matrix model.

7.3 Loop equation as recursion relation

If we can calculate the value $w(\mu, \lambda_1, \cdots, \lambda_b)$, we can get much information about this regularized theory of gravity. There exists a pretty good way to do that, that is *the loop equation*. To let our discussion go, we need to mention about something more about the discretized surface. We can think about the two types of triangulations, the regular triangulation and the unrestricted triangulation (see Figure 9). The regular triangulation means the surface is constructed only by triangles. On the other hand, the unrestricted triangulation has the infinitesimally narrow double links[8] in addition to triangles as the building blocks. These two classes of the triangulation in fact are in the same universality class at the continuum limit. As for the unrestricted triangulation, we can also obtain from the matrix model so that in the following discussions we will focus on the unrestricted triangulation and compare the two.

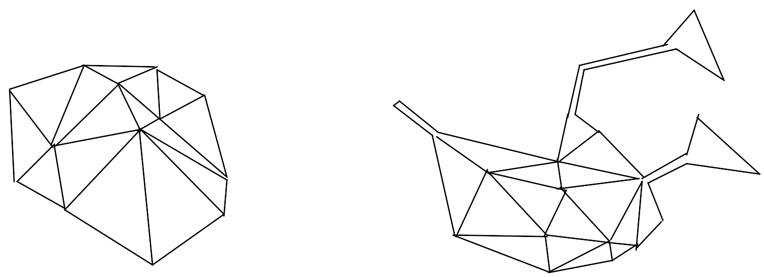


Figure 9: Randomly triangulated surfaces: The left is the regular triangulation, and the right is the unrestricted triangulation.

Now we think about the one boundary case[9]. The generating function is

$$w(g, z) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} w_{k,l} g^k z^{-(l+1)} \equiv \sum_{l=0}^{\infty} \frac{w_l(g)}{z^{l+1}}. \tag{7.22}$$

We normalize the loop function with respect to one point as $\frac{1}{z}$, or in other words:

$$w_0(g) \equiv 1. \tag{7.23}$$

Now we would like to find the recursion relation as for the coefficients of $w(g, z)$. This can be done by considering all possible triangulations of the surface with one marked point. Marked point makes us avoid over-counting the graphs. If we do not mark a point, then we need to product $1/l$ to the surface graph.

We focus on the one link marked by a point. This marked link may belong to two possible origins, those are the triangle and the double link. We can read off by the graphical representation of the recursion relation. The possible number of triangulations corresponds to the coefficient of the

[8] The double link is the 2-gon gluing no triangle so that the double link can only be the boundary.

[9] Since the recursion formula with one boundary can be derived somewhat easily, we take a look at the one boundary case. However, more general multi-boundary case can also be derived by digging holes by the loop insertion operator, as we will see in the later discussion.

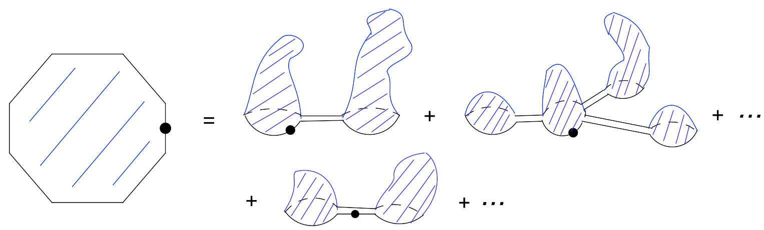


Figure 10: All possible triangulations with one boundary: one of boundary links is marked by a point.

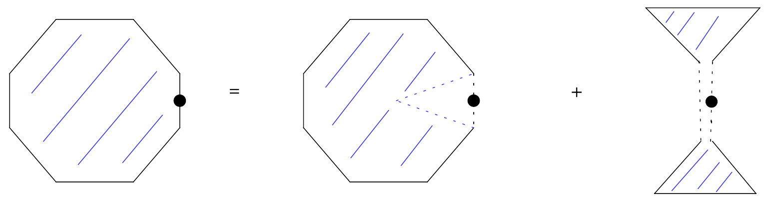


Figure 11: The graphical recursion relation

loop function which is denoted by $[w(g,z)]_{k,l}$. In this notation, we can get the recursion equation explicitly in the following form.

$$\left[w(g,z)-\frac{w_0(g)}{z}\right]_{k,l}=\left[gz\left(w(g,z)-\frac{w_0(g)}{z}-\frac{w_1(g)}{z^2}\right)\right]_{k,l}+\left[\frac{1}{z}w^2(g,z)\right]_{k,l}. \tag{7.24}$$

The left hand side can be understood by noticing that if there exists no link, only one point, then this line of thought breaks down so that we remove one point in advance. As for the first one in the right hand side, one can notice that if we remove one triangle from the surface, which is equivalent to removing the two links, then the surface need to have the two links at least. Thus we remove a point and a link, that is $w(g,z)-\frac{w_0(g)}{z}-\frac{w_1(g)}{z^2}$. After getting rid of a triangle, the residual surface must have one less triangle and one more link, which can be expressed by multiplying gz. More precisely,

$$\begin{aligned} gzw(g,z) &= \sum_{k,l} w_{k,l}\frac{g^{k+1}}{z^{(l+1)-1}} \\ &= \sum_{k',l'} w_{k'-1,l'+1}\frac{g^{k'}}{z^{l'+1}}. \end{aligned}$$

The second term in the right hand side is from the notion that the residual surface has the two less links, which can be checked like:

$$\begin{aligned} \frac{1}{z}w^2(g,z) &= \left(\sum_{k,l} w_{k,l}\frac{g^k}{z^{(l+1)+1}}\right)\left(\sum_{k',l'} w_{k',l'}\frac{g^{k'}}{z^{l'+1}}\right) \\ &= \sum_{k,l}\sum_{k',l''} w_{k,l}w_{k',l''-2}\frac{g^{k+k'}}{z^{(l+l'')+1}}. \end{aligned}$$

In the last case, we need not to remove any object from the surface because $w^2(g,z)$ has $\frac{1}{z^2}$ term at least. From the recursion relation, we can get the so-called loop equation for the generating function:

$$w^2(g,z) - (z - gz^2)w(g,z) + [1 - g(w_1(g) + z)] = 0. \tag{7.25}$$

7.4 Loop equation via matrix model

In fact, the loop equation for the generating function in 2-dimensional discretized gravity has the same form with the one for the resolvent in the matrix model. This coincidence is nontrivial, and furthermore it would be the one of evidence about the duality between the 2-dimensional discretized gravity and the matrix model. We would like to see how the loop equation can emerge in the matrix model at least spherical topology with one boundary, and whether it has really the same form with the one we have derived above. What we need to do is constructing the closed (compact and no-boundary) and connected surface from the matrix model. Then we would like to find out the way of digging holes. This process leads us to get the discrete Hartle-Hawking wave-function in the matrix model context. Taking the correct t'Hooft coupling, we can make the discussion clear (See Appendix C). Thus we take the matrix model action in terms of the $N \times N$ Hermitian matrix M and the t'Hooft couplings g_n like:

$$S_M = -\sum_{n=1}^{\infty} \frac{g_n}{nN^{n/2-1}} \mathrm{tr}(M^n) \equiv \mathrm{tr}[V(M)], \tag{7.26}$$

where $g_2 < 0$, and for other couplings $g_n > 0$. In this form, the diagram has the same factor N^χ at any perturbative order. To factor out N as the overall one, we redefine the matrix as

$$M \to \sqrt{N} M. \tag{7.27}$$

The action can be

$$S_M = N\mathrm{tr}[V(M)]. \tag{7.28}$$

We denote the partition function like:

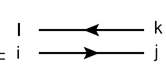
$$Z(g_1, g_2, \cdots) = \int d^{N^2} M e^{-N\mathrm{tr}[V(M)]}. \tag{7.29}$$

This partition function has actually disconnected components. For picking up the only connected components, what we need to do is taking the logarithm of Z.

Getting off the point a little bit, we would like to see how the matrix model corresponds to the 2-dimensional discretized gravity. If we take the matrix model potential like $V(M) = \frac{1}{2}M^2 - \frac{g}{3\sqrt{N}}M^3$ with a positive coupling constant g, then from the perturbative expansion of Z, we can lead

$$\int d^{N^2} M e^{-\frac{1}{2}\mathrm{tr}(M)^2} \frac{1}{k!}\left[\frac{g}{3\sqrt{N}}\mathrm{tr}M^3\right]^k. \tag{7.30}$$

Remember the following Gaussian integral:

$$\int d^{N^2} M e^{-\frac{1}{2}\mathrm{tr}(M)^2} M_i{}^j M_k{}^l = \delta_i^l \delta_k^j \doteq$$ 

$. \quad (7.31)$

By using this as Wick contraction, we can write the (connected or disconnected) vacuum diagrams. $\frac{1}{k!}$ cancels the number of where to put vertices, and $\frac{1}{n}$ cancels the number of contraction patterns. If we take the one of connected diagrams, then it can be

$$\int d^{N^2}Me^{-\mathrm{tr}(M)^2}\frac{1}{k!}\left[\frac{g}{3\sqrt{N}}\mathrm{tr}M^3\right]^k\Bigg|_{conn.}\sim g^kN^\chi C_T. \tag{7.32}$$

where C_T is the symmetry factor of the diagram. Once we describe the Feynman diagram, we notice

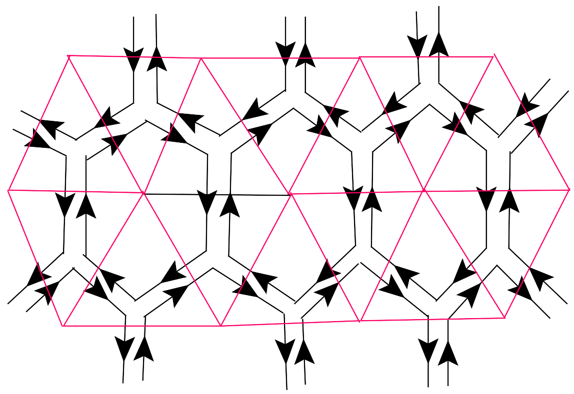


Figure 12: The dual graph

that there is a one-to-one correspondense among the triangulated surface and the Feynman diagram in the double line notation (See Figure 12). More precisely, the number of loops $\mathcal{I}$, propagators $\mathcal{P}$, and vertices $\mathcal{V}$ in the matrix model correspond to the number of vertices V, edges E, and faces F in the 2-dimensional dynamical triangulation, respectively. The Feynman diagrams written by the matrix model picture have the originated from the fact that the matrix M is hermite so that propagators have an orientation. Thus in the dual picture, this orientation must be taken over, which does not conflict with our setting that in the continuum limit the surface must be orientable. C_T corresponds to the symmetry factor of a triangulation T. As we have seen, the duality of the matrix model and the 2-dimensional discretized gravity seems to be true at the graphical level. In fact, by the following discussion, we will see the duality also seems to be true as to the loop equation.

Now, turning our attention to the primary discussion. We define the expectation value of arbitrary observable as

$$\langle\mathcal{O}(M)\rangle\equiv\frac{1}{Z}\int d^{N^2}Me^{-N\mathrm{tr}[V(M)]}\mathcal{O}(M). \tag{7.33}$$

The merit of considering the general potential $\mathrm{tr}[V(M)]$ is that we can take the coupling g_n as the source of $\mathrm{tr}M^n$. That is

$$\frac{d}{dg_n}\log Z(g_1,g_2,\cdots)=\frac{N}{n}\langle\mathrm{tr}M^n\rangle. \tag{7.34}$$

Of course $\langle\mathrm{tr}M^n\rangle$ has only connected graphs. The reason why we consider such an expectation value is that $\langle\frac{1}{N}\mathrm{tr}M^n\rangle$ creates the all possible surfaces with one boundary which consists of n edges (or links). For instance, taking a look at the graph of $\mathcal{V}=10$ ($F=10$) with one boundary (corresponds to removing the 8-sided polygons from the discretized surface). For simplicity, we take the potential as $V(M) = N(\frac{1}{2}M^2 - \frac{g}{3}M^3)$. That is

$$\frac{1}{Z}\int d^{N^2}Me^{-\frac{1}{2}N\mathrm{tr}(M)^2}\frac{1}{10!}\left[\frac{gN}{3}\mathrm{tr}(M)^3\right]^{10}\left[\frac{1}{N}\mathrm{tr}(M^8)\right]. \tag{7.35}$$

The graphical expression can be seen in Figure 13. Similarly, we can add one more boundary:

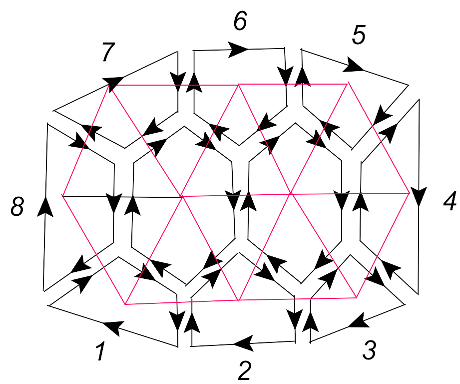


Figure 13: The triangulated surface with one boundary

$$\frac{1}{N^2}\langle \mathrm{tr}M^m \mathrm{tr}M^n\rangle.$$

But this has disconnected parts. The connected parts can be derived by

$$\frac{1}{N^2}\langle \mathrm{tr}M^m \mathrm{tr}M^n\rangle_{conn.} = \frac{1}{N^2}\langle \mathrm{tr}M^m \mathrm{tr}M^n\rangle - \frac{1}{N^2}\langle \mathrm{tr}M^m\rangle\langle \mathrm{tr}M^n\rangle. \tag{7.36}$$

In fact, this is just the discrete analogue of the Hartle-Hawking wave functional with 2 boundaries. In general, we can define the generating function for the b-boundary correlators:

$$\begin{aligned} w(z_1, \cdots, z_b) &\equiv N^{b-2}\sum_{m_1,\cdots,m_b=0}^{\infty}\frac{\langle \mathrm{tr}M^{m_1}\ldots \mathrm{tr}M^{m_b}\rangle_{conn.}}{z_1^{m_1+1}\ldots z_b^{m_b+1}} \\ &= N^{b-2}\left\langle \mathrm{tr}\frac{1}{z_1-M}\ldots \mathrm{tr}\frac{1}{z_b-M}\right\rangle_{conn.}. \end{aligned} \tag{7.37}$$

We introduce the complex z_i because it is good for us to take the analytic continuation when we calculate the generating function. The above one is the general form of the discrete analogue of the Laplace-transformed Hartle-Hawking wave functional. Now we try to get the loop equation for one boundary case. The generating function for one-boundary correlator is

$$w(z) = \frac{1}{N}\left\langle \mathrm{tr}\frac{1}{z-M}\right\rangle = \frac{1}{N}\sum_{i=1}^{N}\left\langle\frac{1}{z-\lambda_i}\right\rangle, \tag{7.38}$$

where λ_i is the eigenvalue of M. This is often called *resolvent* of the matrix M. We also introduce the density of the eigenvalue λ_i of M:

$$\rho(\lambda) = \frac{1}{N}\left\langle\sum_{i=1}^{N}\delta(\lambda-\lambda_i)\right\rangle, \tag{7.39}$$

where

$$1 \equiv \int_{-\infty}^{\infty} d\lambda \rho(\lambda). \tag{7.40}$$

By using this density, the resolvent can be written by

$$w(z) = \int_{-\infty}^{\infty} d\lambda \frac{\rho(\lambda)}{z-\lambda}. \tag{7.41}$$

In the following discussion, we take the potential as

$$V(M) = N\left[\frac{1}{2}M^2 - \frac{g}{n}M^n\right]. \tag{7.42}$$

For convenience, we factor out the 'tHooft coupling, and the correct shift is

$$M \to g^{\frac{1}{2-n}} M. \tag{7.43}$$

This leads

$$S_M = \frac{N}{g^{\frac{2}{n-2}}}\left[\frac{1}{2}\mathrm{tr}(M^2) - \frac{1}{n}\mathrm{tr}(M^n)\right] \tag{7.44}$$

$$\equiv \frac{N}{g^{\frac{2}{n-2}}}[\mathrm{tr}V(M)]. \tag{7.45}$$

The partition function is then

$$\begin{aligned}
Z(g) &= \int d^{N^2} M e^{-S_M} \\
&= \int \prod_i d\lambda_i \left[\prod_{i<j}(\lambda_i - \lambda_j)^2\right] \exp\left[-\frac{N}{g^{\frac{2}{n-2}}}\mathrm{tr}V(M)\right] \\
&= \int \prod_i d\lambda_i \exp\left[-\frac{N}{g^{\frac{2}{n-2}}}\mathrm{tr}V(M) + \sum_{i\neq j}\log \mid \lambda_i - \lambda_j \mid\right] \\
&\equiv \int \prod_i d\lambda_i e^{S_{eff}}.
\end{aligned} \tag{7.46}$$

Our interest is about the spherical topology (genus 0), and such a condition is equivalent to the planar limit in the matrix model. Thus we take the large-N limit, and the leading contribution can be evaluated by the saddle point method, that is $\frac{dS_{eff}}{d\lambda_i} = 0$. $1/N$ expansion in the large-N matrix model corresponds to $\hbar$ expansion in the quantum mechanics so that this saddle point approximation can be called *WKB approximation*. This leads

$$\frac{1}{g^{\frac{2}{n-2}}} V^{'}(\lambda_i) = \frac{2}{N}\sum_{j\neq i}\frac{1}{\lambda_i - \lambda_j}, \tag{7.47}$$

where $V^{'}(\lambda_i)$ is the derivative of $V(\lambda_i)$ with respect to λ_i. We try to rewrite this equation of motion into the more useful form. To do that, multiplying the equation of motion by $\frac{1}{N}\frac{1}{\lambda_i - z}$, and summing up with respect to i, we get

$$\frac{1}{Ng^{\frac{2}{n-2}}}\sum_i \frac{V^{'}(\lambda_i)}{\lambda_i - z} = \frac{2}{N^2}\sum_i\sum_{j\neq i}\frac{1}{\lambda_i - z}\frac{1}{\lambda_i - \lambda_j}. \tag{7.48}$$

The left hand side is

$$\frac{1}{Ng^{\frac{2}{n-2}}}\sum_i \frac{V^{'}(\lambda_i)}{\lambda_i - z} = -\frac{1}{Ng^{\frac{2}{n-2}}}\sum_i\left(\frac{V^{'}(\lambda_i)}{z-\lambda_i} - \frac{V^{'}(z)}{z-\lambda_i}\right) - \frac{1}{Ng^{\frac{2}{n-2}}}\sum_i \frac{V^{'}(z)}{z-\lambda_i}$$
$$= -\frac{1}{Ng^{\frac{2}{n-2}}}\sum_i \frac{V^{'}(\lambda_i)-V^{'}(z)}{z-\lambda_i} - \frac{1}{g^{\frac{2}{n-2}}}w(z)V^{'}(z).$$

The right hand side is

$$\frac{2}{N^2}\sum_i\sum_{j\neq i}\frac{1}{\lambda_i - z}\frac{1}{\lambda_i-\lambda_j} = \frac{1}{N^2}\sum_i\sum_{j\neq i}\left(\frac{1}{\lambda_i - z}\frac{1}{\lambda_i-\lambda_j} - \frac{1}{\lambda_j - z}\frac{1}{\lambda_i-\lambda_j}\right)$$
$$= -\frac{1}{N^2}\sum_i\sum_{j\neq i}\frac{1}{(\lambda_i - z)(\lambda_j - z)}$$
$$= -\frac{1}{N^2}\left[\sum_{i,j}\frac{1}{(\lambda_i - z)(\lambda_j - z)} - \sum_{i=i}\frac{1}{(\lambda_i - z)^2}\right]$$
$$= -w^2(z) + \frac{1}{N}w^{'}(z).$$

Combining the above results, we get the loop equation for the resolvent in the matrix model:

$$w^2(z) - \frac{1}{N}w^{'}(z) - \frac{1}{g^{\frac{2}{n-2}}}w(z)V^{'}(z) + \frac{1}{Ng^{\frac{2}{n-2}}}\sum_i \frac{V^{'}(z)-V^{'}(\lambda_i)}{z-\lambda_i} = 0. \tag{7.49}$$

If we use the density of eigenvalues, then we also have

$$w^2(z) - \frac{1}{N}w^{'}(z) - \frac{1}{g^{\frac{2}{n-2}}}w(z)V^{'}(z) + \frac{1}{g^{\frac{2}{n-2}}}\int d\lambda\rho(\lambda)\frac{V^{'}(z)-V^{'}(\lambda)}{z-\lambda} = 0. \tag{7.50}$$

For instance, let us calculate $n=3$ case, $V(M) = \frac{1}{2}M^2 - \frac{1}{3}M^3$, in the large-$N$ limit. Under the shift, $z \to gz$ and $\lambda \to g\lambda$, the loop equation becomes

$$w^2(z) - (z - gz^2)w(z) + \left[1 - g\left(\int d\lambda\rho(\lambda)(\lambda + z)\right)\right] = 0. \tag{7.51}$$

We can also notice

$$\int d\lambda\rho(\lambda)\lambda = \frac{1}{N}\sum_{i=1}^{N}\langle\lambda_i\rangle = \frac{1}{N}\langle \mathrm{tr}M\rangle. \tag{7.52}$$

We have already known that $\frac{1}{N}\langle \mathrm{tr}M\rangle$ shows the all possible connected and discretized surfaces with one boundary. Remember the loop equation for the generating function in the 2-dimensional discretized gravity (7.25):

$$w^2(g,z) - (z-gz^2)w(g,z) + [1 - g(w_1(g) + z)] = 0.$$

Comparing the two, if

$$w_1(g) = \frac{1}{N}\langle \mathrm{tr}M\rangle,$$

then the two loop equations are exactly the same. Of course we can understand that the above equality is true. Thus at last we can see the two loop equations derived by different roots are just the same form at the level of genus 0 and one boundary. This is the strong evidence of the duality between the 2-dimensional discretized gravity and the matrix model.

7.5 Loop insertion operator

Once we can solve the generating function for one-boundary correlator, we can derive the generating function for any boundary correlators via *the loop insertion operator*. In other words, we can punch any number of holes on the surface as we desire. Such a great operator can be defined by

$$\frac{d}{dV(z)} \equiv \sum_{k=1}^{\infty} \frac{k}{z^{k+1}} \frac{d}{dg_k}. \tag{7.53}$$

The resolvent can be written down as follows:

$$\begin{aligned} w(z_1) &= \frac{1}{N} \sum_{k_1=0}^{\infty} \frac{\langle \mathrm{tr} M^{k_1} \rangle}{z_1^{k_1+1}} \\ &= \frac{1}{N} \frac{1}{Z(g_1, g_2, \cdots)} \sum_{k_1=0}^{\infty} \left[\int \frac{d^{N^2} M}{z_1^{k_1+1}} e^{-\frac{1}{2}M^2} \sum_{l_1, l_2, \cdots}^{\infty} \left\{ \frac{1}{l_1!} (\sqrt{N} g_1 \mathrm{tr} M)^{l_1} \right\} \left\{ \frac{1}{l_2!} \left(\frac{g_2}{2} \mathrm{tr} M^2 \right)^{l_2} \right\} \right. \\ &\quad \left. \times \cdots \times \left\{ \frac{1}{l_m!} \left(\frac{g_m}{m N^{m/2-1}} \mathrm{tr} M^m \right)^{l_m} \right\} \times \cdots (\mathrm{tr} M^{k_1}) \right]. \end{aligned} \tag{7.54}$$

Then we act the loop insertion operator to $w(z_1)$:

$$\begin{aligned} \frac{d}{dV(z_2)} w(z_1) &= \frac{1}{N} \frac{1}{Z(g_1, g_2, \cdots)} \sum_{k_1=0}^{\infty} \left[\int \frac{d^{N^2} M}{z_1^{k_1+1}} e^{-\frac{1}{2}M^2} \right. \\ &\quad \left. \times \sum_{l_1, l_2, \cdots}^{\infty} \left\{ \frac{1}{l_1!} (l_1 g_1^{l_1-1}) (\sqrt{N} \mathrm{tr} M)^{l_1} \right\} \frac{1}{z_2} \cdots (\mathrm{tr} M^{k_1}) \right] + \cdots \\ &\quad + \frac{1}{N} \frac{1}{Z(g_1, g_2, \cdots)} \sum_{k_1=0}^{\infty} \left[\int \frac{d^{N^2} M}{z_1^{k_1+1}} e^{-\frac{1}{2}M^2} \sum_{l_1, l_2, \cdots}^{\infty} \cdots \right. \\ &\quad \left. \times \left\{ \frac{1}{l_m!} (l_m g_m^{l_m-1}) \left(\frac{1}{m N^{m/2-1}} \mathrm{tr} M^m \right)^{l_m} \right\} \frac{m}{z_2^m} \times \cdots (\mathrm{tr} M^{k_1}) \right] \\ &= \sum_{k_1=0}^{\infty} \frac{1}{N z_1^{k_1+1}} \left\langle \left[\frac{\sqrt{N} \mathrm{tr} M}{z_2} + \frac{\mathrm{tr} M^2}{z_2^2} + \cdots + \frac{\mathrm{tr} M^m}{N^{m/2-1} z_2^m} + \cdots \right] (\mathrm{tr} M^{k_1}) \right\rangle_{conn.} \\ &= \sum_{k_1, k_2=0}^{\infty} \left\langle \frac{tr M^{k_1} tr M^{k_2}}{z_1^{k_1+1} z_2^{k_2+1}} \right\rangle_{conn.}. \end{aligned} \tag{7.55}$$

In the last line, we have shifted, $M \to \sqrt{N} M$ and $z_1 \to N^{\frac{k_1}{k_1+1}} z_1$. We derive the surface with two boundaries from the connected surface with one boundary so that derived surface is also connected. By the same discussion, we can get the generating function for any numbers of loops:

$$w(z_1, \cdots, z_b) = \frac{d}{dV(z_b)} \frac{d}{dV(z_{b-1})} \cdots \frac{d}{dV(z_2)} w(z_1). \tag{7.56}$$

We would like to dwell on the relation between $w(z)$ in the matrix model and $w(g, z)$ in the 2-dimensional discretized gravity. For now we know

$$w(z) = \frac{1}{N} \left\langle \mathrm{tr} \frac{1}{z - M} \right\rangle = \sum_{k,l=0}^{\infty} w_{k,l} \frac{g^k}{z^{l+1}} = w(g, z). \tag{7.57}$$

where the potential of the matrix model is $\frac{1}{2}M^2 - \frac{g}{\sqrt{N}}M^3$. How about the case with arbitrary boundary? To check this, operating the loop insertion operator to $w(g,z)$, we would like to consider about the outcome. However if we try to construct the multi-boundary case $w(g, z_1, \cdots, z_b)$ via the loop insertion operator, then we are forced to extend our unrestricted triangulation to the more general form. Namely, we discretize the surface with not only triangles and the double links but also multi-polygons. To make this procedure go, we redefine the generating function in the 2-dimensional discretized gravity like:

$$w(g,z) \equiv \sum_{l,k_1,\cdots,k_n=0}^{\infty} w_{\{k_j\},l} \frac{\prod_{j=1}^{\infty} g_j^{k_j}}{z^{l+1}}, \tag{7.58}$$

where g_i means the coupling constant of i-gon. In this case, we find

$$\begin{aligned} \frac{d}{dV(z_2)} w(g,z_1) &= \sum_{j=1} \frac{j}{z_2^{j+1}} \frac{d}{dg_j} w(g,z_1) \\ &= \sum_{l,\{k_j\}} \sum_{j} \frac{j}{z_2^{j+1}} \frac{k_j}{z_1^{l+1}} \left[\prod_{i \neq j}^{\infty} g_i^{k_i} \right] g_j^{k_j-1} w_{\{k_j\},l}. \end{aligned} \tag{7.59}$$

This can be lead as follows. Acting the loop insertion operator corresponds to removing the j-gon from the k_j possible j-gons and choosing the marked link from the j possible links. Thus $\frac{d}{dV(z_2)} w(g,z_1)$ is really deserved to $w(g,z_1,z_2)$. By the recursive procedure, we get that the generating function of the multi-boundaries can be derived by the loop insertion operator. From this consequence, it is valid to suppose that

$$\begin{aligned} w(z_1,\cdots,z_b) &= N^{b-2} \left\langle \mathrm{tr}\frac{1}{z_1 - M} \cdots \mathrm{tr}\frac{1}{z_b - M} \right\rangle_{conn.} \\ &= \sum_{\{l_j\},\{k_j\}} \left[\prod_{i=1}^{b} \frac{1}{z_i^{l_i+1}} \right] \left[\prod_{m=1}^{\infty} g_m^{k_m} \right] w_{\{l_j\},\{k_j\}} = w(g,z_1,\cdots,z_b). \end{aligned} \tag{7.60}$$

7.6 Solving the loop equation

We would like to uncover the secrets embedded in the structure of the loop equation. At first, we focus on the spherical topology with one boundary. As we have seen, the loop equation of the 2-dimensional dynamical triangulation has the same form as that of the matrix model with the potential $V(M) = \frac{1}{2}M^2 - \frac{g}{3\sqrt{N}}M^3$. In fact, in the continuum limit, the solution of the loop equation does not depend on what types of polygons we use in the regularized theory, *the universality.* Although one may feel a kind of uneasy, we try to solve the loop equation for the 2-dimensional gravity regularized by m-gons ($\frac{g_m}{m}M^m$) and the double links ($\frac{1}{2}M^2$) by using the matrix model. As the conclusion, we will get to know the continuum limit is truely universal. This time, we explicitly distinguish the propagator $\frac{1}{2}M^2$ and the 2-point interaction $\frac{g_2}{2}M^2$. As our starting action we take

$$S_M = N\left[\frac{1}{2}\mathrm{tr}M^2 - \sum_{i=1}^{n} \frac{g_n}{n}\mathrm{tr}M^n\right] \equiv N\mathrm{tr}[V(M)].$$

In this setting, the loop equation in the large N-limit is

$$w^2(z) - V^{'}(z)w(z) + Q(z) = 0, \tag{7.61}$$

where

$$Q(z) = \frac{1}{N}\sum_{i=1}^{N} \frac{V^{'}(z) - V^{'}(\lambda_i)}{z - \lambda_i}. \tag{7.62}$$

Sinse in the large-N limit we can ignore the differential of $w(z)$, the abstract solution is easily derived by

$$w(z) = \frac{1}{2}\Big(V^{'}(z) \pm \sigma(z)\Big), \tag{7.63}$$

where

$$\sigma(z) = \sqrt{V'^2(z) - 4Q(z)}. \tag{7.64}$$

There are three points when we get the explicit form of the above solution:

- $\sigma^2(z)$ is the polynomial of degree $2(n-1)$.
- $w(z)$ is analytic in the complex z-plane except for the vicinities of cuts and infinity.
- $w(z) \sim \frac{1}{z}$ in the large z.

Above three items are the key ideas for the determination of the solution.

Let us see the first one. We can notice that $V^{'2}(z)$ is the polynomial of degree $2(n-1)$, and $Q(z)$ is also polynomial of degree $n-2$. Thus we get

$$\begin{aligned} \sigma^2(z) &= V^{'2}(z) - 4Q(z) \\ &= A_1 z^{2(n-1)} + A_2 z^{2(n-2)} + \cdots + f_1 z^{n-2} + f_2 z^{n-3} + \ldots, \end{aligned} \tag{7.65}$$

where A_is and f_is are the coefficients. This tells us the very interesting feature:

$$\sigma(z) \equiv C\sqrt{\prod_{i=1}^{n-1}(z - c_{i+})(z - c_{i-})} \tag{7.66}$$

$$\equiv D(z)\sqrt{\prod_{i=1}^{n-2}(z - c_{i+})(z - c_{i-})} \tag{7.67}$$

$$\vdots \tag{7.68}$$

$$\equiv M(z)\sqrt{(z - c_+)(z - c_-)}.$$

where C is some constant, $D(z)$ is the polynomial of degree one, and $M(z)$ is the polynomial of degree $n-2$. That is to say, we can adjust the number of branch points($z = c_+,\ c_-$) by hand. In the following discussion, we treat only with the one-cut solutions.

Next, if the support of $\rho(\lambda)$ is confined to a finite interval $[c_-, c_+]$, then $w(z)$ is the analytic function in the complex z-plane except the vicinity of cuts and infinity. This can be checked as follows. We introduce the formula of the δ-function:

$$\delta(\lambda - \lambda_i) = \frac{1}{2\pi i}\left(\frac{1}{\lambda_i - \lambda - i0} - \frac{1}{\lambda_i - \lambda + i0}\right). \tag{7.69}$$

Using this formula, the density can be rewritten by

$$\begin{aligned}\rho(\lambda) &= \frac{1}{N}\left\langle \sum_{i=1}^{N} \delta(\lambda - \lambda_i) \right\rangle \\ &= \frac{1}{2\pi i}\int d\lambda^{'}\left(\frac{\rho(\lambda^{'})}{\lambda^{'} - \lambda - i0} - \frac{\rho(\lambda^{'})}{\lambda^{'} - \lambda + i0}\right) \\ &= \frac{1}{2\pi i}[w(\lambda + i0) - w(\lambda - i0)]. \end{aligned} \tag{7.70}$$

Thus $w(z)$ really has the discontinuity near the eigenvalues.

The last, if we expand $w(z)$ around $|z| \gg |c_{\pm}|$, then we get

$$\begin{aligned} w(z) &= \int_{-\infty}^{\infty} \rho(\lambda)\frac{d\lambda}{z - \lambda} \\ &= \frac{1}{z}\int_{-\infty}^{\infty} \rho(\lambda)\left(1 - \frac{\lambda}{z}\right)^{-1} d\lambda \\ &= \frac{1}{z} + \frac{1}{z^2}\frac{1}{N}\langle \mathrm{tr} M\rangle + \mathcal{O}\left(\frac{1}{z^3}\right). \end{aligned} \tag{7.71}$$

In fact, this condition restricts sign in front of $\sigma(z)$ to minus one, and the structure of it as we will see.

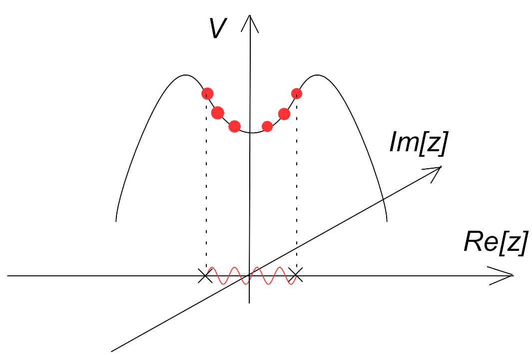


Figure 14: The one-cut graphical relation for $n = 4$

In this thesis, we focus on the 1-cut solution as we mentioned before. As a starter, we try to find out the structure of $M(z)$, which is the great useful tool when we take the continuum limit. First, we would like to introduce three types of paths in the complex ω-plane. Those are c_1, c_2, and c_3, which are the contour along the cut, the one around z, and the one to infinity, respectively.

Going off a tangent a little bit, by using the contour argument, we rewrite the loop equation. Remembering the loop equation in the large-N limit:

$$w^2(z) - V^{'}(z)w(z) + Q(z) = 0,$$

and its solution:

$$w(z) = \frac{1}{2}\left(V^{'}(z) - M(z)\sqrt{(z - c_+)(z - c_-)}\right),$$

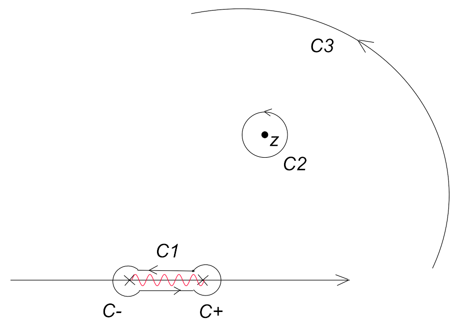


Figure 15: Three paths in the complex ω-plane

where we take the minus sign in front of $M(z)$, in short we shift the ambiguity of the sign to $M(z)$, then we can notice that

$$\oint_{c_1} \frac{d\omega}{2\pi i} \frac{V'(\omega)}{z-\omega} w(\omega) = w^2(z). \tag{7.72}$$

This is because

$$\begin{aligned}
\oint_{c_1} \frac{d\omega}{2\pi i} \frac{V'(\omega)}{z-\omega} w(\omega) &= \oint_{c_3 - c_2} \frac{d\omega}{2\pi i} \frac{V'(\omega)}{z-\omega} w(\omega) \\
&= V'(z) w(z) + \oint_{c_3} \frac{d\omega}{2\pi i} \frac{V'(\omega)}{z-\omega} w(\omega) \\
&\sim V'(z) w(z) + \oint_{c_3} \frac{d\omega}{2\pi i} \frac{V'(\omega)}{z-\omega} \left[\frac{Q(\omega)}{V'(\omega)} \right] \\
&= V'(z) w(z) - Q(z).
\end{aligned} \tag{7.73}$$

The above equation is the another form of the loop equation in the large-N limit.

Now let us find out the more detailed form of $M(z)$ [20]. If we transpose the solution of $w(z)$ in $|z| \gg |c_\pm|$, we get

$$\begin{aligned}
M(z) &= \frac{-2w(z) + V'(z)}{\sqrt{(z-c_+)(z-c_-)}} \\
&\sim \frac{V'(z)}{\sqrt{(z-c_+)(z-c_-)}}.
\end{aligned}$$

Using this, we find

$$M(z) = \oint_{c_3} \frac{d\omega}{2\pi i} \frac{M(\omega)}{\omega - z} \sim \oint_{c_3} \frac{d\omega}{2\pi i} \frac{1}{\omega - z} \left(\frac{V'(\omega)}{\sqrt{(\omega-c_+)(\omega-c_-)}} \right). \tag{7.74}$$

In the following, we treat above $\sim$ as $=$. It is good for us to introduce two new functions:

$$M_k(c_-, c_+, g) \equiv \oint_{c_1} \frac{d\omega}{2\pi i} \frac{V'(\omega)}{(\omega-c_+)^{k+1/2}(\omega-c_-)^{1/2}}, \tag{7.75}$$

$$J_k(c_-, c_+, g) \equiv \oint_{c_1} \frac{d\omega}{2\pi i} \frac{V'(\omega)}{(\omega-c_+)^{1/2}(\omega-c_-)^{k+1/2}}, \tag{7.76}$$

where $g_i \equiv g t_i$ with some variable t_i. These functions are called *moments*. By using M_k and J_k, we can write down $w(z)$:

$$
\begin{aligned}
\sum_{k=1}^{n-1} M_k (z-c_+)^{k-1} &= \oint_{c_1} \frac{d\omega}{2\pi i} \frac{V'(\omega)}{\sqrt{(\omega-c_+)(\omega-c_-)}} \left[\sum_{k=1}^{n-1} \frac{1}{\omega-c_+} \left(\frac{z-c_+}{\omega-c_+} \right)^{k-1} \right] \\
&= \oint_{c_1} \frac{d\omega}{2\pi i} \frac{V'(\omega)}{\sqrt{(\omega-c_+)(\omega-c_-)}} \left[\frac{1}{\omega-z} \left(1 - \left(\frac{z-c_+}{\omega-c_+} \right)^{n-1} \right) \right] \\
&= \oint_{c_3} \frac{d\omega}{2\pi i} \frac{V'(\omega)}{\omega-z} \frac{1}{\sqrt{(\omega-c_+)(\omega-c_-)}} \\
&= M(z),
\end{aligned} \tag{7.77}
$$

and

$$
\sum_{k=1}^{n-1} J_k (z-c_-)^{k-1} = M(z). \tag{7.78}
$$

Finally, we get

$$
\begin{aligned}
w(z) &= \frac{1}{2} \left(V'(z) - \sum_{k=1}^{n-1} M_k (z-c_+)^{k-1} \sqrt{(z-c_+)(z-c_-)} \right) &(7.79)\\
&= \frac{1}{2} \left(V'(z) - \sum_{k=1}^{n-1} J_k (z-c_-)^{k-1} \sqrt{(z-c_+)(z-c_-)} \right). &(7.80)
\end{aligned}
$$

Further we can combine the two:

$$
w(z) = \frac{1}{2} \left(V'(z) - \sum_{k=1}^{n-1} \sqrt{(z-c_+)(z-c_-)} \left[M_k (z-c_+)^{k-1} + J_k (z-c_-)^{k-1} \right] \right). \tag{7.81}
$$

We need to determine M_k and J_k to scale $\frac{1}{z}$ at $z \gg |c_\pm|$. To carry it out, we need to rewrite $w(z)$ in the different form. First, we notice

$$
\begin{aligned}
M(z)\sqrt{(z-c_+)(z-c_-)} &= \oint_{c_3} \frac{d\omega}{2\pi i} \frac{V'(\omega)}{\omega-z} \left(\frac{\sqrt{(z-c_+)(z-c_-)}}{\sqrt{(\omega-c_+)(\omega-c_-)}} \right) \\
&= \oint_{c_1} \frac{d\omega}{2\pi i} \frac{V'(\omega)}{\omega-z} \left(\frac{\sqrt{(z-c_+)(z-c_-)}}{\sqrt{(\omega-c_+)(\omega-c_-)}} \right) + V'(z).
\end{aligned}
$$

In the second line, we changed the contour from c_3 to c_1 and c_2, and $V'(z)$ is the residue at z. Putting this into the solution of $w(z)$, we get

$$
w(z) = \frac{1}{2} \oint_{c_1} \frac{d\omega}{2\pi i} \frac{V'(\omega)}{z-\omega} \left(\frac{\sqrt{(z-c_+)(z-c_-)}}{\sqrt{(\omega-c_+)(\omega-c_-)}} \right). \tag{7.82}
$$

We can also notice

$$
\frac{\sqrt{(z-c_+)(z-c_-)}}{z-w} = 1 + \frac{1}{z} \left[(w-c_+) + (w-c_-) \right] + \mathcal{O}\left(\frac{1}{z^2} \right). \tag{7.83}
$$

Using this, we get

$$w(z)=\frac{1}{2}M_0+\frac{1}{z}\left(\frac{M_{-1}}{4}+\frac{J_{-1}}{4}\right)+\mathcal{O}\left(\frac{1}{z^2}\right). \tag{7.84}$$

Letting $w(z)$ be asymptotically $\frac{1}{z}$ in $z\gg|c_\pm|$, for example we should take

$$M_0=0,\quad M_{-1}=J_{-1}=2. \tag{7.85}$$

In the subsequent discussions, we would like to see three interesting solutions, and figure out its physics.

7.6.1 No-interaction case -*branched polymer*-

What is the universe with no surface? Such a universe is often called *branched polymer*. This can be generated by the following action:

$$S_M=N\left[\frac{1}{2}\mathrm{tr}M^2\right]. \tag{7.86}$$

The corresponding loop equation in the spherical limit is

$$w^2(z)-zw(z)+1=0. \tag{7.87}$$

This expression may be somewhat confusing because the generating function is just the same expression $w(z)$ in the matrix model and the 2-dimensional discretized gravity. Thus we emphasize that $w(z)$ means the generating function with no interaction in the matrix model. The solution is

$$w(z)=\frac{1}{2}\left(z-\sqrt{(z-2)(z+2)}\right), \tag{7.88}$$

where we choose the minus sign in front of $\sqrt{(z-2)(z+2)}$ because of the asymptotic behavior, $z\sim 1/z$ in $z\gg 1$. If we change the variable $x\equiv 1/z^2$, then we get

$$\begin{aligned} w(z) &= \frac{1}{\sqrt{x}}\left[\frac{1-\sqrt{1-4x}}{x}\right] \\ &= \frac{1}{\sqrt{x}}\sum_{l=0}^{\infty}C_l x^l, \end{aligned} \tag{7.89}$$

where

$$C_l\equiv\frac{(2l)!}{(l+1)!l!}. \tag{7.90}$$

C_l is called *Catalan number*. The number l now represents the number of links. Let us consider the meaning of C_l. For example we can take the Catalan number as the number of possible ways to go from A to B in the lower-triangular region (see Figure 16). By using the combinations of U and R in Figure 17, we can generate the all possible branched polymers based on the double links if we assign R with going down the dotted line by one in Figure 18, similarly U with going up. Thus we can understand $C_l=w_{2l}$ where w_{2l} means the possible ways of branched polymers. We again rewrite $w(x)$ in the z coordinate, then we get

$$w(z)=\sum_{l=0}^{\infty}w_{2l}\frac{1}{z^{2l+1}}. \tag{7.91}$$

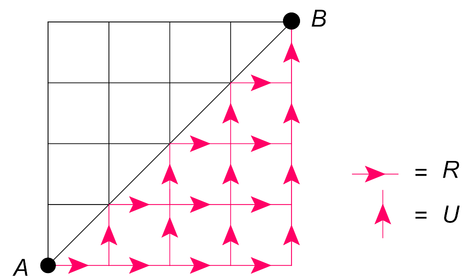


Figure 16: The graphical interpretation of the Catalan number with $l = 4$

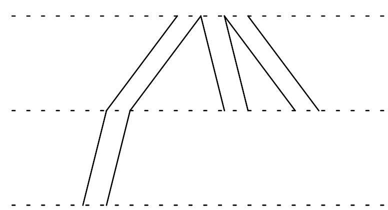


Figure 17: UURRURUR-type branched polymer with $2l = 8$

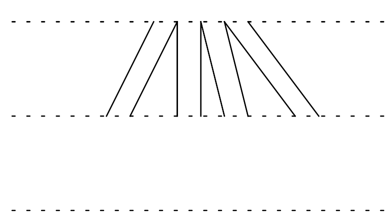


Figure 18: URURURUR-type branched polymer with $2l = 8$

From the discussion of the Catalan number, we can notice that in advance we need to choose the one link so that the graphical representation of the branched polymer must have the one marked link (see Figure 18). The double links in the branched polymer must be infinitesimaly narrow

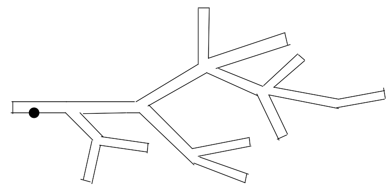


Figure 19: The general branched polymer

because the branched polymer is just the boundary so that all the links must be connected.

7.6.2 Surface induced by quartic vertices

This time, by using the quartic potential we would like to see how we can solve the loop equation as to the one-boundary case. Furthermore, we try to investigate what happens at the critical point through this example. We take our action as:

$$S_M = N\left[\frac{1}{2}M^2 - \frac{g}{4}M^4\right].$$

The general solution derived before is

$$w(z) = \frac{1}{2}\left(V^{'}(z) - \sum_{k=1}^{n-1} M_k (z-c_+)^{k-1}\sqrt{(z-c_+)(z-c_-)}\right).$$

We introduce the finite size eigenvalue distribution like $c \equiv c_+ = -c_- > 0$. Using this information and expanding around the large $|z|$ region, we have

$$\begin{aligned} w(z) \sim \frac{1}{2}\Bigg[& z^3(-g - M_3) + z^2(-M_2 + 2cM_3) + z\left(1 - M_1 + M_2 c - \frac{c^2}{2}M_3\right) \\ &+ \left(\frac{c^2}{2}M_2 - M_3 c^3\right) + \frac{1}{z}\left(\frac{c^2}{2}M_1 - \frac{c^3}{2}M_2 + \frac{5c^4}{8}M_3\right)\Bigg]. \end{aligned} \tag{7.92}$$

What we need to impose on this solution is $w(z) \sim \frac{1}{z}$ in $|z| \gg |c_\pm|$. From the vanishment of cubic terms,

$$M_3 = -g. \tag{7.93}$$

From quadratic terms

$$M_2 = -2cg. \tag{7.94}$$

This does not contradict with the vanishment of zero-th terms. From linear terms

$$M_1 = -\frac{3}{2}c^2 g + 1. \tag{7.95}$$

Plugging these values, we get

$$w(z) = \frac{1}{2}\left[z - gz^3 + \left(\frac{1}{2}c^2 g + gz^2 - 1\right)\sqrt{z^2 - c^2}\right]. \tag{7.96}$$

The last requirement that the coefficient of $1/z$ is one restricts the above solution by

$$3gc^4 - 4c^2 + 16 = 0. \tag{7.97}$$

We can solve this with respect to c^2, and the solution is

$$c^2 = \frac{2 - 2\sqrt{1-12g}}{3g}. \tag{7.98}$$

In the above, we have taken the signature of the singular part of c^2 to be analytic at $g = 0$ (Thr branched polymer) where the branched polymer phase exists. From this result, we can understand that the resolvent $w(z)$ is analytic in the vicinity of $g = 0$, and this becomes singular at

$$g_c = \frac{1}{12}. \tag{7.99}$$

This is the coupling constant at the critical point. Such a singularity is originated with $c^2(g)$. We know that the value $\pm c(g_c)$ means the branch point, the end of the cut, and the value is

$$c_c \equiv c(g_c) = \sqrt{\frac{2}{3g_c}} = 2\sqrt{2}. \tag{7.100}$$

Let us clarify what happens at this critical point more precisely. At the critical point, M_1 becomes

$$M_1(g_c) = 0. \tag{7.101}$$

This implies that the analytic structure of $w(z)$ changes at the critical point:

$$w(z(g_c)) = \frac{1}{2}\left[z - g_c z^3 + g_c(z^2 - c_c^2)^{3/2}\right]. \tag{7.102}$$

For any potential, this does take place. Namely, at the critical point M_1 becomes zero, which means that the variation of the fractional power in $w(z)$. If we go over the critical value g_c, then the end of the cut $c(g)$ turns to be imaginary, that is that eigenvalues have no more real support.

7.6.3 Even-sided-multi-boundary case

The last example is about the surface discretized by the even-sided polygons. First we derive the one-boundary case, and then by using the loop insertion operator we find out the multi-boundary case. We choose the one cut as $c_+ = -c_- = c$. In this case, we find

$$w(z) = \frac{1}{2}\left(V'(z) - M(z)\sqrt{z^2 - c^2}\right), \tag{7.103}$$

where

$$\begin{aligned} M(z) &= \oint_{c_3} \frac{d\omega}{2\pi i}\frac{1}{\omega - z}\left(\frac{V'(\omega)}{\sqrt{\omega^2 - c^2}}\right) \\ &\sim \oint_{c_3} \frac{d\omega}{2\pi i}\frac{V'(\omega)}{\omega^2 - z^2}\frac{\omega}{\sqrt{\omega^2 - c^2}}. \end{aligned}$$

Again, we take $\sim$ as $=$ in the following discussion. If we notice

$$\frac{1}{\omega^2-c^2}\sum_{k=0}^{\infty}\left(\frac{z^2-c^2}{\omega^2-c^2}\right)^k=\frac{1}{\omega^2-z^2},$$

then we get

$$\begin{aligned} M(z) &= \sum_{k=1}^{\infty}\oint_{c_1}\frac{d\omega}{2\pi i}(z^2-c^2)^{k-1}\frac{\omega V^{'}(\omega)}{(\omega^2-c^2)^{k+1/2}} \\ &= \sum_{k=1}^{\infty}\tilde{M}_k(c^2,g)(z^2-c^2)^{k-1}, \end{aligned} \tag{7.104}$$

where

$$\tilde{M}_k(c^2,g)\equiv\oint_{c_1}\frac{d\omega}{2\pi i}\frac{\omega V^{'}(\omega)}{(\omega^2-c^2)^{k+1/2}}. \tag{7.105}$$

We must determine some value of $\tilde{M}_k$ for $w(z)$ to be $1/z$ in $z\gg|c|$ as before. By using the similar way as before, we get

$$w(z)=\frac{1}{2}\oint_{c_1}\frac{d\omega}{2\pi i}\frac{\omega V^{'}(\omega)}{z^2-\omega^2}\sqrt{\frac{z^2-c^2}{\omega^2-c^2}}. \tag{7.106}$$

If we notice

$$\frac{\sqrt{z^2-c^2}}{z^2-\omega^2}=\frac{1}{z}+\mathcal{O}\left(\frac{1}{z^2}\right), \tag{7.107}$$

then we find

$$w(z)=\frac{1}{z}\left(\frac{\tilde{M}_0(c^2,g)}{2}\right)+\mathcal{O}\left(\frac{1}{z^2}\right). \tag{7.108}$$

This must be equivalent to $1/z+\mathcal{O}(1/z^2)$. Thus we can find

$$\tilde{M}_0(c^2,g)=2. \tag{7.109}$$

Next we would like to construct the multi-boundary case by using the loop insertion operator. To begin with, we rewrite the loop insertion operator into the more useful form:

$$\frac{d}{dV(z)}=\sum_{j=1}^{\infty}\frac{2j}{z^{2j+1}}\frac{d}{dg_{2j}}=\frac{\partial}{\partial V(z)}+\frac{dc^2}{dV(z)}\frac{\partial}{\partial c^2}, \tag{7.110}$$

where

$$\frac{\partial}{\partial V(z)}=\sum_{j=1}^{\infty}\frac{2j}{z^{2j+1}}\frac{\partial}{\partial g_{2j}}. \tag{7.111}$$

For preparation, we derive some quantities for a while. Now we take the potential as $\frac{1}{2}M^2-\sum_j\frac{g_{2j}}{2j}M^{2j}$, which leads

$$\begin{aligned} \frac{\partial V^{'}(\omega)}{\partial V(z)} &= -\frac{2}{z\omega}\left[\sum_{j=i}^{\infty}j\left(\frac{\omega}{z}\right)^{2j}\right] \\ &= \frac{-2z\omega}{(z^2-\omega^2)^2}. \end{aligned} \tag{7.112}$$

In the similar manner, we get

$$\frac{\partial \tilde{M}_k}{\partial V(z)} = \frac{\partial}{\partial z}\left[\frac{-z}{(z^2-c^2)^{k+1/2}}\right]. \tag{7.113}$$

From the definition of $\tilde{M}_k$, we get

$$\frac{\partial}{\partial V(z)}\tilde{M}_k = (k+1)\oint_{c_3}\frac{d\omega}{2\pi i}\frac{\omega V'(\omega)}{(\omega^2-c^2)^{k+3/2}}.$$

This leads the recursion relation:

$$\frac{\partial}{\partial V(z)}\tilde{M}_k = (k+1)\tilde{M}_{k+1}. \tag{7.114}$$

By using the above equations, we get

$$\begin{aligned}
\frac{dc^2}{dV(z)} &= \left(\frac{\partial \tilde{M}_0}{\partial c^2}\right)^{-1}\left(\frac{\partial \tilde{M}_0}{\partial V(z)}\right) \\
&= \frac{2}{\tilde{M}_1}\frac{\partial}{\partial z}\left(\frac{-z}{\sqrt{z^2-c^2}}\right) \\
&= \frac{2}{\tilde{M}_1}\frac{c^2}{(z^2-c^2)^{3/2}}.
\end{aligned} \tag{7.115}$$

Thus the loop insertion operator can be expressed by

$$\frac{d}{dV(z)} = \frac{\partial}{\partial V(z)} + \frac{2}{\tilde{M}_1}\frac{c^2}{(z^2-c^2)^{3/2}}\frac{\partial}{\partial c^2}. \tag{7.116}$$

Now we are ready to calculate the multi-boundary case. First, we try to get the 2-boundary case:

$$\begin{aligned}
w(z_1,z_2) &= \frac{d}{dV(z_2)}w(z_1) \\
&= \frac{1}{2(z_1^2-z_2^2)^2}\left[2z_1z_2 + z_2^2\sqrt{\frac{z_1^2-c^2}{z_2^2-c^2}} + z_1^2\sqrt{\frac{z_2^2-c^2}{z_1^2-c^2}}\right].
\end{aligned} \tag{7.117}$$

In the above calculation, we took a kind of technical deformation:

$$\begin{aligned}
&\sum_{k=1}^{\infty}\left[(k+\frac{1}{2})\tilde{M}_{k+1}(z_1^2-c^2)^{k-\frac{1}{2}} - (k-\frac{1}{2})\tilde{M}_k(z_1^2-c^2)^{k-\frac{3}{2}}\right] \\
&= \sum_{k=1}^{\infty}\left[(k-\frac{1}{2})\tilde{M}_k(z_1^2-c^2)^{k-\frac{3}{2}} - (k-\frac{1}{2})\tilde{M}_k(z_1^2-c^2)^{k-\frac{3}{2}}\right] + \frac{1}{2}(z_1^2-c^2)^{-\frac{1}{2}} \\
&= \frac{1}{2(z_1^2-c^2)^{\frac{1}{2}}}.
\end{aligned} \tag{7.118}$$

Similarly we get the 3-boundary case:

$$w(z_1,z_2,z_3) = \frac{c^4}{2\tilde{M}_1(c^2)}\left[\frac{1}{(z_1^2-c^2)(z_2^2-c^2)(z_3^2-c^2)}\right]^{3/2}. \tag{7.119}$$

By the induction, we can find the b-boundary case [20]:

$$w(z_1,\cdots,z_b) = \left(\frac{2}{\tilde{M}_1(c^2)}\frac{d}{dc^2}\right)^{b-3}\frac{1}{2c^2\tilde{M}_1(c^2)}\prod_{k=1}^{b}\frac{c^2}{(z_k^2-c^2)^{3/2}}. \tag{7.120}$$

7.7 Structure of singularity

As we have seen in the case of the surface discretized by the even-sided polygons, the singularity of the generating function is originated from $\tilde{M}_1 = 0$, or equivalently from the non-analyticity of eigenvales $\pm c$. This is because the derivative of c^2 with respect to $V(z)$ diverge for $\tilde{M}_1 = 0$, which implies that there exists the first order phase transition with respect to eigenvalues. Since $\tilde{M}_1 = \tilde{M}_1(g, c^2(g))$, there must be some critical coupling g_c satisfying $\tilde{M}_1(g_c, c(g_c)) = 0$. To figure out the singularity structure of the generating function (of the surface discretized by the even-sided polygons) at the critical point mentioned above, we define

$$\Delta g \equiv g_c - g, \quad \Delta c^2 \equiv c_c^2 - c^2(g), \quad \tilde{M}_2^c \equiv \tilde{M}_2(g_c, c_c^2). \tag{7.121}$$

First, we expand $\tilde{M}_1(g, c^2)$ around the critical values:

$$\begin{aligned}\tilde{M}_1(g, c^2) &= \tilde{M}_1(g_c, c_c^2) - \left.\frac{\partial \tilde{M}_1(g, c^2)}{\partial c^2}\right|_{c=c_c} \Delta c^2 - \left.\frac{\partial \tilde{M}_1(g, c^2)}{\partial g}\right|_{g=g_c} \Delta g + \dots \\ &= -\frac{3}{2}\tilde{M}_2^c \Delta c^2 + \mathcal{O}(\Delta g).\end{aligned} \tag{7.122}$$

In the last line, we used the recursion relation $\frac{\partial}{\partial V(z)}\tilde{M}_k = (k+1)\tilde{M}_{k+1}$, and $\tilde{M}_1(g_c, c(g_c)) = 0$. Subsequently, expanding $\tilde{M}_0(g_c, c_0^2)$, we get

$$\begin{aligned}\tilde{M}_0(g_c, c_c^2) &= \tilde{M}_0(g + \Delta g, c^2 + \Delta c^2) \\ &= \tilde{M}_0(g, c^2) + \left.\frac{\partial \tilde{M}_0(g + \Delta g, c^2 + \Delta c^2)}{\partial g}\right|_{\Delta g = 0} \Delta g \\ &\quad + \left.\frac{\partial \tilde{M}_0(g + \Delta g, c^2 + \Delta c^2)}{\partial c^2}\right|_{\Delta c^2 = 0} \Delta c^2 + \dots .\end{aligned}$$

Noticing $\tilde{M}_0 = 2$ at any argument, we get

$$\frac{\partial \tilde{M}_0(g, c^2)}{\partial g}\Delta g + \frac{\partial \tilde{M}_0(g, c^2)}{\partial c^2}\Delta c^2 \sim 0. \tag{7.123}$$

Now let us evaluate the above relation. As for the second term on the left-hand side, by using the recursion relation and the value of $\tilde{M}_1(g, c^2)$, we can rewrite:

$$\begin{aligned}\frac{\partial \tilde{M}_0(g, c^2)}{\partial c^2} &= \tilde{M}_1(g, c^2)\Delta c^2 \\ &= -\frac{3}{4}\tilde{M}_2^c (\Delta c^2)^2 + \mathcal{O}(\Delta g \Delta c^2).\end{aligned} \tag{7.124}$$

Additionally, the explicit form of $\tilde{M}_0(g_c, c_c^2)$ can be

$$\begin{aligned}2 &= \tilde{M}_0(g_c, c_c^2) \\ &= \oint_{c_1} \frac{d\omega}{2\pi i} \frac{\omega}{(\omega^2 - c_c^2)^{1/2}} \left[\omega - \sum_{j=1}^{n} g_c t_j \omega^{j-1}\right].\end{aligned}$$

Or equivalently,

$$\oint_{c_1} \frac{d\omega}{2\pi i} \frac{\omega^2}{(\omega^2 - c_c^2)^{1/2}} = 2 + \sum_{j=1}^{n} \oint_{c_1} \frac{d\omega}{2\pi i} \frac{g_c t_j \omega^j}{(\omega^2 - c_c^2)^{1/2}}.$$

The left-hand side is

$$\begin{aligned}\oint_{c_1} \frac{d\omega}{2\pi i} \frac{\omega^2}{(\omega^2 - c_c^2)^{1/2}} &= \oint_{c_3} \frac{d\omega}{2\pi i} \frac{\omega^2}{(\omega^2 - c_c^2)^{1/2}} \\ &\sim \oint_{c_3} \frac{d\omega}{2\pi i} \left(\omega + \frac{1}{2}\frac{c_c^2}{\omega}\right) \\ &= \frac{c_c^2}{2}.\end{aligned}$$

Thus we get

$$\sum_{j=1}^{n} \oint_{c_1} \frac{d\omega}{2\pi i} \frac{t_j \omega^j}{(\omega^2 - c_c^2)^{1/2}} = \frac{1}{2g_c}(c_c^2 - 4).$$

Using this, we have

$$\begin{aligned}\frac{\partial \tilde{M}_0(g, c^2)}{\partial g} \Delta g &= \frac{\partial \tilde{M}_0(g_c, c_c^2)}{\partial g} + \mathcal{O}((\Delta g)^2) \\ &\sim -\sum_{j=1}^{n} \oint_{c_1} \frac{d\omega}{2\pi i} \frac{t_j \omega^j}{(\omega^2 - c_c^2)^{1/2}} \\ &= -\frac{1}{2g_c}(c_c^2 - 4). \end{aligned} \tag{7.125}$$

Putting all the results into $\frac{\partial \tilde{M}_0(g,c^2)}{\partial g}\Delta g + \frac{\partial \tilde{M}_0(g,c^2)}{\partial c^2}\Delta c^2 \sim 0$, we get

$$(\Delta c^2)^2 = -\frac{2}{3\tilde{M}_2^c}(c_c^2 - 4)\left(\frac{\Delta g}{g_c}\right) + \mathcal{O}(\Delta g). \tag{7.126}$$

This leads

$$\tilde{M}_1(g, c^2) = \sqrt{\frac{3}{2}|\tilde{M}_2^c|(c_c^2 - 4)}\sqrt{\frac{\Delta g}{g_c}} + \mathcal{O}(\sqrt{\Delta g}). \tag{7.127}$$

Next we would like to find out the singular part of $w(z_1, \cdots, z_b)$ in the limit, $g \to g_c$. Remembering

$$w(z_1, \cdots, z_b) = \left(\frac{2}{\tilde{M}_1(c^2)} \frac{d}{dc^2}\right)^{b-3} \frac{1}{2c^2 \tilde{M}_1(c^2)} \prod_{k=1}^{b} \frac{c^2}{(z_k^2 - c^2)^{3/2}},$$

the most singular part is

$$\begin{aligned}\left(\frac{1}{\tilde{M}_1} \frac{d}{dc^2}\right)^{b-3} \frac{1}{\tilde{M}_1} &\sim (\Delta g)^{\frac{3-b}{2}} \left(\frac{1}{\tilde{M}_1}\right)^{b-2} \frac{\partial \tilde{M}_1}{\partial c^2} \\ &\sim \left(\frac{1}{\Delta g}\right)^{b-\frac{5}{2}}. \end{aligned} \tag{7.128}$$

More precisely,

$$\left(\frac{1}{\Delta g}\right)^{b-\frac{5}{2}} = g_c^{\frac{5}{2}-b} \sum_{k=0}^{\infty} \left[{}_{(b-\frac{7}{2}+k)}C_k\right] \left(\frac{g}{g_c}\right)^k \equiv g_c^{\frac{5}{2}-b} \sum_{k=0}^{\infty} g^k \mathcal{D}_k,$$

where we have used ${}_{(-n)}C_r = (-1)^r {}_{(n+r-1)}C_r$ for $n > 0$. If we take some diagram with fixed links l_i and with the limit $k \to \infty$, then by using the Stirling's formula, $N! \sim N^N$ for $N \gg 1$, we get

$$\mathcal{D}_k \sim k^{b-\frac{7}{2}} \left(\frac{1}{g_c}\right)^k. \tag{7.129}$$

In the context of the 2-dimensional gravity, the asymptotic value of $\mathcal{D}_k$ is supposed to be the asymptotic value of $w_{k,l_1,\cdots,l_b}$ which represents the number of polygonizations, that is

$$w_{k,l_1,\cdots,l_b} \sim k^{b-\frac{7}{2}} \left(\frac{1}{g_c}\right)^k = A^{b-\frac{7}{2}} \left(\frac{1}{g_c}\right)^A, \tag{7.130}$$

where we have used the fact that the value k is the total area A if the each polygon has the unit area. In fact, letting g close to g_c with taking the limit $k \to \infty$ corresponds to taking the continuum limit of the discretized theory as we will see more detail. Even at this stage, we can read off the string susceptibility γ_{str}:

$$A^{b-\frac{7}{2}} = A^{\gamma_{str}-3}. \tag{7.131}$$

In the zero-boundary case, we have

$$\gamma_{str} = -\frac{1}{2}. \tag{7.132}$$

This result does agree with the one derived by the Liouville field theory ((6.52) with $h = 0$ and $d = 0$)! By this striking coincidence, we can confirm that the matrix model is a non-perturbative formulation of the 2-dimensional quantum gravity.

7.8 Continuum limit

We would like to take the continuum limit. To perform this conduct, we need to know the critical hyper surface. In fact the continuum limit is the lattice-spacing-zero limit, $a \to 0$, in the fixed volume, which is equal to the coarse graining of the lattice spacing corresponding to the expansion of the volume. In the renormalization group theory, the one-step coarse graining means the one-time-renormalization-group transformation. To make the infinite coarse graining possible, we must move the coupling constants closer to their critical values. Now let us find out the critical hyper surface. For simplicity, we consider the surface regularized by tetragons with one boundary. However the choice of building blocks is independent of quantities in the continuum limit so that our choice does not loss of generality. First, we define the average of the volume in 2-dimensional discretized surface:

$$\langle N_t \rangle \equiv \frac{1}{w(z)} \left[g \frac{d}{dg} w(z) \right]. \tag{7.133}$$

where $w(z)$ is the resolvent, and g is the coupling constant. This is because the power of the coupling constant is the number of tetragons which is just the volume of the surface. At the critical point, $\langle N_t \rangle$ must go to the infinity. We have already derived $w(z)$ before so that we can read off such a point satisfying

$$1 - 12g = 0. \tag{7.134}$$

In the above calculation, the singular part is from $\frac{dc^2}{dg}$ in the root if that helps. We can check that the critical coupling g_c is $\frac{1}{12}$ as we have said before. The critical value of z must be the edge of the branch point:

$$z_c \equiv c(g_c) = 2\sqrt{2}. \tag{7.135}$$

We assume the canonical scaling:

$$g = g_c e^{-\Lambda a^2} \to \frac{1}{12}(1 - \Lambda a^2), \tag{7.136}$$

and

$$z = z_c e^{aX} \to 2\sqrt{2}\left(1 + \frac{a}{2}X\right), \tag{7.137}$$

where Λ is the renormalized bulk cosmological constant and X is the renormalized boundary cosmological constant. The arrow means the redefinition of cosmological constants. Remembering

$$w(z) = \frac{1}{2}\left[z - gz^3 + \left(\frac{1}{2}c^2 g + gz^2 - 1\right)\sqrt{z^2 - c^2}\right],$$

and putting the canonical scalings into $w(z)$, we get

$$w(z) = \frac{1}{2}\left[z - gz^3 + \frac{4}{3}a^{3/2}W_\Lambda + \mathcal{O}(a^{5/2})\right], \tag{7.138}$$

where

$$W_\Lambda \equiv (X - \sqrt{\Lambda})\sqrt{X + \sqrt{\Lambda}}. \tag{7.139}$$

This is because

$$\sqrt{z^2 - c^2} = 2\sqrt{2}a^{1/2}\sqrt{X + \sqrt{\Lambda}} + \mathcal{O}(a), \tag{7.140}$$

and

$$\frac{1}{2}c^2 g + gz^2 - 1 = \frac{2}{3}a(X - \sqrt{\Lambda}) + \mathcal{O}(a^2). \tag{7.141}$$

The leading non-analytic contribution comes from $a^{3/2}$. Differentiating $w(z)$ two times with respect to a, we find a divergence in the limit, $a \to 0$. Therefore, the continuum generating function is really $W_\Lambda(X)$. The Hartle-Hawking wave function $W_\Lambda(L)$ can also be derived by the inverse Laplace transformation:

$$W_\Lambda(L) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{XL} W_\Lambda(X) \equiv \mathcal{L}^{-1}[W_\Lambda(X)]. \tag{7.142}$$

Calculating by parts:

$$\mathcal{L}^{-1}\left[\sqrt{X + \sqrt{\Lambda}}\right] = -\frac{2}{3}\frac{L^{-3/2}}{\Gamma(-3/2)}e^{-\sqrt{\Lambda}L}\theta(L), \tag{7.143}$$

and

$$\mathcal{L}^{-1}\left[(X + \sqrt{\Lambda})\sqrt{X + \sqrt{\Lambda}}\right] = \frac{L^{-5/2}}{\Gamma(-3/2)}e^{-\sqrt{\Lambda}L}\theta(L). \tag{7.144}$$

where $\theta(L)$ is the Heaviside step function. Using the above, we get

$$\begin{aligned} W_\Lambda(L) &= \mathcal{L}^{-1}[W_\Lambda(X)] \\ &= \mathcal{L}^{-1}\left[(X + \sqrt{\Lambda})\sqrt{X + \sqrt{\Lambda}}\right] - \frac{3}{2}\sqrt{\Lambda}\mathcal{L}^{-1}\left[\sqrt{X + \sqrt{\Lambda}}\right] \\ &= \left(\frac{\theta(L)}{\Gamma(-3/2)}\right)\frac{1}{L^{5/2}}(1 + \sqrt{\Lambda}L)e^{-\sqrt{\Lambda}L}. \end{aligned} \tag{7.145}$$

This is the Hartle-Hawking wave function for the spherical topology with one boundary.

8 (1+1)-dimensional causal dynamical triangulation

The dynamical triangulation in 2 dimensions has been succeeded in the sense that its continuum limit is coincident with the $c = 0$ Liouville field theory. In this thesis, we have confirmed the coincidence of the string susceptibility. However it is known that the dynamical triangulation gets into the trouble in the dimension larger than 2. One of possible reasons to this problem is that the summation over configurations in the path-integral is not correct. Namely we may adopt the too much configurations. If it is true, we need to restrict configurations. As such a restriction, we pose the causality on the dynamical triangulation, which is called *causal dynamical triangulation.* I n the following, we will review this causal dynamical triangulation in $1 + 1$ dimensions [10].

8.1 What is causality?

We first define the causal structure. The space-time manifold has the causal structure if it has the global proper time structure. How can we construct the global proper time structure in the lattice-regularized gravity? To do that, we define the two flat simplices as building blocks (see Figure 20). We assign the time direction to the time-like link. Noting that the length of the time-like links

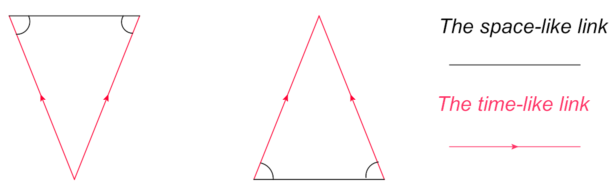


Figure 20: Simplices in the causal dynamical triangulation

in each triangle is same by definition. If we take the different lengths of the time-like links (not isosceles triangles), then the time has the nonuniform flow, which must not be the proper time. The space-time manifold is intrinsically Lorentzian so that we also define that these two building blocks (isosceles triangles) are Lorentzian. This can be done by defining the lengths of the time-like link a_t and of the space-like link a_s as follows:

$$a_t^2 = -\alpha a^2, \quad a_s^2 = a^2, \tag{8.1}$$

where a is the lattice spacing of the space-like link, and α is the anisotropic factor between space and time. In the lattice gravity, each triangle stands for each space-time point in the continuum theory, which means that each space-time point has the uniform time. In other words, we can construct the discretized surface with the local proper time structure (the light-cone structure) if we use the above two isosceles triangles as building blocks.

The next question we would like to handle is how we can construct the global proper time structure by using these two triangles. This can be done by gluing these two triangles in agreement with (1) aligning the time-like directions and (2) gluing the time-like link to the time-like link and for the space-like link to the space-like link. With this prescription, the (1+1)-dimensional surface has the equal time foliations, that is that the space-time simplicial manifold has the global time structure (see Figure 21).

[10]This section is based on [21, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45].

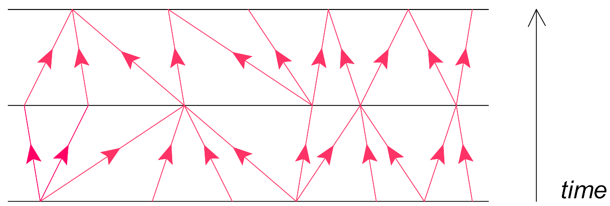


Figure 21: The time foliation

In the following discussion, we will consider the universe that the 1-dimensional space-like loop S^1 propagates along the time R. Due to the causality, we can only consider the cylindrical universe, $S^1 \times R$. If we violate this assignment, for example taking the trouser topology, then the light-cone structure degenerates, which leads the violation of the causality.

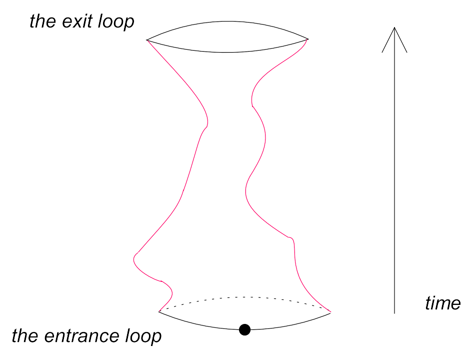


Figure 22: The $(1+1)$-dimensional universe: we call the initial loop *the entrance loop* which is labeled by one point, and we call the final loop *the exit loop*

8.2 Constructing discretized universe with causal structures

In the pure general relativity in $1+1$ dimensions, the gravitational action with 2 boundaries is given by

$$S = \frac{1}{4\pi G_N}\left(\int_M d^2\xi\sqrt{-g}R + \sum_{i=1}^{2}\oint_{\partial_i M} ds(\xi)2k_g\right) + \Lambda\int_M d^2\xi\sqrt{-g}, \tag{8.2}$$

where M denotes the space-time manifold, G_N is the Newton's gravitational coupling constant, R is the curvature, k_g is the geodesic curvature, and Λ is the bulk cosmological constant. As we have mentioned before, the curvature term is just the constant in $1+1$ dimensions (Gauss-Bonnet's theorem). Further, now we take the cylindrical topology, and its Euler character is zero. This implies that in the 2-dimensional causal dynamical triangulation the curvature term is always absent. The corresponding lattice action can be defined as follows:

$$S_T = \lambda a^2 N_t, \tag{8.3}$$

where λ is the bare bulk cosmological constant, a is the lattice spacing of the space-like link, and N_t is the total number of triangles in the triangulation T. Our target universe is that the space-like loop propagates along the time with no interaction so that what we need to know is the cylinder amplitude. By definition, we call the initial loop *the entrance loop*, and call the final loop *the exit loop*. Let us introduce the cylinder amplitude:

$$G_\lambda(l_1, l_2) = \sum_{t=1}^{\infty} G_\lambda(l_1, l_2; t), \tag{8.4}$$

where

$$G_\lambda(l_1, l_2; t) = \sum_{\mathcal{T}} \frac{1}{C_T} e^{iS_T} = \sum_{\mathcal{T}} \frac{1}{C_T} e^{i\lambda a^2 N_t}. \tag{8.5}$$

$\mathcal{T}$ denotes the triangulation, l_1 is the length of the entrance loop, l_2 is the length of the exit loop, and C_T is the volume of global discrete symmetries. Now a height of a triangle corresponds to (the discrete-analog of) the "lapse function" $N(a, \alpha)$, and we set it 1:

$$N(a, \alpha) = a\sqrt{\alpha + \frac{1}{4}} \equiv 1. \tag{8.6}$$

By this treatment, each time step is normalized to 1. Further, we set the lattice spacing of the space-like link, a, to 1. This assignment implies that l_1 counts the number of links in the entrance loop, and l_2 does the number of links in the exit loop. In the following discussions, we mark a point on the entrance loop to get the simple expression. We define the propagator marked on the entrance loop $G_\lambda^{(1)}(l_1, l_2; t)$ as

$$G_\lambda^{(1)}(l_1, l_2; t) = l_1 G_\lambda(l_1, l_2; t). \tag{8.7}$$

If we also would like to mark on the exit loop, then the corresponding propagator $G_\lambda^{(1,2)}(l_1, l_2; t)$ is

$$G_\lambda^{(1,2)}(l_1, l_2; t) = l_2 G_\lambda^{(1)}(l_1, l_2; t) = l_1 l_2 G_\lambda(l_1, l_2; t). \tag{8.8}$$

Next we assume the significant relation:

$$G_\lambda^{(1)}(l_1, l_2; t_1 + t_2) = \sum_{l} G_\lambda^{(1)}(l_1, l; t_1) G_\lambda^{(1)}(l, l_2; t_2). \tag{8.9}$$

This decomposition law plays an important role when we determine the scaling of the propagator as we will see. For the convenience for the calculation, we introduce the generating function for the propagator:

$$G_\lambda^{(1)}(x, y; t) = \sum_{l_1, l_2} x^{l_1} y^{l_2} G_\lambda^{(1)}(l_1, l_2; t), \tag{8.10}$$

where

$$x = e^{i\lambda_I a}, \quad y = e^{i\lambda_F a}. \tag{8.11}$$

λ_I is the boundary cosmological constant for the entrance loop, and λ_F is the boundary cosmological constant for the exit loop. Let us introduce the decomposition law in terms of the generating function:

$$G_\lambda^{(1)}(x, y; t_1, t_2) = \oint \frac{dz}{2\pi z} G_\lambda^{(1)}(x, z^{-1}; t_1) G_\lambda^{(1)}(z, y; t_2), \tag{8.12}$$

where the contour is encircling around the pole of $G_\lambda^{(1)}(x, z^{-1}; t_1)$. We also introduce the intriguing function, that is *the one-step-causal propagator* $G_\lambda^{(1)}(l_1, l_2; 1)$. By using the decomposition law, we

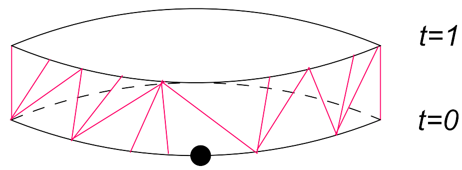


Figure 23: The graphical expression for the one-step-causal propagator $G_\lambda^{(1)}(l_1, l_2; 1)$

can get the general propagator in terms of the one-step-causal propagators:

$$\begin{aligned} G_\lambda^{(1)}(l_1, l_2; t) &= \sum_{l_{1.1}} G_\lambda^{(1)}(l_1, l_{1.1}; 1) G_\lambda^{(1)}(l_{1.1}, l_2; t-1) \\ &= \sum_{l_{1.1}, l_{1.2}} G_\lambda^{(1)}(l_1, l_{1.1}; 1) G_\lambda^{(1)}(l_{1.1}, l_{1.2}; 1) G_\lambda^{(1)}(l_{1.2}, l_2; t-2) \\ &\vdots \\ &= \sum_{l_{1.1}, \cdots l_{1.t-1}} G_\lambda^{(1)}(l_1, l_{1.1}; 1) \cdots G_\lambda^{(1)}(l_{1.t-2}, l_{1.t-1}; 1) G_\lambda^{(1)}(l_{1.t-1}, l_2; 1). \end{aligned} \tag{8.13}$$

Thus if we can get the explicit form of the one-step-causal propagator, then we can derive the general propagator. In the following, we would like to construct this one-step-causal propagator by combinatorics. As a preparation, it is good to introduce the weight g:

$$g = e^{i\lambda a^2} \tag{8.14}$$

Using g, the generating function can be written as

$$\begin{aligned} G_\lambda^{(1)}(x, y; t) &= \sum_{l_1, l_2} \sum_{\mathcal{T}} \frac{1}{C_T} x^{l_1} y^{l_2} g^{N_t} \\ &= \sum_{l_1, l_2, k} w_{k, l_1, l_2} x^{l_1} y^{l_2} g^k \end{aligned} \tag{8.15}$$

where k is the number of triangles, and w_{k,l_1,l_2} is the number of triangulations. This implies that we can get $G_\lambda^{(1)}(x, y; t)$ by summing up all triangulations with the proper weights; g, x, and y (see Figure 24). As for the generating function for the one-step-causal propagator, we can implement

Figure 24: The weights of the up-type triangle and the down-type triangle

its caluculus exactly, that is that it is easy to find out w_{k,l_1,l_2} (see Figure 25). We can read off the exact form of the generating function for the one-step-causal propagator from the combinatrics:

$$G_\lambda^{(1)}(x, y; 1) = \sum_{i=0}^{\infty} \left(gx \sum_{j=0}^{\infty} (gy)^j \right)^i - \sum_{i=0}^{\infty} (gx)^i = \frac{g^2 xy}{(1-gx)(1-gx-gy)}. \tag{8.16}$$

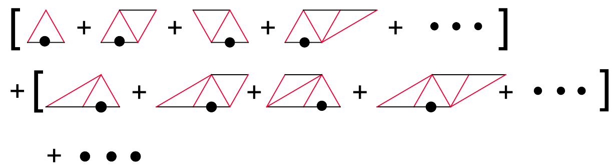


Figure 25: The combinatric calculus of the generating function for the one-step-propagator

In the above expression, we do subtract the situation that the finite entrance loop shrinks to a point or a one point evolves to the finite exit loop. This is because the point becomes the singular point in the continuum limit. The above generating function is asymmetric as to x and y, which is originated with the mark on the entrance loop. If one would like to get the symmetric one $G^{(1,2)}_{\lambda}(x,y;1)$, what we need to do is differentiating it with respect to y:

$$G^{(1,2)}_{\lambda}(x,y;1) = y\frac{d}{dy}G^{(1)}_{\lambda}(x,y;1) = \frac{g^2xy}{1-gx-gy}. \tag{8.17}$$

Putting (8.16) into eq.(8.12), we get

$$\begin{aligned} G^{(1)}_{\lambda}(x,y;t) &= \oint \frac{dz}{2\pi z}\left[\frac{g}{1-gx}\frac{gxz^{-1}}{1-gx-gz^{-1}}\right]G^{(1)}_{\lambda}(z,y;t-1) \\ &= \frac{gx}{1-gx}G^{(1)}_{\lambda}\left(\frac{g}{1-gx},y;t-1\right). \end{aligned} \tag{8.18}$$

In the last line, we have used the fact that the countor is around $z = \frac{1}{1-gx}$. In a similar manner, we can easily construct the generating function in terms only of $G^{(1)}_{\lambda}(x,y;1)$:

$$G^{(1)}_{\lambda}(x,y;t) = F_1^2(x)F_2^2(x)\cdots F_{t-1}^2(x)\frac{g^2xy}{[1-gF_{t-1}(x)][1-gF_{t-1}(x)-gy]}, \tag{8.19}$$

where

$$F_t(x) = \frac{g}{1-gF_{t-1}(x)}, \quad F_0(x) = x. \tag{8.20}$$

One can derive the fixed point, $F_t = F_{t-1}$, which we call F:

$$F = \frac{1-\sqrt{1-4g^2}}{2g}, \tag{8.21}$$

where we have chosen the minus sign because F_0 must converge at $g = 0$. After some computations, we get

$$F_t(x) = F\frac{1-xF+F^{2t-1}(x-F)}{1-xF+F^{2t+1}(x-F)}. \tag{8.22}$$

By inserting (8.22) into (8.19), we get

$$G^{(1)}_{\lambda}(x,y;t) = \frac{F^{2t}(1-F^2)^2xy}{[(1-xF)-F^{2t+1}(F-x)][(1-xF)(1-yF)-F^{2t}(F-x)(F-y)]}. \tag{8.23}$$

Now we consider the convergence region of $G^{(1)}_{\lambda}(x,y;t)$. For letting it converge for any t, we need to take $|F| < 1$, or equivalently $|g| < 1/2$. In addition, remembering (8.15), we get the convergence region:

$$|g| < \frac{1}{2}, \quad |x| < 1, \quad |y| < 1. \tag{8.24}$$

8.3 Continuum limit

In order to take the continuum limit, we need to find out the critical hyper-surface as a first step. To do this, we introduce the average volume of the (1+1)-dimensional universe, $\langle N_t \rangle$:

$$\langle N_t \rangle \equiv g \frac{d}{dg} G_\lambda^{(1)}(x, y; t) \tag{8.25}$$

This expression can be understood by noticing that the power of g is the number of triangles. This average volume diverges, $\langle N_t \rangle \to \infty$, on the critical hyper-surface. In such a place, we get a relation:

$$|g(x+y)| = 1. \tag{8.26}$$

Under the convergence region, the couplings satifying this relation are called *the critical couplings*, and those are

$$g_c = \frac{1}{2}, \quad x_c = 1, \quad y_c = 1. \tag{8.27}$$

We assume the canonical scalings:

$$g = e^{i\lambda a^2} \equiv \frac{1}{2} e^{i\Lambda a^2}, \quad x = e^{i\lambda_I a} \equiv e^{iXa}, \quad y = e^{i\lambda_F a} \equiv e^{iYa}, \tag{8.28}$$

where Λ, X, and Y are the renormalized bulk cosmological constant, the renormalized boundary cosmological constant for the entrance loop, and the renormalized boundary cosmological constant for the exit loop, respectively. Taking the convergence of $G_\lambda^{(1)}(x, y; t)$ into account, it is natural to implement the analytic continuations[11] :

$$\Lambda \equiv i\Lambda, \quad X \equiv iX, \quad Y \equiv iY. \tag{8.29}$$

With this treatment the scaling of coupling constants yields

$$g = \frac{1}{2}(1 - \Lambda a^2) + \mathcal{O}(a^4) \equiv \frac{1}{2}\left(1 - \frac{1}{2}\Lambda a^2\right), \tag{8.30}$$

$$x = 1 - Xa + \mathcal{O}(a^2) \equiv 1 - Xa, \tag{8.31}$$

$$y = 1 - Ya + \mathcal{O}(a^2) \equiv 1 - Ya. \tag{8.32}$$

The last step in each equation means the redefinition of the coupling constant. Just for reference, if we take the set of critical coupling constants as $(g_c, x_c, y_c) = (-1/2, -1, -1)$, the continuum limit has the same form with our choice.

Next, we would like to find out the scaling of the generating function. Before doing that, we need to define the finite variables in the continuum limit. When the lattice spacing of the space-like link a goes to zero, the time t and the boundary lengths l_1, l_2 must diverge. We define the finite variables in this situation:

$$T \equiv at, \quad L_1 \equiv al_1, \quad L_2 \equiv al_2. \tag{8.33}$$

[11] If we take those cosmological constants as really the cosmological constants appeared in the action, in that case there exists the problem by considering this analytic continuation as the Wick rotation, $t \to -it$. Namely, constructing the consistent theory it seems that we have two choices: (1) considering the weights $e^{-\lambda_I a}$ and $e^{-\lambda_F a}$ as just the Boltzmann weights or (2) extending the notion of the Wick rotation.

In the above expressions, we defined the canonical scaling so as to exist no anomalous dimension in both space and time. The validity of our setting will be checked later. Now turn our attention to the scaling of the generating function. What we need to do in the concrete is finding out the critical exponent η in the following relation:

$$G_\Lambda^{(1)}(X,Y;T) = \lim_{a\to 0} a^\eta G_\lambda^{(1)}(x,y;t), \tag{8.34}$$

where $G_\Lambda^{(1)}(X,Y;T)$ is the generating function for the propagator $G_\Lambda^{(1)}(L_1,L_2;T)$ in the continuum limit. We determine it by requiring that the decomposition law (8.9) works out in the continuum limit, that is

$$G_\Lambda^{(1)}(L_1,L_2;T_1+T_2) = \int_0^\infty dL G_\Lambda^{(1)}(L_1,L;T_1) G_\Lambda^{(1)}(L,L_2;T_2). \tag{8.35}$$

As a first step, we determine the critical exponent ξ as follows:

$$G_\Lambda^{(1)}(L_1,L_2;T) = \lim_{a\to 0} a^\xi G_\lambda^{(1)}(l_1,l_2;t). \tag{8.36}$$

Putting this into the decomposition law in the continuum limit (8.35), we get

$$\lim_{a\to 0} a^\xi G_\lambda^{(1)}(l_1,l_2;t_1+t_2) = \lim_{a\to 0}\left[a\sum_l\right]\left[a^\xi G_\lambda^{(1)}(l_1,l;t_1)\right]\left[a^\xi G_\lambda^{(1)}(l,l_2;t_2)\right], \tag{8.37}$$

where we use $\lim_{a\to 0} a\sum_l = \int dL$. We can read off the fact that $\xi = -1$. Thus we get

$$G_\Lambda^{(1)}(L_1,L_2;T) = \lim_{a\to 0} a^{-1} G_\lambda^{(1)}(l_1,l_2;t). \tag{8.38}$$

To derive the critical exponent η, we implement the inverse Laplace transformation:

$$G_\Lambda^{(1)}(L_1,L_2;T) = \int_{-i\infty}^{i\infty} dX \int_{-i\infty}^{i\infty} dY e^{XL_1} e^{YL_2} G_\Lambda^{(1)}(X,Y;T). \tag{8.39}$$

Putting (8.34) and (8.38) into the above, we get

$$\lim_{a\to 0} a^{-1} G_\lambda^{(1)}(l_1,l_2;t) = \lim_{a\to 0}\left[-a^{-1}\oint \frac{dx}{2\pi i x}\right]\left[-a^{-1}\oint \frac{dy}{2\pi i y}\right] x^{-l_1} y^{-l_2}\left[a^\eta G_\lambda^{(1)}(x,y;t)\right], \tag{8.40}$$

where we used the fact that $x = e^{-aX}$ and $y = e^{-aY}$, and the contour was chosen for $G_\lambda^{(1)}(x,y;t)$ to be analytic. This tells us that $\eta = 1$, that is

$$G_\Lambda^{(1)}(X,Y;T) = \lim_{a\to 0} a G_\lambda^{(1)}(x,y;t). \tag{8.41}$$

For a convenience in the later discussion, we also introduce the following relation:

$$G_\Lambda^{(1)}(X,L_2;T) = \lim_{a\to 0} G_\lambda^{(1)}(x,l_2;t). \tag{8.42}$$

This can be checked by a similar manner.

Now we are ready to take the continuum limit. By tuning the coupling constants to their critical values, $(g,x,y)\to(1/2,1,1)$, we get the continuum generating function for the propagator:

$$G_\Lambda^{(1)}(X,Y;T) = \frac{4\Lambda e^{-2\sqrt{\Lambda}T}}{\left[(\sqrt{\Lambda}+X) + e^{-2\sqrt{\Lambda}T}(\sqrt{\Lambda}-X)\right]}$$

$$\times \frac{1}{\left[(\sqrt{\Lambda}+X)(\sqrt{\Lambda}+Y)-e^{-2\sqrt{\Lambda}T}(\sqrt{\Lambda}-X)(\sqrt{\Lambda}-Y)\right]}. \tag{8.43}$$

By carrying out the inverse Laplace transformation of the above solution, we get

$$G_\Lambda^{(1)}(L_1,L_2;T)=\frac{e^{-[\coth\sqrt{\Lambda}T]\sqrt{\Lambda}(L_1+L_2)}}{\sinh\sqrt{\Lambda}T}\frac{\sqrt{\Lambda L_1L_2}}{L_2}I_1\left(\frac{2\sqrt{\Lambda L_1L_2}}{\sinh\sqrt{\Lambda}T}\right), \tag{8.44}$$

where I_1 is a modified Bessel function of the first kind. The quite remarkable thing is that this continuum propagator coincides with the one derived by the Liouville field theory with $c=0$ in the proper-time gauge [34].

9 Generalized causal dynamical triangulation

9.1 "Mild" causality violation

Let us consider the violation of the causality. In the Euclidean gravity, it seems that too much configurations are taken into account in the path-integral, which may be the reason why the Euclidean discretized geometry comes into the trouble in the dimension greater than 2 [12]. However, the causal dynamical triangulation seems to have less configurations. This is because in the path-integral it is reasonable that the geometries which violate the causality at least "mildly". We use the word, "mildly", in the sense that the universe can branch into several universes under the requirement that all universes except for the main channel vanish into the vacuum. We tune these branches by introducing the coupling constant g_s. If we take this coupling constant as small in the continuum context, then we can suppress the birth of universes to the minimum.

To make our setup possible, we define *the generalized-one-step-causal propagator* up to leading contributions of g_s:

$$G_\lambda^{(1)}(l_1,l_2;1)=\tilde{G}_\lambda^{(1)}(l_1,l_2;1)+2g_s\sum_{l=1}^{l_1-1}l_1w(l_1-l,g)\tilde{G}_\lambda^{(1)}(l,l_2;1). \tag{9.1}$$

where $\tilde{G}_\lambda^{(1)}(l_1,l_2;1)$ is the bare one-step-causal propagator we have used in the causal dynamical triangulation, and $w(l_1-l,g)$ is the disc amplitude. The second term in the right-hand side has l_1 because the pinching may occur at l_1 vertices. The disc amplitude plays a role of the Euclidean cap on the baby universe. For graphical interpretation, see Figure 26. The new propagator $G_\lambda^{(1)}(l_1,l_2;t)$

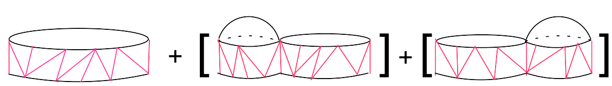


Figure 26: The generalized-one-step-causal propagator

[12]More precisely, in the dimension greater than 2, only two phases can exist, *the crumpled phase* and *the branched-polymer phase*. Those phases are separated by a first-order phase transition line. Taking the continuum limit, we need to find out the critical point where the first-order line becomes the second-order. However, in the dynamical triangulation with the dimension greater than 2, there did not exist such a point.

satisfies the decomposition laws:

$$G_\lambda^{(1)}(l_1,l_2;t_1+t_2)=\sum_l G_\lambda^{(1)}(l_1,l;t_1)G_\lambda^{(1)}(l,l_2;t_2), \tag{9.2}$$

$$G_\lambda^{(1)}(l_1,l_2;t)=\sum_l G_\lambda^{(1)}(l_1,l;1)G_\lambda^{(1)}(l,l_2;t-1). \tag{9.3}$$

The Laplace transformation of (9.3) leads

$$\begin{aligned}
G_\lambda^{(1)}(x,y;t) &= \oint \frac{dz}{2\pi i z}\left[\sum_{l_1,l_2,l} x^{l_1} z^{-l} G_\lambda^{(1)}(l_1,l;1) y^{l_2} z^{l} G_\lambda^{(1)}(l,l_2;t-1)\right] \\
&= \oint \frac{dz}{2\pi i z}\sum_{l_1,l_2,l} x^{l_1} z^{-l} y^{l_2} z^{l}\Bigg[\tilde{G}_\lambda^{(1)}(l_1,l;1) \\
&\quad + 2g_s\sum_{l'=1}^{l_1-1} l_1 w(l_1-l',g)\tilde{G}_\lambda^{(1)}(l',l;1)\Bigg] G_\lambda^{(1)}(l,l_2;t-1) \\
&= \oint \frac{dz}{2\pi i z}\Bigg[\tilde{G}_\lambda^{(1)}(x,z^{-1};1) \\
&\quad + 2g_s\sum_{l_1}\sum_{l'=1}^{l_1-1} l_1 x^{l_1-l'} w(l_1-l',g)\tilde{G}_\lambda^{(1)}(l',z^{-1};1)x^{l'}\Bigg] G_\lambda^{(1)}(z,y;t-1) \\
&= \oint \frac{dz}{2\pi i z}\Bigg[\tilde{G}_\lambda^{(1)}(x,z^{-1};1) \\
&\quad + 2g_s\sum_{l_1}\sum_{l'=1}^{l_1-1} x\frac{\partial}{\partial x}\left(x^{l_1-l'} w(l_1-l',g)\tilde{G}_\lambda^{(1)}(l',z^{-1};1)x^{l'}\right)\Bigg] G_\lambda^{(1)}(z,y;t-1).
\end{aligned}$$

Thus we get

$$G_\lambda^{(1)}(x,y;t)=\oint \frac{dz}{2\pi i z}\left[\tilde{G}_\lambda^{(1)}(x,z^{-1};1)+2g_s x\frac{\partial}{\partial x}\left(w(x,g)\tilde{G}_\lambda^{(1)}(x,z^{-1};1)\right)\right]G_\lambda^{(1)}(z,y;t-1), \tag{9.4}$$

where

$$w(x,g)\equiv\sum_{l_1=0}^{\infty} x^{l_1} w(l_1,g). \tag{9.5}$$

Alternatively, putting (8.16) into the above expression, we can get the following straightforwardly:

$$G_\lambda^{(1)}(x,y;t)=\left[1+2g_s x\frac{\partial w(x,g)}{\partial x}+2g_s x w(x,g)\frac{\partial}{\partial x}\right]\frac{gx}{1-gx}G_\lambda^{(1)}\left(\frac{g}{1-gx},y;t-1\right). \tag{9.6}$$

Now let us find out the nontrivial scaling of the coupling constant g_s so as to realize the branch which violates the causality mildly under the same scaling relations with (not the generalized but) the causal dynamical triangulation:

$$T=at,\quad L_1=al_1,\quad L_2=al_2. \tag{9.7}$$

In addition to the above, we impose that the scaling of the generating function for the disc amplitude $w(x,g)$ has also the same scaling one in the causal dynamical triangulation $\tilde{w}(x,g)$. To do this, we need to know the scaling of $\tilde{w}(x,g)$ in advance. The generating function for the bare disc amplitude is defined in an alternative way:

$$\tilde{w}(x,g) \equiv \sum_t \tilde{G}^{(1)}_\lambda(x, l_2 = 1; t) \equiv \tilde{G}^{(1)}_\lambda(x, l_2 = 1; g). \tag{9.8}$$

In the continuum limit, the disc amplitude $\tilde{W}_\Lambda(L_1)$ and its generating function $\tilde{W}_\Lambda(X)$ can be defined by

$$\begin{aligned}\tilde{W}_\Lambda(X) &\equiv \int_0^\infty dL_1 e^{-XL_1} \tilde{W}_\Lambda(L_1) \\ &\equiv \int_0^\infty dL_1 e^{-XL_1} \left[\int_0^\infty dT \tilde{G}^{(1)}_\Lambda(L_1, L_2 = 0; T)\right] \\ &= \int_0^\infty dT \tilde{G}^{(1)}_\Lambda(X, L_2 = 0; T). \end{aligned}\tag{9.9}$$

What we need to do is determining the critical exponent β:

$$\lim_{a\to 0} a^\beta \tilde{w}(x,g) = \tilde{W}_\Lambda(X). \tag{9.10}$$

We have already derived the scaling of $\tilde{G}^{(1)}_\lambda(x,l;t)$, that is (8.42). Putting (8.42) with $l_2 = 1$ and (9.10) into (9.9), we get

$$\lim_{a\to 0} a^\beta \tilde{w}(x,g) = \lim_{a\to 0}\left[a\sum_t\right] G^{(1)}_\lambda(x, l_2 = 1; t), \tag{9.11}$$

where we used the fact that $\int_0^\infty dT = \lim_{a\to 0} a \sum_t$. By comparing with (9.8), we get $\beta = 1$, that is

$$\lim_{a\to 0} a\tilde{w}(x,g) = \tilde{W}_\Lambda(X). \tag{9.12}$$

Now we are ready to get the nontrivial scaling of the coupling constant g_s. In fact, the scaling of $G^{(1)}_\lambda(x,l;t)$ has been already known. Because the decomposition law has just the same form between the generalized and (the bare) causal dynamical triangulation. Namely we have

$$G^{(1)}_\Lambda(X, L_2; T) = \lim_{a\to 0} G^{(1)}_\lambda(x, l_2; t). \tag{9.13}$$

Let us find out the continuum form of (9.6). Using the scalings, $x = 1 - aX$ and $g = \frac{1}{2}(1 - \frac{1}{2}\Lambda a^2)$, we get

$$\frac{g}{1-gx} = 1 - aX - \Lambda a^2 + a^2X^2 + \mathcal{O}(a^3) = x - \Lambda a^2 + a^2X^2 + \mathcal{O}(a^3), \tag{9.14}$$

and

$$\frac{gx}{1-gx} = 1 - aX - \Lambda a^2 + 2a^2X^2 + \mathcal{O}(a^3). \tag{9.15}$$

Additionally, we get

$$\frac{\partial}{\partial X} G^{(1)}_\Lambda(X, Y; T) = \lim_{a\to 0} a \frac{\partial}{\partial x} G^{(1)}_\lambda(x, y; t)\frac{dx}{dX} = \lim_{a\to 0}(-a^2)\frac{\partial}{\partial x} G^{(1)}_\lambda(x, y; t). \tag{9.16}$$

In a similar manner, we get

$$\frac{\partial}{\partial T} G^{(1)}_\Lambda(X, Y; T) = \lim_{a\to 0} \frac{\partial}{\partial t} G^{(1)}_\lambda(x, y; t), \tag{9.17}$$

and

$$a^{-3}\frac{\partial}{\partial X}\left[W_\Lambda(X)G_\Lambda^{(1)}(X,Y;T)\right]=\lim_{a\to 0}\frac{\partial}{\partial x}\left[w(x,g)G_\lambda^{(1)}(x,y;t)\right]. \tag{9.18}$$

Summing up all the results, we can take the continuum limit of (9.6):

$$\begin{aligned}
&\lim_{a\to 0}G_\lambda^{(1)}(x,y;t)\\
&\quad=\lim_{a\to 0}\left[\frac{gx}{1-gx}G_\lambda^{(1)}\left(\frac{g}{1-gx},y;t-1\right)+2g_s x\frac{\partial}{\partial x}\left\{w(x,g)\frac{gx}{1-gx}G_\lambda^{(1)}\left(\frac{g}{1-gx},y;t-1\right)\right\}\right]\\
&\quad=\lim_{a\to 0}\left[G_\lambda^{(1)}-2aXG_\lambda^{(1)}+\left(X^2-\Lambda\right)a^2\frac{\partial}{\partial x}G_\lambda^{(1)}-\frac{\partial}{\partial t}G_\lambda^{(1)}+2g_s\frac{\partial}{\partial x}\left\{w(x,g)G_\lambda\right\}\right].
\end{aligned}$$

Thus we get

$$\frac{\partial}{\partial T}G_\Lambda^{(1)}(X,Y;T)=-\lim_{a\to 0}\frac{\partial}{\partial X}\left[\left(X^2-\Lambda\right)+2g_s a^{-3}W_\Lambda(X)\right]G_\Lambda^{(1)}(X,Y;T). \tag{9.19}$$

From this expression, we can read off the nontrivial scaling of the coupling constant g_s:

$$\lim_{a\to 0}a^{-3}g_s\equiv G_s. \tag{9.20}$$

Introducing the above G_s, we get the complete form of the differential equation:

$$\frac{\partial}{\partial T}G_\Lambda^{(1)}(X,Y;T)=-\frac{\partial}{\partial X}\left[\left(X^2-\Lambda\right)+2G_sW_\Lambda(X)\right]G_\Lambda^{(1)}(X,Y;T). \tag{9.21}$$

Of course, $G_s\to 0$ limit reproduces the differential equation of the (bare) causal dynamical triangulation.

Next, we try to derive the Hamiltonian of the generalized causal dynamical triangulation by considering the propagator as the transfer matrix:

$$G_\Lambda^{(1)}(L_1,L_2;T)\equiv\langle L_1|e^{-\hat{\mathcal{H}}T}|L_2\rangle. \tag{9.22}$$

where $\langle L_1|$ and $|L_2\rangle$ are the boundary states as to the entrance loop and the exit loop, respectively, and $\hat{\mathcal{H}}$ is the Hamiltonian operator. If we take the shift $T\to T+\Delta T$ and the limit $\Delta T\to 0$, then we get

$$\frac{\partial}{\partial T}G_\Lambda^{(1)}(L_1,L_2;T)=-\langle L_1|\hat{\mathcal{H}}e^{-\hat{\mathcal{H}}T}|L_2\rangle. \tag{9.23}$$

Taking the inverse Laplace transformation of (9.21), we get

$$\begin{aligned}
\frac{\partial}{\partial T}G_\Lambda^{(1)}(L_1,L_2;T)&=\int_{-i\infty}^{i\infty}dX\int_{-i\infty}^{i\infty}dXe^{XL_1}e^{YL_1}L_1\left[\left(X^2-\Lambda\right)+2G_sW_\Lambda(X)\right]G_\Lambda^{(1)}(X,Y;T)\\
&=L_1\left[\left(\frac{\partial^2}{\partial L_1^2}-\Lambda\right)+2G_sW_\Lambda(L_1)\right]G_\Lambda^{(1)}(L_1,L_2;T).
\end{aligned}$$

Thus we find

$$\hat{\mathcal{H}}(L_1)=\hat{\mathcal{H}}_0(L_1)-2G_sL_1\hat{W}_\Lambda(L_1), \tag{9.24}$$

where

$$\hat{\mathcal{H}}_0(L_1)=-L_1\frac{\partial^2}{\partial L_1^2}+L_1\Lambda. \tag{9.25}$$

In fact, this Hamiltonian can be understood via the string field theory of noncritical strings as we will see.

9.2 String field theory

In the generalized causal dynamical triangulation, there exists the annihilation of the spacial boundary loop. If we consider this process seriously, the string field theory of noncritical strings may be a good help. This is based on the two reasons. First, a noncritical string theory can be seen as the 2-dimensional pure gravity, and the $(1+1)$-dimensional pure gravity with the causality is supposed to be realized by the noncritical string theory under a certain condition. Second if the noncritical string theory with a certain condition duplicates the causal dynamical triangulation in the continuum limit, then its generalization, the generalized causal dynamical triangulation, which includes the annihilation of the boundary loop needs the noncritical string theory including the annihilation process of the string, that is the string field theory of noncritical strings. This statement can be checked by comparing the Hamiltonians.

The closed string states in the noncritical string field theory correspond to the boundary loops in the continuum limit of the (generalized) causal dynamical triangulation. Thus at first we would like to define the creation and annihilation operators of the closed string in the following sense:

$$[\Psi(L), \Psi^\dagger(L^{'})] = L\delta(L - L^{'}), \quad \Psi(L)|0\rangle = \langle 0|\Psi^\dagger(L) = 0, \tag{9.26}$$

where $|0\rangle$ is the vacuum state, the creation operator $\Psi^\dagger(L)$ and the annihilation operator $\Psi(L)$ are both dimension-less. The boundary state of the closed string with the length L is defined by

$$|L\rangle \equiv \Psi^\dagger(L)|0\rangle. \tag{9.27}$$

Considering the scaling dimension, we can almost determine the second-quantized Hamiltonian [40]:

$$\begin{aligned}\hat{H}(L) = &\int_0^\infty \frac{dL}{L}\Psi^\dagger(L)K(L)\Psi(L)\\ &- G_s\int_0^\infty dL_1\int_0^\infty dL_2\Psi^\dagger(L_1)\Psi^\dagger(L_2)\Psi(L_1+L_2)\\ &- \alpha G_s\int_0^\infty dL_1\int_0^\infty dL_2\Psi^\dagger(L_1+L_2)\Psi(L_1)\Psi(L_2) - \int_0^\infty \frac{dL}{L}\rho(L)\Psi(L),\end{aligned} \tag{9.28}$$

where $K(L)$ is the kernel, G_s is the string coupling, α is some constant, and $\rho(L)$ is the tadpole term. The tadpole term means the amplitude that the closed string with length L disappears into the vacuum. In the case of $K(L) = 0$, this second-quantized Hamiltonian reproduces the 2-dimensional Euclidean pure gravity up to the over-all factor. Let us restrict the above second-quantized Hamiltonian so as to match it to the one in the generalized causal dynamical triangulation. First, we take the kernel as

$$K(L) = \mathcal{H}_0(L) = -L\frac{\partial^2}{\partial L^2} + L\Lambda. \tag{9.29}$$

Then our universe can not merge but branch in the time evolution so that we need to take $\alpha = 0$. Further one of universes in the branching process must disappear into the vacuum, which leads

$$\Psi^\dagger(L_1)\Psi^\dagger(L_2) \to W_\Lambda(L_1)\Psi^\dagger(L_2) + \Psi^\dagger(L_1)W_\Lambda(L_2), \tag{9.30}$$

where $W_\Lambda(L_1)$ and $W_\Lambda(L_2)$ are the disc amplitudes that the closed strings with the length L_1 and L_2 disappear into the vacuum. By these restrictions, the second-quantized Hamiltonian becomes

$$\hat{H}_{incl} = \int_0^\infty \frac{dL}{L}\Psi^\dagger(L)\mathcal{H}_0(L)\Psi(L) - 2G_s\int_0^\infty dL_1\int_0^\infty dL_2 W_\Lambda(L_1)\Psi^\dagger(L_2)\Psi(L_1+L_2). \tag{9.31}$$

We call this second-quantized Hamiltonian *the inclusive Hamiltonian* (See Figure 27). By the

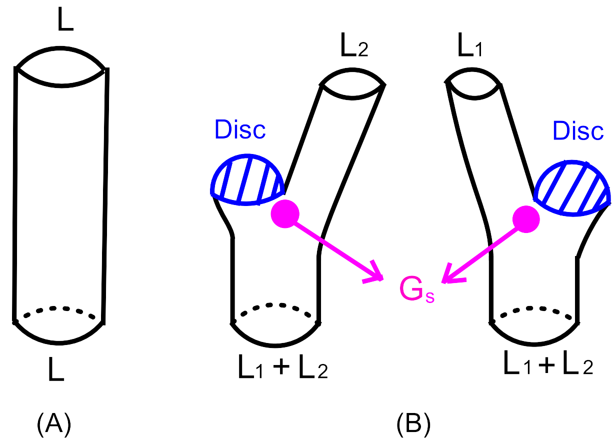


Figure 27: The graphical expression of the inclusive Hamiltonian: (A) The free part, (B) The interaction part

inclusive Hamiltonian, we can read off the first-quantized Hamiltonian:

$$\hat{\mathcal{H}}(L_1) = -L_1 \frac{\partial^2}{\partial L_1^2} + L_1 \Lambda - 2G_s L_1 \hat{W}_\Lambda(L_1). \tag{9.32}$$

This is exactly the same form with the one derived in the context of the generalized causal dynamical triangulation. We can write the propagator in terms of the inclusive Hamiltonian:

$$G_\Lambda^{(1)}(L_1, L_2; T) = \langle 0|\Psi(L_2) e^{-T\hat{H}_{incl}} \Psi^\dagger(L_1)|0\rangle. \tag{9.33}$$

By differentiating the above equation with respect to T and using $\hat{H}_{incl}|0\rangle = 0$, we can reproduce the differential equation (9.21).

10 Discussions

As we have seen, the 2-dimensional gravity has the well discrete-vs.-continuum structure. In the Euclidean case, the dynamical triangulation of the 2-dimensional surface has the matrix model dual. This duality has been checked at the level of the loop equation at least spherical topology. It is known that this duality is also true for the higher-genus case. The continuum limit of those theories is the Liouville field theory with $c = 0$. We have comfirmed this statement by deriving the same string susceptibility, $\gamma_{str} = -\frac{1}{2}$.

In the last half of this part, we have discussed the dynamical triangulation with the causality. In fact, we can also calculate the string susceptibility of the causal dynamical tiangulation via the disc amplitude, and its valus is $\frac{1}{2}$ [29], which is different from the Euclidean case. This implies that the causal dynamical triangulation is really different from the Euclidean theory. The continuum theory of the causal dynamical triangulation is unclear although we have gotten the same propagator with the one in [34]. Further we have defined the generalized causal dynamical triangulation by allowing some kind of baby universes, and derived its Hamiltonian via the string field theory. It is difficult to give the physical meaning to this generalized theory. However we can understand the relation between the causal dynamical triangulation and its generalization to some extent via the stochastic quantization. The introduction of the generalized theory is somewhat handmade. But if we assume the time variable in the generalized theory as the fictitious time in the context of the stochastic quantization, the differential equation (9.21) can be seen as the Fokker-Planck equation [45]. In this line of thought, we can treat the causal dynamical triangulation as the classical limit where no stochastic processe is present. Also in a different point of view, the generalized theory is interesting, that is that we can think of the generalized theory as the grand-unified thoery between the Euclidean dynamical triangulation and the causal dynamical triangulation. This is because in the limit $G_s \to 0$ the differential equation (9.21) becomes the one in the causal dynamical triangulation as we have mentioned, and further in the large-G_s region the differential equation is being the one derived in the Euclidean dynamical triangulation (good derivation can be seen in [20]).

Part III

4 dimensions

11 Causal dynamical triangulation in $3+1$ dimensions

In this section, we review the basic construction of the causal dynamical triangulation in $3+1$ dimensions somewhat briefly [13]. Our goal is understanding the restriction of the asymmetric parameter α, which is the characteristic feature in the dimension higher than 2.

11.1 Regge action

In the lattice regularized gravity, the Einstein-Hilbert action goes over its range of application. Fortunately, the discrete-analog of the Einstein-Hilbert action has been introduced by Regge in 1960 [18]. His great idea still be the powerful tool in the recent lattice gravity. For the derivation of the Regge action, see the apppendix B. In the following, we will construct the Regge action in the $(3+1)$-dimensional causal dynamical triangulation. First, we need to define the building blocks in $3+1$ dimensions. In the Euclidean dynamical triangulation, the building block is the 4-simplex (the pentatope). In the causal dynamical triangulation, the 4-simplex is also the building block but this statement is not exactly true. This is because the causal structure requires the Lorentzian signature in the sense that:

$$l_s^2 = a^2, \quad l_t^2 = -\alpha a^2, \tag{11.1}$$

where l_s and l_t are the lengths of the space-like link and the time-like link, and α stands for the anisotropy between space and time satisfying $\alpha > 0$. Namely there exists two different 4-simplices depending on which links we take as the time-like links (see Figure 28). We call those simplices as

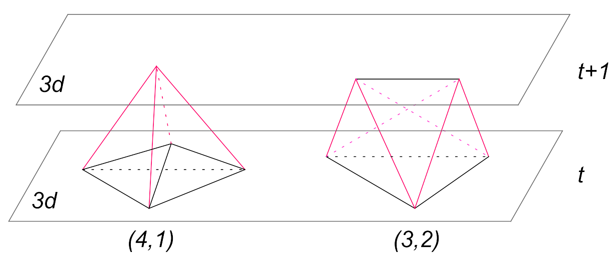


Figure 28: The 4-simplices, $(4,1)$ and $(3,2)$: The red line means the time-like link, and the black one is the space-like link.

$(4,1)$ and $(3,2)$, which is defined by the following requirement:

$$(a, b) \equiv (\#[\text{vertices at time t}], \# \ [\text{vertices at time t+1}]).$$

Of course there also exists $(1,4)$ and $(2,3)$. Let us calculate the volume of $(4,1)$ and $(3,2)$. What we need to pay attention to is that the volume is Lorentzian. As a practice, we calculate the volume

[13] This section is based on [21, 18, 46, 47].

of the lower-dimensional simplices. First, we set that the volume of 0-simplex (node) is equal to zero. The 1-simplices are the time-like link and the space-like link. Those volumes are

$$\mathrm{Vol}(\text{time-like link}) = |i\sqrt{\alpha}| = \sqrt{\alpha}, \quad \mathrm{Vol}(\text{space-like link}) = 1, \tag{11.2}$$

where we set that the lattice spacing of the space-like link is 1 by definition. Noting that we implement the calculation under the assignment that $l_t = i\sqrt{\alpha}$, and then for taking the volume as real and positive we define all the volumes in the absolute values. As for the 2-simplices, we have

$$\mathrm{Vol}(\text{time-like triangle}) = \frac{1}{4}\sqrt{4\alpha+1}, \quad \mathrm{Vol}(\text{space-like triangle}) = \frac{\sqrt{3}}{4}. \tag{11.3}$$

The 3-simplices are $(3,1)$ and $(2,2)$ given by Figure 29. In fact, there is a useful formula to calculate

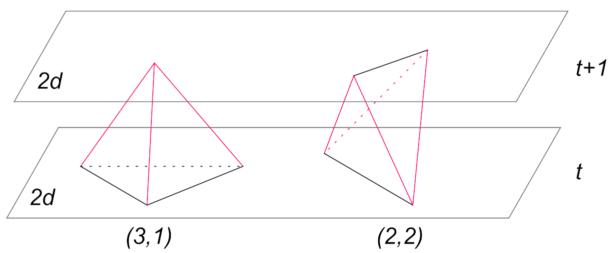


Figure 29: The 3-simplices, $(3,1)$ and $(2,2)$: The red line means the time-like link, and the black one is the space-like link.

the volume of the n-simplex recursively, that is

$$\mathrm{Vol}(\text{n-simplex}) = \frac{1}{n} h_n \mathrm{Vol}(\text{(n-1)-simplex}), \tag{11.4}$$

where h_n is a height of a n-simplex. Thus finding the value h_n, we can calculate the volume of the n-simplex via the volume of the $(n-1)$-simplex. Returning to the discussion, and taking a look at $(3,1)$, we can find the relation:

$$h_2^2 = h_3^2 + s_2^2. \tag{11.5}$$

where $s_2 = \frac{1}{3}h_2$. Taking the Lorentzian signature into account, we can calculate the height of $(3,1)$ as

$$\left(i\sqrt{\alpha+\frac{1}{4}}\right)^2 = h_3^2 + \left(\frac{1}{3}\left[\frac{\sqrt{3}}{2}\right]\right)^2.$$

Thus we find

$$h_3 = \left|i\sqrt{\alpha+\frac{1}{3}}\right| = \sqrt{\alpha+\frac{1}{3}}. \tag{11.6}$$

Noting that h_3 is not the height of $(2,2)$ but $(3,1)$. Using the formula (11.4), we get

$$\mathrm{Vol}(3,1) = \frac{1}{12}\sqrt{3\alpha+1}. \tag{11.7}$$

In a similar manner, we can derive the volume for $(2,2)$. Now we terminate the lower-dimensional calculus because we can find out the height of any simplex by extending the equation (11.5) as

$$h_{n-1}^2 = h_n^2 + s_{n-1}^2, \tag{11.8}$$

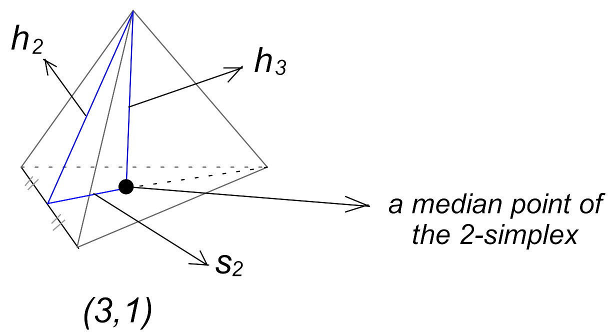


Figure 30: The height-relation of $(3,1)$

where $s_n \equiv \frac{1}{n+1}h_n$. Using the equation (11.8), we can calculate h_4:

$$\left(i\sqrt{\alpha+\frac{1}{3}}\right)^2 = h_4^2 + \left(\frac{1}{6}\sqrt{\frac{3}{2}}\right)^2. \tag{11.9}$$

Thus we get

$$h_4 = \left|i\sqrt{\alpha+\frac{3}{8}}\right| = \sqrt{\alpha+\frac{3}{8}}. \tag{11.10}$$

Consequently, from (11.4) we find

$$\mathrm{Vol}(4,1) = \frac{1}{96}\sqrt{8\alpha+3} \tag{11.11}$$

In a similar manner, it is known that the volume of $(3,2)$ is

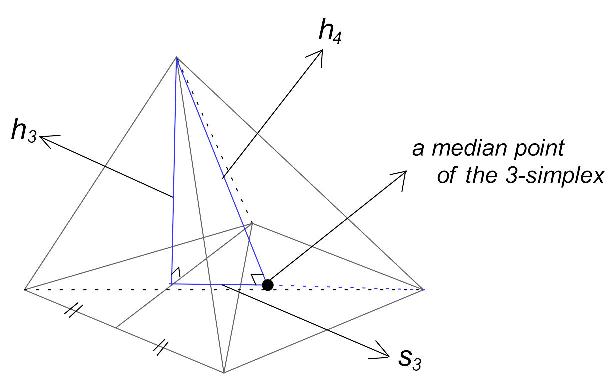


Figure 31: The height-relation of (4,1)

$$\mathrm{Vol}(3,2) = \frac{1}{96}\sqrt{12\alpha+7} \tag{11.12}$$

Summing up all the results, we can write down the Regge action for the causal dynamical triangulation in 3+1 dimensions:

$$S_T = k\sum_{h_s}\left[\frac{\sqrt{3}}{4}\right]\frac{1}{i}(2\pi - \sum_{@h_s}\Theta) + k\sum_{h_t}\left[\frac{1}{4}\sqrt{4\alpha+1}\right](2\pi - \sum_{@h_t}\Theta)$$

$$- \lambda \sum_{(1,4),(4,1)} \left[\frac{1}{96}\sqrt{8\alpha+3} \right] - \lambda \sum_{(2,3),(3,2)} \left[\frac{1}{96}\sqrt{12\alpha+7} \right], \tag{11.13}$$

where $\sum_{h_s}$ and $\sum_{h_t}$ are the summation over space-like hinges (triangles) and time-like hinges respectively, $(2\pi - \sum_{@h_s} \Theta)$ and $(2\pi - \sum_{@h_t} \Theta)$ are the total deficit angles at space-like hinges and time-like hinges each, and k and λ are the gravitational coupling constant and the bulk cosmological constant. Now we would like to consider the rotation from the Lorentzian signature to the Euclidean signature:

$$e^{iS_T} \to e^{-S_T^E}, \tag{11.14}$$

where S_T^E is the Euclideanized action. This can be done by the following shift:

$$\alpha \to -\alpha. \tag{11.15}$$

From $\mathrm{Vol}(4,1)$ and $\mathrm{Vol}(3,2)$, we can notice that α has the lower bound if we implement the above rotation:

$$\alpha > \frac{7}{12}. \tag{11.16}$$

This constraint is originated with the fact that the Euclideanized action must still be real. The restriction of the asymmetry factor α is the characteristic feature in the dimension higher than 2. (for 3 dimensions, $\alpha > \frac{1}{2}$ [46].)

12 Spectral dimension

This section is based on [4, 21, 47, 48, 49, 50, 51, 52].

12.1 Effective dimensions

In general, the notion of the dimension depends on what aspects of the geometry we observe. We call such dimensions defined by some geometrical aspects *effective dimensions*. A good example of effective dimensions is the so-called *Hausdorff dimension* d_h. This is defined by the thickness of an annulus at a geodesic distance R:

$$dV \sim R^{d_h - 1} dR, \tag{12.1}$$

where V denotes the d-dimensional volume.

The other way of defining the effective dimension is known as *the spectral dimension*, which we will focus on in the following discussion. The spectral dimension can be applied to general manifolds such as not only smooth manifolds but also fractals. Thus in this sense, the spectral dimension is a good tool to investigate an a priori unknown geometry. Let us define the spectral dimension of a d-dimensional Euclidean geometry by the dimension which describes a random walk of a trial particle induced by the diffusion equation:

$$\frac{\partial}{\partial \sigma} K_g(\xi, \xi_0, \sigma) = \Delta_g K_g(\xi, \xi_0, \sigma), \tag{12.2}$$

where σ is the diffusion time (which is not the time equipped with the manifold), $K_g(\xi, \xi_0, \sigma)$ is the heat kernel (or the probability density of diffusion from ξ_0 to ξ) with fixed metric g, and Δ_g is the Laplace operator compatible with the metric g. This expresses the diffusion process of a trial particle from $(\xi_0, \sigma = 0)$ to $(\xi, \sigma = \sigma)$. The initial condition of the above equation is

$$K_g(\xi, \xi_0, \sigma = 0) = \frac{\delta^{(d)}(\xi - \xi_0)}{\sqrt{g}}. \tag{12.3}$$

The solution of the equation (12.2) is given by

$$K_g(\xi, \xi_0, \sigma) = \langle \xi | e^{\sigma \Delta_g} | \xi_0 \rangle, \tag{12.4}$$

where $\langle \xi |$ and $| \xi_0 \rangle$ are the final state and the initial state, respectively. For the necessity to define the diffeomorphism-invariant function, we introduce the heat trace (or in other words the average return probability):

$$P_g(\sigma) = \frac{1}{V_g} \int d^d\xi \sqrt{g} K_g(\xi, \xi, \sigma), \tag{12.5}$$

where $V_g = \int d^d\xi \sqrt{g}$ is the space-time volume. This means the average probability density of a particle's diffusion for returning to the starting point ξ in the diffusion time σ. Taking a look at the flat-space example, we would like to derive the spectral dimension. In the flat space ($g = \delta$), the heat kernel is

$$K_\delta(\xi, \xi_0, \sigma) = \frac{e^{-(\xi_0 - \xi)^2/4\sigma}}{(4\pi\sigma)^{d/2}}. \tag{12.6}$$

The corresponding average return probability is

$$P_\delta(\sigma) = \frac{1}{V_\delta} = \frac{1}{\sigma^{d/2}}. \tag{12.7}$$

Thus we can extract the dimension d:

$$-2 \frac{d \log P_\delta(\sigma)}{d \log \sigma} = d. \tag{12.8}$$

In the curved case, the heat trace can be written as the heat-trace expansion:

$$P_g(\sigma) = \frac{1}{\sigma^{d/2}} \sum_{n=0}^{\infty} C_n \sigma^n, \tag{12.9}$$

where C_n is a series of curvature invariants [49, 50]. For example, we have

$$C_0 = \int d^d\xi \sqrt{g}, \tag{12.10}$$

$$C_1 = \frac{1}{6} \int d^d\xi \sqrt{g} R, \tag{12.11}$$

$$C_2 = \frac{1}{360} \int d^d\xi \sqrt{g} (5R^2 - 2R_{ab}R_{ab} + 2R_{abcd}R_{abcd}), \tag{12.12}$$

where R_{abcd}, R_{ab}, and R are the Riemann tensor, the Ricci tensor, and the Ricci scalar respectively. In general, we define the spectral dimension $d_s(\sigma)$ as

$$d_s(\sigma) \equiv -2 \frac{d \log P_g(\sigma)}{d \log \sigma} = d - 2 \frac{\sum_m m C_m \sigma^m}{\sum_n C_n \sigma^n}. \tag{12.13}$$

This means that the spectral dimension has the possibility to deviate from its topological dimension d. Further in the limit $\sigma \to 0$, the spectral dimension is coincident with the topological dimension, that is

$$\lim_{\sigma \to 0} d_s(\sigma) = d. \tag{12.14}$$

This makes us understand the relation between the energy scale and the value of σ. Namely, the small value of σ implies that we probe the short range of the geometry, and on the other hand, by the large value of σ we can know the global information about the geometry. Thus $\sigma \to 0$ corresponds to the high-energy limit, and $\sigma \to \infty$ does the low-energy limit.

12.2 Spectral dimension in causal dynamical triangulation

In $3+1$ dimensions, there exists no analytic method for the causal dynamical triangulation up to January 2010. Thus the investigation is implemented by the computer simulation, the Monte Carlo simulation. As the setting, the Euclideanized action (11.13) is taken, and the topology is chosen as not $R \times S^3$ but $S^1 \times S^3$ for convenience. Furthermore, the space-time volume N (the number of 4-simplices) is fixed in the simulation. Following [51], we define the spectral dimension of the causal dynamical triangulation, and quote the consequence. The partition function is written as

$$Z_N = \sum_{T_N} \frac{1}{C_{T_N}} e^{-S_T}, \tag{12.15}$$

where C_{T_N} is the volume of the discrete symmetry group in the triangulation T_N with fixed N, and S_T is the Euclideanized action of the causal dynamical triangulation. The expectation value of the observable $\mathcal{O}$ is

$$\langle \mathcal{O} \rangle_N = \frac{1}{Z_N} \sum_{T_N} \frac{1}{C_{T_N}} e^{-S_T} \mathcal{O}. \tag{12.16}$$

Labeling the 4-simplex as x, we can write down the discretized diffusion equation:

$$K_T(x, x_0, \sigma+1) = (1-\chi) K_T(x, x_0, \sigma) + \frac{\chi}{4} \sum_{x' \in G(x)} K_T(x', x_0, \sigma), \tag{12.17}$$

where K_T is the discretized heat kernel, χ is the diffusion constant with the range $0 \leq \chi \leq 1$, and $G(x)$ is the set of 4-simplices glued to x. The discrete-analog of the heat trace is

$$P_T(\sigma) = \frac{1}{N} \sum_{x \in T} K_T(x, x, \sigma). \tag{12.18}$$

We define the spectral dimension $D_s(\sigma)$ as the ensemble average of $d_s(\sigma)$ introduced before:

$$D_s(\sigma) \equiv \langle d_s(\sigma) \rangle_N. \tag{12.19}$$

Among the range $\sigma \in [40, 400]$, the data agrees with

$$D_s(\sigma) = a - \frac{b}{\sigma + c}, \tag{12.20}$$

where a, b, and c are free parameters. Superimposing the data, three parameters are determined from the entire range $\sigma \in [40, 400]$:

$$D_s(\sigma) = 4.02 - \frac{119}{54 + \sigma}. \tag{12.21}$$

Considering the various cuts $[\sigma_{\min}, \sigma_{\max}]$ as well as different weights of errors, the asymptotic value can be derived by

$$D_s(\sigma = \infty) = 4.02 \pm 0.1, \tag{12.22}$$

and

$$D_s(\sigma = 0) = 1.80 \pm 0.25. \tag{12.23}$$

These values show that the dynamical dimensional reduction occurs in the causal dynamical triangulation in 4 dimensions. Namely, at the low-energy region $D_s \sim 4$, and at the high-energy region $D_s \sim 2$.

12.3 Spectral dimension in Lifshitz-type gravity

In the Lifshitz-type gravity, there exists the anisotropy between space and time as we have seen. This is expressed by the critical exponent Z in the anisotropic scaling:

$$x^i \to bx^i, \quad t \to b^Z t,$$

where x^i and t denote space and time, respectively, and b is some constant. If $Z \neq 1$, the Lorentz symmetry is broken. Remembering the Lifshitz scalar in d dimensions, the action is given by

$$S = -\frac{1}{2}\int d^d x dt \bigg((\partial_t \phi)^2 - \frac{1}{2}[(\nabla_i)^Z \phi]^2 \bigg), \tag{12.24}$$

where $\nabla_i = \partial_i \partial_i$. The important point here is the derivative with respect to t is independent of the critical exponent Z. Contrary to that, the spacial derivative can vary according to Z. Just the same thing happens in the Lifshitz-type gravity. In this line of thought, we can extend the diffusion equation in d dimensions to the one with the anisotropic scaling [4]:

$$\frac{\partial}{\partial \sigma} K_g(x, x_0, \sigma) = \bigg(\frac{\partial^2}{\partial \tau^2} + (-1)^{Z+1} \Delta_g^Z \bigg) K_g(x, x_0, \sigma) \equiv \tilde{\Delta}_g K_g(x, x_0, \sigma). \tag{12.25}$$

where Δ_g is the spacial Laplace operator $\Delta_g = \partial_i \partial_i$, and its power Z has been determined by the dimensional analysis. In the above equation, we have implemented the Wick rotation, $t = -i\tau$, and set $x = (x_i, \tau)$. The sign $(-1)^{Z+1}$ ensures the ellipticity of the diffusion operator $\tilde{\Delta}_g$. Noting that we have taken the proper-time gauge in the flat space:

$$g_{ij} = \delta_{ij}, \quad N = 1, \quad N_i = 0, \tag{12.26}$$

where N and N_i are the lapse function and the shift vector, respectively. The anisotropic diffusion equation (12.25) can be solved, and the solution is

$$K_g(x, x_0, \sigma) = \int \frac{d\omega d^d k}{(2\pi)^{d+1}} e^{i\omega(\tau - \tau_0) + ik(x^i - x_0^i)} e^{-\sigma(\omega^2 + |k^i|^{2Z})}. \tag{12.27}$$

The corresponding heat trace is

$$P_g(\sigma) = K_g(x, x, \sigma) = \int \frac{d\omega d^d k}{(2\pi)^{d+1}} e^{-\sigma(\omega^2 + |k^i|^{2Z})} = \frac{C}{\sigma^{(1+d/Z)/2}}, \tag{12.28}$$

where C is some constant. Using (12.13), we can derive the spectral dimension of the theory with an anisotropic scaling:

$$d_s = 1 + \frac{d}{Z}. \tag{12.29}$$

Thus in the case of the Z=3 Lifshitz-type gravity, the spectral dimension can be lead for the ultraviolet region:

$$d_s|_{\mathrm{UV}} = 2, \tag{12.30}$$

and for the infrared region:

$$d_s|_{\mathrm{IR}} = 4. \tag{12.31}$$

This result behaves in the similar way as that of the causal dynamical triangulation.

13 Discussions

In 4 dimensions, we have confirmed the similar behaviour of the spectral dimensions in the causal dynamical triangulation and the Lifshitz-type gravity. The spectral dimension is the reliable observable in the lattice simulation so that at least even the asymptotic coincidence is seemingly-meaningful. However we have to say that this coincidence has lots of ambiguities. In the Lifshitz-type gravity, we have calculated the spectral dimension with the proper-time gauge at the classical level, and on the other hand in the causal dynamical triangulation the spacial foliations have not been flat and its spectral dimension has been evaluated by summing over the geometries so that the quantum fluctuations have been taken into account. In [4], P. Horava has pointed out that the spectral dimension in the $Z = 4$ Lifshitz-type gravity might have the more closer value with that of the causal dynamical triangulation, which needs to be reconsidered for the reasons mentioned above.

At this stage, the calculation of the causal dynamical triangulation in $3+1$ dimensions depends only on the numerical simulation. This is because the construction of the analytic method is hard, as we have seen, and its difficulty may be originated with the fact that the 4-dimensional gravity is no more topological.

Part IV
Concluding remarks

Now we discuss the question as mentioned in the openning, "Is the continuum limit of the causal dynamical triangulation the Lifshitz-type gravity?" In the case of the $Z = 3$ Lifshitz-type gravity in $3 + 1$ dimensions and the causal dynamical triangulation in $3 + 1$ dimensions (if we implement the inverse Wick rotation), this seems to be the case from the behaviour of spectral dimensions.

However, we can come up with the different possibility from the lessons in 2 dimensions. In the generalized causal dynamical triangulation in 2 dimensions, the proper "time" can be considered to be the fictitious time in the stochastic quantization. On the other hand, in the Lifshitz-type gravity with the detailed balance condition, the "time" can be also the fictitious time in the stochastic quantization. This implies that it is natural that if we can construct the $(3 + 1)$-dimensional generalized causal dynamical triangulation, its continuum limit may be the Lifshitz-type gravity.

In any case, from the numerical method we can only read off quantitative aspects of theories, which is of course meaningful, though. Thus if we would like to know qualitative aspects of theories, we need to construct the corresponding analytic method. Taking over the virtue in the 2 dimensions, such an analytic method may be the matrix model. On present showing, in the dimension greater than 2, there exists the matrix model of 3-dimensional causal dynamical triangulation [52]. The 4-dimensional candidate does not exist. If we can find out the matrix model corresponding to the causal dynamical triangulation or the generalized theory, it is possible that we can derive lots of qualitative aspects embedded in those, and further we may understand what those continuum limits are. We hope it does the Lifshitz-type gravity. This may lead to the possibility to judge what the true quantum gravity is from lots of candidates. However, the matrix model describing the 4-dimensional causal dynamical triangulation is obviously hard, but it does not mean "impossible.

Acknowledgements

I would like to thank T. Sakai for his hospital guidance and the well-honed indications. And I appreciate H. Fuji for fruiteful comments and lots of discussions. Despite being in a different location, Y. Watabiki has checked my thesis with good grace. I would like to do thank him. I would also like to thank H. Yoda, U. Emori, and K. Sakakibara for discussions.

Part V

Appendices

A ADM formalism

We would like to review the Arnowitt-Deser-Misner, or ADM decomposition of the pure general relativity[14]. The central idea of this decomposition is that by slicing the space-time manifold into the space-like hyper-plane and describing how the sliced hyper-plane propagates along the time direction, we can understand the evolution of the space-time geometry. Another way of saying the former idea, slicing the space-time manifold, is designating how the hyper-plane sits in the space-time manifold, which can be described by the extrinsic curvature. As for the latter one, expressing the way of propagation of the hyper-plane, we can carry it out by using the lapse function and the shift vector. In the following subsections, we will see the above ideas precisely, and then we will construct its Hamiltonian, and investigate the structure of constraints in general relativity.

A.1 Extrinsic curvature

To begin with, we let our space-time manifold be paracompact one denoted by M. In addition to that, we assume M is diffeomorphic to $S \times R$ where S denotes the space, and R stands for the time. We can take the diffeomorphism

$$\phi : M \to S \times R. \tag{A.1}$$

This gives us the time coordinate τ on M, that is that we get τ by the pullback of t on $S \times R$

$$\tau = \phi^* t. \tag{A.2}$$

Especially, we are interested in the 3+1 dimension so that we let M be a space-time manifold in four dimensions, and let Σ_t be the slice of $S \times R$ which is the spacial hyper-plane equipped with the 3-metric, $g^{(3)}$ defined by

$$g(v, w) = g^{(3)}_{ij} v^i w^j, \qquad (i = 1, 2, 3), \tag{A.3}$$

where vectors v and w sit in $T_p\Sigma_t$, the tangent space of Σ_t at $p \in M$. We also restrict the 3-metric to Riemannian one as usual in a following sense:

$$g(v, v) > 0, \tag{A.4}$$

for all nonzero $v \in T_p\Sigma_t$. In this line of the discussion, we can introduce the timelike unite vector, n, normal to Σ_t:

$$g(n, n) = -1, \qquad g(v, n) = 0, \tag{A.5}$$

for any $v \in T_p\Sigma_t$. Now we can think of the decomposition of vectors in T_pM into the normal and tangential direction of Σ_t:

$$v = -g(v, n)n + (v + g(v, n)n). \tag{A.6}$$

For graphical expression, see Figure 32.

[14] This appendix is based on [53, 54, 55, 56]

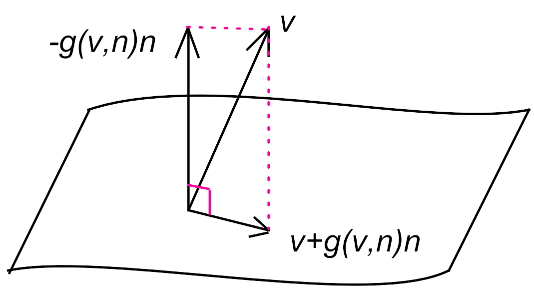


Figure 32: The decomposition of the vector $v \in T_pM$

We are ready to construct the very meaningful tool, the extrinsic curvature. Given any vectors v, u in Σ_t, it can be derived by projecting the covariant derivative of v in the direction u onto the normal and tangential direction of Σ_t:

$$\nabla_u v = -g(\nabla_u v, n)n + \Big[\nabla_u v + g(\nabla_u v, n)n\Big]. \tag{A.7}$$

We define the first term as

$$-g(\nabla_u v, n)n \equiv K(u, v)n, \tag{A.8}$$

where $K(u, v)$ is just *the extrinsic curvature*. It tells us how the tangent vector v goes off the tangent by parallel translation. In other words, we can see the way of embedding of Σ_t in M by the extrinsic curvature. As for the second term, we can write as follows:

$$\nabla^{(3)}_u v = \nabla_u v + g(\nabla_u v, n)n, \tag{A.9}$$

where $\nabla^{(3)}$ is the Levi-Civita connection on Σ_t associated to the metric $g^{(3)}$. We show that this definition of $\nabla^{(3)}$ becomes really a connection by checking the nontrivial three properties: the Leibnitz law, the metric preserving, and the torsion free. At first, we think about the Leibnitz law,

$$\nabla^{(3)}_v (fw) = v(f)w + f\nabla^{(3)}_v w, \qquad (\forall v, w \in T_p\Sigma_t, \quad f \in C^\infty), \tag{A.10}$$

This can be checked by using the definition of $\nabla^{(3)}$, $(A.9)$, and its propaty, $g^{(3)}(v, n) = 0$. Secondly, what we need to check is the metric preserving

$$ug(v, w) = g(\nabla^{(3)}_u v, w) + g(v, \nabla^{(3)}_u w), \tag{A.11}$$

where $\forall u, v, w \in T_p\Sigma_t$. This can be shown similarly by using the definition of $\nabla^{(3)}$ and $g(v, n) = g(w, n) = 0$. The final task to check is the torsion free:

$$\begin{aligned} \nabla^{(3)}_u v - \nabla^{(3)}_v u &= \nabla_u v - \nabla_v u \\ &= [u, v], \end{aligned} \qquad \text{(A.12)} \quad \text{(A.13)}$$

where we have used the two facts that ∇ is torsion free, and the extrinsic curvature is the symmetric which should be checked, though. Thus we understand that the $\nabla^{(3)}$ is really the Levi-Civita connection on Σ_t associated to the metric $g^{(3)}$. Thus our final task to be checked is the symmetry of the extrinsic curvature. If the extrinsic curvature eats the local coordinates, then we get

$$K(\partial_i, \partial_j) = K_{ij}. \tag{A.14}$$

By using this expression, we can get

$$\begin{aligned} K_{ij} - K_{ji} &= K(\partial_i, \partial_j) - K(\partial_j, \partial_i) \\ &= -g([\partial_i, \partial_j], n) \\ &= 0. \end{aligned} \tag{A.15}$$

This means that our discussion is consistent. In addition, it is worthwhile to rewrite the extrinsic curvature in the following form:

$$g(\nabla_u n, v) = K(u, v). \tag{A.16}$$

This expression can be derived by the metric preserving of ∇. In this expression we can understand the extrinsic curvature in a different point of view, that is that $K(u, v)$ measures how the normal vector, n, rotates in the v-direction by parallel translating in the direction u.

A.2 Lapse function and shift vector

In this subsection, we would like to see how the hyper-plane can flow along the time direction. To achieve this, it is good for us to think of the time-like vector ∂_τ on M which can be derived by the push forward of ∂_t on $\Sigma \times R$:

$$\partial_\tau = \phi^{-1}\partial_t = -g(\partial_\tau, n)n + (\partial_\tau + g(\partial_\tau, n)n). \tag{A.17}$$

We denote the above expression as follows:

$$\partial_\tau = Nn + \vec{N}, \tag{A.18}$$

where N is the lapse function, and $\vec{N}$ is the shift vector. By using this, we get the normal vector:

$$n = \frac{1}{N}(\partial_\tau - \vec{N}). \tag{A.19}$$

Before getting the explicit Hamiltonian form, we can get some constraints on the general relativity

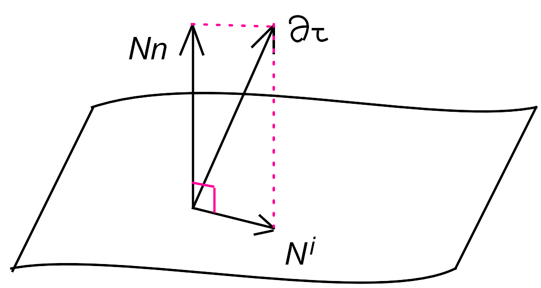


Figure 33: The ADM decomposition

in the ADM decomposition. In the following discussions, we will see the constraint equations, called *the Gauss-Codazzi equations*.

A.3 Gauss-Codazzi equations

Einstein equation has 10 independent equations, and in ADM formalism it is well known that 4 of 10 are constraints and 6 of 10 are the evolutionary equations. In this section, we concentrate on the constraints, and get them all. For simplicity, at first we take the local coordinates on Σ_0 such that we take x^0 and x^i (i denotes the spacial direction) as $x^0 = \tau$, $\partial_0 = \partial_\tau$, and ∂_i's are tangent to Σ_0. In terms of these coordinates, we get the constraints, and then we extend them to the general forms. To accomplish our work, we need to calculate the following quantity:

$$R(\partial_i, \partial_j)\partial_k = \nabla_i \nabla_j \partial_k - \nabla_j \nabla_i \partial_k, \tag{A.20}$$

We can conduct this calculation just by using two relations. The first one is

$$\begin{aligned} \nabla_i^{(3)} \partial_j &= K_{ij} n + \nabla_i^{(3)} \partial_j \\ &= K_{ij} n + \Gamma_{ij}^{m(3)} \partial_m. \end{aligned} \tag{A.21}$$

The second one is

$$\nabla_i n = K_i^m \partial_m. \tag{A.22}$$

This can be derived as follows:

$$\begin{aligned} K_{ij} &= g(\nabla_i n, \partial_j) \\ &= g((\nabla_i n)^k \partial_k, \partial_j) \\ &= (\nabla_i n)_j. \end{aligned}$$

By using the two, we get

$$\begin{aligned} \nabla_i \nabla_j \partial_k &= \nabla_i (K_{jk} n + \Gamma_{jk}^{m(3)} \partial_m) \\ &= (\partial_i K_{jk} + \Gamma_{jk}^{m(3)} K_{im}) n + K_{jk} K_i^m \partial_m + (\partial_i \Gamma_{jk}^{m(3)} + \Gamma_{jk}^{l(3)} \Gamma_{il}^{m(3)}) \partial_m. \end{aligned}$$

And then interchanging the indices i and j, we can derive

$$R(\partial_i, \partial_j)\partial_k = (\nabla_i^{(3)} K_{jk} - \nabla_j^{(3)} K_{ik}) n + (R_{ijk}^{m(3)} + K_{jk} K_i^m - K_{ik} K_j^m) \partial_m. \tag{A.23}$$

The equations above are *Gauss-Codazzi equations*. To see the role of these equations, we impose a little more restriction, that is $\partial_0 = \partial_\tau = n$. In this situation, we act the 1-form dx^0 on the Gauss-Codazzi equations:

$$\begin{aligned} (\text{LHS}) &= R_{ijk}^{\mu} dx^0(\partial_\mu) \\ &= R_{ijk}^{\mu} \delta_\mu^0 \\ &= R_{ijk}^0, \end{aligned}$$

where the Greek letters range from 0 to 3.

$$\begin{aligned} (\text{RHS}) &= (\nabla_i^{(3)} K_{jk} - \nabla_j^{(3)} K_{ik}) dx^0(n) + (R_{ijk}^{m(3)} + K_{jk} K_m^i - K_{ik} K_m^j) x^0(\partial_m) \\ &= \nabla_i^{(3)} K_{jk} - \nabla_j^{(3)} K_{ik}. \end{aligned}$$

Comparing the two sides, we get

$$R^0_{ijk} = \nabla^{(3)}_i K_{jk} - \nabla^{(3)}_j K_{ik}. \tag{A.24}$$

This is called *Gauss equation*. Similarly, if we let the 1-form dx^m eat the Gauss-Codazzi equation, then we get

$$R^m_{ijk} = R^{m(3)}_{ijk} + K_{jk}K^m_i - K_{ik}K^m_j. \tag{A.25}$$

This is called *Codazzi equation*. We find the fact that the intrinsic curvature $R^{m(3)}_{ijk}$ is equivalent to the spacial component of the Riemann curvature, R^m_{ijk}, when the extrinsic curvature vanishes.

Now let us calculate the 4 of Einstein equations. Our starting equation is $G^\mu_\nu = R^{\mu\alpha}{}_{\nu\alpha} - \frac{1}{2}\delta^\mu_\nu R^{\alpha\beta}{}_{\alpha\beta}$. We put the zero in the two indices of G^μ_ν, and we get

$$\begin{aligned}
G^0_0 &= \frac{1}{2}(R^{0\alpha}{}_{0\alpha} - R^{1\alpha}{}_{1\alpha} - R^{2\alpha}{}_{2\alpha} - R^{3\alpha}{}_{3\alpha}) \\
&= -R^{12}{}_{12} - R^{23}{}_{23} - R^{31}{}_{31}.
\end{aligned}$$

Then we apply the Codazzi equation

$$\begin{aligned}
G^0_0 &= -(R^{(3)12}{}_{12} + R^{(3)23}{}_{23} + R^{(3)31}{}_{31}) - (K^1_2K^2_1 - K^2_2K^1_1) - (K^2_3K^3_2 - K^2_2K^3_3) - (K^3_1K^1_3 - K^1_1K^3_3) \\
&= -\frac{1}{2}(R^{(3)} + (\mathrm{tr}K)^2 - \mathrm{tr}(K^2)).
\end{aligned} \tag{A.26}$$

This equation shows that the Einstein equation $G^0_0 = 8\pi\kappa T^0_0$ means the constraint equation given above. The rest of 3 constraints are derived by G^0_i. If we take G^0_1, then we have

$$\begin{aligned}
G^0_1 &= R^{(3)0}{}_{21}{}^2 + R^{(3)0}{}_{31}{}^3 \\
&= \nabla^{(3)}_j K^j_1 - \nabla^{(3)}_1 K^j_j.
\end{aligned}$$

Similarly, we also get the rest directions, 1 and 2, that is

$$G^0_i = \nabla^{(3)}_j K^j_i - \nabla^{(3)}_i K^j_j. \tag{A.27}$$

This equation shows that the Einstein equations $G^0_i = 8\pi\kappa T^0_i$ are constraints.

Till now we restricted our discussion as $\partial_0 = n$. However we can generalize the above derivation of constraints. Although we skip the details of calculation, it is well know that the general forms of the constraints are given by

$$G_{\mu\nu}n^\mu n^\nu = \frac{1}{2}(R^{(3)} + (\mathrm{tr}K)^2 - \mathrm{tr}(K^2)), \tag{A.28}$$

and

$$G_{\mu i}n^\mu = \nabla^{(3)}_j K^j_i - \nabla^{(3)}_i K^j_j. \tag{A.29}$$

The remaining 6 equations $G_{ij} = 8\pi\kappa T_{ij}$ are dynamical equations which describe the time evolution of $g^{(3)}$.

A.4 Hamiltonian formalism

We have already had enough tools to write down the Hamiltonian. In this subsection, we see that the Hamiltoniam is given by the lapse function and the shift vector. It is very natural because the Hamiltonian itself is just the generator of the time translation.

For preparation, we need to rewrite the Einstein-Hilbert action in terms of the extrinsic curvature, the lapse function, and the shift vector. Again for simplicity, we choose the local coordinate as $x^0 = \tau$, $\partial_0 = \partial_\tau$, and $\partial_i^{'}$s are tangent to Σ_0. At first, we would like to rewrite the extrinsic curvature in the following expression:

$$K_{ij} = \frac{1}{2}N^{-1}(\dot{q_{ij}} - \nabla_i^{(3)}N_j - \nabla_j^{(3)}N_i), \tag{A.30}$$

where $q_{ij} \equiv g_{ij}^{(3)}$, and the dot denotes the time derivative. By using the two relations

$$K_{ij} = -g(\nabla_i \partial_j, n),$$

and

$$n = \frac{1}{N}(\partial_0 - N^i \partial_i),$$

K_{ij} is computed explicitly as follows:

$$\begin{aligned}
K_{ij} &= -g(\nabla_i \partial_j, n) \\
&= -\frac{1}{N}\Big[g(\nabla_i \partial_j, [\partial_0 - N^m \partial_m])\Big] \\
&= \frac{1}{N}\Big[g(\partial_j, \nabla_i[\partial_0 - N^m \partial_m])\Big] \qquad (\because g(n, \partial_i) = 0) \\
&= \frac{1}{2N}\Big[g(\partial_j, \nabla_i[\partial_0 - N^m \partial_m] + g(\partial_i, \nabla_j[\partial_0 - N^m \partial_m])\Big] \\
&= \frac{1}{2N}\Big[\dot{q_{ij}} - \partial_i[q_{jm}N^m] + \Gamma_{ij}^{m(3)}N_m - \partial_j[q_{im}N^m] + \Gamma_{ji}^{m(3)}N_m\Big] \\
&= \frac{1}{2}N^{-1}(\dot{q_{ij}} - \nabla_i^{(3)}N_j - \nabla_j^{(3)}N_i).
\end{aligned}$$

Next, we try to write the 4-spacetime metric expricitely for our later discussions. Those components are

$$\begin{aligned}
g(\partial_0, \partial_0) &= g(Nn + N^m \partial_m, Nn + N^l \partial_l) \\
&= -N^2 + N^m N_m,
\end{aligned}$$

$$\begin{aligned}
g(\partial_0, \partial_i) &= g(Nn + N^m \partial_m, \partial_i) \\
&= N_i,
\end{aligned}$$

and

$$\begin{aligned}
g(\partial_i, \partial_j) &= g_{ij}^{(3)} \\
&= q_{ij}.
\end{aligned}$$

By combining the above, we get

$$g_{\mu\nu} = \begin{pmatrix} -N^2 + N^m N_m & N_i \\ N_i & q_{ij} \end{pmatrix}. \tag{A.31}$$

We can get the volume element $\sqrt{-g}$, s.t. g is the determinant of the 4-metric, by using the above matrix and the identity:

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(D)\det(A - BD^{-1}C), \tag{A.32}$$

where A, B, C, and D are the invertible matrices. The volume element can be rewritten like:

$$\sqrt{-g} = N\sqrt{q}. \tag{A.33}$$

To rewrite the Einstein-Hilbert action, we need two more tools. The first one is

$$K(\partial_\nu, \partial_\mu) = \nabla_\nu n_\mu. \tag{A.34}$$

This can be checked as:

$$\begin{aligned}
(\text{LHS}) &= g(\nabla_\nu n, \partial_\mu) \\
&= \partial_\nu(g(n, \partial_\mu)) - g(n, \nabla_\nu \partial_\mu) \\
&= \partial_\nu(n^\lambda g_{\lambda\mu}) - g(n^\lambda \partial_\lambda, \Gamma^\tau_{\nu\mu}\partial_\tau) \\
&= \nabla_\nu n_\mu \\
&= (\text{RHS}).
\end{aligned}$$

The next one is

$$R = 2(G_{\mu\nu}n^\mu n^\nu - R_{\mu\nu}n^\mu n^\nu), \tag{A.35}$$

where $G_{\mu\nu}$, $R_{\mu\nu}$, R shows the Einstein tensor, the Ricci tensor, and the Ricci scalar, respectively. This is obviously trivial from the definition. Thus we get

$$\begin{aligned}
R_{\mu\nu}n^\mu n^\nu &= R^\lambda_{\mu\lambda\nu}n^\mu n^\nu \\
&= -n^\mu(\nabla_\mu\nabla_\lambda - \nabla_\lambda\nabla_\mu)n^\lambda \\
&= (\text{tr}K)^2 - \text{tr}K^2 - \nabla_\mu(n^\mu\nabla_\lambda n^\lambda) + \nabla_\lambda(n^\mu\nabla_\mu n^\lambda) \\
&= (\text{tr}K)^2 - \text{tr}K^2 - 2\nabla_\mu(n^\mu trK) + \nabla_\mu\nabla_\lambda(n^\mu n^\lambda).
\end{aligned}$$

By above results and eq. $(A.28)$, we find

$$\begin{aligned}
R &= 2(G_{\mu\nu}n^\mu n^\nu - R_{\mu\nu}n^\mu n^\nu) \\
&= R^{(3)} + \text{tr}K^2 - (\text{tr}K)^2 + (\text{surface terms}).
\end{aligned} \tag{A.36}$$

Now we can write down the Einstein-Hilbert action, or its Lagrangian density explicitly:

$$\begin{aligned}
\mathcal{L}_{EH} &= \sqrt{-g}R \\
&= N\sqrt{q}(R^{(3)} + \text{tr}K^2 - (\text{tr}K)^2) + (\text{surface terms}).
\end{aligned} \tag{A.37}$$

Finally, we construct the Hamiltonian. As usual, we start with the momentum, p^{ij}, defined by

$$p^{ij} \equiv \frac{\partial \mathcal{L}_{EH}}{\partial \dot{q_{ij}}} = \sqrt{q}(K^{ij} - q^{ij}\mathrm{tr}K). \tag{A.38}$$

This is because

$$\mathrm{tr}(K^2) = \frac{1}{2N}(\dot{q}_{ij} - \nabla^{(3)}_i N_j - \nabla^{(3)}_j N_i)K^{ij},$$

and

$$(\mathrm{tr}K)^2 = \frac{1}{2N}(q^{ij}\dot{q}_{ij} - \nabla^{(3)}_i N^i - \nabla^{(3)}_i N^i)(\mathrm{tr}K).$$

The Hamiltonian density can be derived by Legendre transformation:

$$\mathcal{H}(p^{ij}, q_{ij}) = p_{ij}\dot{q^{ij}} - \mathcal{L}_{EH}. \tag{A.39}$$

The calculation is straightforward, and by using the relation, $\mathrm{tr}(K) = -\frac{1}{2\sqrt{q}}\mathrm{tr}(p)$, which is checked easily, we get

$$\mathcal{H} = \sqrt{q}(NC + N^i C_i), \tag{A.40}$$

where

$$C \equiv -R^{(3)} + q^{-1}\left(\mathrm{tr}(p^2) - \frac{1}{2}\mathrm{tr}(p)^2\right), \tag{A.41}$$

and

$$C_i \equiv -2\nabla^{(3)j}\left(q^{-\frac{1}{2}}p_{ij}\right). \tag{A.42}$$

The above expression shows that the Hamiltonian is really proportional to the lapse function and the shift vector. Additionally, we can find the surprising result, that is that C and C_i are the constraints derived in the previous subsection. We introduce this fact without showing as follows:

$$C = -2G_{\mu\nu}n^\mu n^\nu, \tag{A.43}$$

and

$$C_i = -2G_{\mu i}n^\mu. \tag{A.44}$$

This is the puzzling result that if we have the vacuum Einstein equation, then the corresponding Hamiltonian is equal to zero! In the next subsection, we would like to show C and C_i are the first class constraints that are expected to be the generators of diffeomorphism.

A.5 Constraint algebra

We would like to see the structure of constraints in the context of ADM decomposition of the general relativity in this subsection. What we need to do at first is setting the Poisson brackets:

$$\{q_{ij}(x), p^{kl}(y)\}_{PB} = \frac{1}{2}(\delta^k_i\delta^l_j + \delta^k_j\delta^l_i)\delta^{(3)}(x-y), \tag{A.45}$$

$$\{N(x), p_N(y)\}_{PB} = \delta^{(3)}(x-y), \tag{A.46}$$

$$\{N_i(x), p^j_{\vec{N}}(y)\}_{PB} = \delta^j_i\delta^{(3)}(x-y). \tag{A.47}$$

where

$$p_N \equiv \frac{\partial \mathcal{L}_{EH}}{\partial \dot{N}}, \qquad p^i_{\vec{N}} \equiv \frac{\partial \mathcal{L}_{EH}}{\partial \dot{N}_i}. \tag{A.48}$$

In this situation, to derive constraints, we need to remember the Lagrangian density

$$\mathcal{L}_{EH} = N\sqrt{q}(R^{(3)} + \mathrm{tr}K^2 - (\mathrm{tr}K)^2),$$

up to total derivatives. We notice that both momenta conjugate to the lapse function and the shift vector are the primary constraints, those are

$$p_N = \frac{\partial \mathcal{L}_{EH}}{\partial \dot{N}} = 0, \tag{A.49}$$

and

$$p^i_{\vec{N}} = \frac{\partial \mathcal{L}_{EH}}{\partial \dot{N}_i} = 0. \tag{A.50}$$

Those are true for some fixed time slice. We can understand the above two because the Lagrangian has no time derivative as to the lapse function and the shift vector. We also need to check the consistency of the above constraints with time evolution, which can be done by calculating the Poisson brackets with Hamiltonian:

$$\frac{d}{dt}p_N(x) = \{p_N(x), H\}_{PB} = -\sqrt{q}C(x), \tag{A.51}$$

and

$$\frac{d}{dt}p^i_{\vec{N}}(x) = \{p^i_{\vec{N}}(x), H\}_{PB} = -\sqrt{q}C^i(x), \tag{A.52}$$

where

$$H \equiv \int d^3x \mathcal{H}(x) = \int dx^3 \sqrt{q}(N(x)C(x) + N^i(x)C_i(x)). \tag{A.53}$$

For the consistency with time evolution, we should impose the following weak equalities:

$$C(x) \approx 0, \qquad C^i(x) \approx 0. \tag{A.54}$$

These are secondary constraints. Next, we need to calculate the Poisson brackets among these constraints and Hamiltonian because we have to impose the consistency of secondary constraints under the time evolution. For simplicity, we introduce the constraint functions:

$$C(N) \equiv \int d^3x \sqrt{q} N(x) C(x), \tag{A.55}$$

and

$$C(\vec{N}) \equiv \int d^3x \sqrt{q} N^i(x) C_i(x), \tag{A.56}$$

where

$$\begin{aligned}
C(\vec{N}) &= -2\int d^3x N_i(x)\nabla_j(p^{ij}) \\
&= -\int d^3x (N_i(x)\nabla_j + N_j(x)\nabla_i)(p^{ij}) \equiv -\int dx^3 \mathcal{L}_{\vec{N}}(p^{ij}) q_{ij} && \text{(A.57)} \\
&= \int d^3x (\nabla_j N_i(x) + \nabla_i N_j(x))(p^{ij}) \equiv \int dx^3 \mathcal{L}_{\vec{N}}(q_{ij}) p^{ij}. && \text{(A.58)}
\end{aligned}$$

$\mathcal{L}_{\vec{N}}$ denotes the Lie derivative along the N^i direction. Equivalently we can write the above equations as follows:

$$\frac{\delta C(\vec{N})}{\delta q_{ij}} = -\mathcal{L}_{\vec{N}} p^{ij}, \tag{A.59}$$

$$\frac{\delta C(\vec{N})}{\delta p^{ij}} = \mathcal{L}_{\vec{N}} q_{ij}. \tag{A.60}$$

At this point, it is worthwhile that $C(\vec{N})$ is just the generator of the spacial translation along the hyper-plane, which is same with the gauge theory. We can easily check this by calculating the Poisson brackets:

$$\{q_{ij}, C(\vec{N})\}_{PB} = \mathcal{L}_{\vec{N}} q_{ij}, \tag{A.61}$$

$$\{p^{ij}, C(\vec{N})\}_{PB} = \mathcal{L}_{\vec{N}} p^{ij}. \tag{A.62}$$

The next one that we would like to know is about $C(N)$:

$$C(N) = \int d^3x N \left[-\sqrt{q}R^{(3)} + q^{-1/2}\left(p^{ij}p_{ij} - \frac{1}{2}p^2\right)\right],$$

where $p = p^i_i$. Let us consider the variation of $C(N)$ as to the 3-metric:

$$\begin{aligned}\delta_q C(N) = \int d^3x \delta q_{ij} N \Bigg[& q^{1/2}\left(R^{(3)ij} - \frac{1}{2}q^{ij}R^{(3)}\right) - \frac{1}{2}q^{-1/2}q^{ij}\left(p^{kl}p_{kl} - \frac{1}{2}p^2\right) \\ & + 2q^{-1/2}\left(p^{ik}p^j_k - \frac{1}{2}pp^{ij}\right)\Bigg] + \int d^3x N \left[-\sqrt{q}q^{ij}\delta R^{(3)}{}_{ij}\right].\end{aligned}$$

As for the last term, we can calculate by using

$$v^i \equiv -\nabla^i(q^{kl}\delta q_{kl}) + \nabla^k(q^{il}\delta q_{kl}), \tag{A.63}$$

That is

$$\begin{aligned}\int d^3x N\left[-\sqrt{q}q^{ij}\delta R^{(3)}{}_{ij}\right] &= \int dx^3 N\left[-\sqrt{q}\nabla_i v^i\right] \\ &= \int d^3x (\nabla_i N)\sqrt{q}v^i \\ &= \int d^3x \left[(\nabla^k\nabla_k N)\sqrt{q}q^{ij} - (\nabla^j\nabla_l N)\sqrt{q}q^{li}\right]\delta q_{ij}.\end{aligned}$$

Thus we get

$$\begin{aligned}\delta_q C(N) = \int d^3x \delta q_{ij} \Bigg[& N q^{1/2}\left(R^{(3)ij} - \frac{1}{2}q^{ij}R^{(3)}\right) - \frac{1}{2}Nq^{-1/2}q^{ij}\left(p^{kl}p_{kl} - \frac{1}{2}p^2\right) \\ & + 2q^{-1/2}\left(p^{ik}p_k{}^j - \frac{1}{2}pp^{ij}\right) - \sqrt{q}(\nabla^i\nabla^j N - q^{ij}\nabla^k\nabla_k N)\Bigg].\end{aligned} \tag{A.64}$$

Alternatively we have:

$$\frac{\delta C(N)}{\delta q_{ij}} = Nq^{1/2}\left(R^{(3)ij} - \frac{1}{2}q^{ij}R^{(3)}\right) - \frac{1}{2}Nq^{-1/2}q^{ij}\left(p^{kl}p_{kl} - \frac{1}{2}p^2\right)$$

$$+ 2q^{-1/2}\left(p^{ik}p_k{}^j - \frac{1}{2}pp^{ij}\right) - \sqrt{q}(\nabla^i\nabla^j N - q^{ij}\nabla^k\nabla_k N). \tag{A.65}$$

Similarly we can derive the variation as to p^{ij} easily, and the consequence is

$$\begin{aligned}\frac{\delta C(N)}{\delta p^{ij}} &= 2Nq^{-1/2}\left(p_{ij} - \frac{1}{2}q_{ij}p\right)\\ &\equiv \mathcal{L}_{\vec{t}}q_{ij} - (\nabla_i N_j + \nabla_j N_i),\end{aligned} \tag{A.66}$$

where $\mathcal{L}_{\vec{t}}$ denotes the Lie derivative along the time direction ($\mathcal{L}_{\vec{t}}q_{ij} = \dot{q_{ij}}$). If we take the gauge, $N^i = 0$, then we get

$$\{q_{ij}, C(N)\}_{PB} = \mathcal{L}_{\vec{t}}q_{ij}. \tag{A.67}$$

By using the secondary constraint, $C(x) \approx 0$, we get

$$\{p_{ij}, C(N)\}_{PB} \approx \mathcal{L}_{\vec{t}}p_{ij}. \tag{A.68}$$

We notice that $C(N)$ becomes the generator of the time translation only in the gauge, $N^i = 0$, which means that the time is perpendicular to the hyper-surface.

For the evaluation of the constraint algebra, it is good for us to introduce the real-valued function:

$$f(M) \equiv \int d^3x M^{a\cdots b}{}_{c\cdots d}\tilde{f}^{c\cdots d}_{a\cdots b}(q,p), \tag{A.69}$$

where $M^{a\cdots b}{}_{c\cdots d}$ is a tensor field on the hyper-surface, and independent of q^{ij} and p_{ij}. Then we consider about the following bracket:

$$\begin{aligned}\{C(\vec{N}), f(M)\}_{PB} &= \int d^3x\left[-\mathcal{L}_{\vec{N}}p^{ab}\left(\frac{\delta f(M)}{\delta p^{ab}}\right) - \mathcal{L}_{\vec{N}}q_{ab}\left(\frac{\delta f(M)}{\delta q^{ab}}\right)\right]\\ &= -\int d^3x M^{a\cdots b}{}_{c\cdots d}\mathcal{L}_{\vec{N}}\tilde{f}^{c\cdots d}_{a\cdots b}(q,p)\\ &= -\mathcal{L}_{\vec{N}}\left[\int dx^3 M^{a\cdots b}{}_{c\cdots d}\tilde{f}^{c\cdots d}_{a\cdots b}(q,p)\right] + \int d^3x \mathcal{L}_{\vec{N}}(M^{a\cdots b}{}_{c\cdots d})\tilde{f}^{c\cdots d}_{a\cdots b}(q,p)\\ &= \int d^3x \mathcal{L}_{\vec{N}}(M^{a\cdots b}{}_{c\cdots d})\tilde{f}^{c\cdots d}_{a\cdots b}(q,p)\\ &= f(\mathcal{L}_{\vec{N}}M).\end{aligned} \tag{A.70}$$

By using the above expression, we get

$$\{C(\vec{N}), C(M)\}_{PB} = C(\mathcal{L}_{\vec{N}}M), \tag{A.71}$$

$$\{C(\vec{N}), C(\vec{M})\}_{PB} = C(\mathcal{L}_{\vec{N}}\vec{M}) = C([\vec{N},\vec{M}]). \tag{A.72}$$

Let us calculate the last Poisson bracket:

$$\begin{aligned}\{C(N), C(M)\}_{PB} &= \int d^3x\left[-2M(\nabla^a\nabla^b N - q^{ab}\nabla^c\nabla_c N)\left(p_{ab} - \frac{1}{2}q_{ab}p\right) - (N \leftrightarrow M)\right]\\ &= \int d^3x\left[2M(-p\nabla^c\nabla_c N - \nabla^a\nabla^b N)p_{ab} + (\nabla^c\nabla_c N)p - (N \leftrightarrow M)\right]\\ &= \int d^3x\left[-2M(\nabla^a\nabla^b N)p_{ab} + 2N(\nabla^a\nabla^b M)p_{ab}\right]\end{aligned}$$

$$= -2\int d^3x\left[(N\partial^b M - M\partial^b N)q_{bc}\nabla_a p^{ac}\right]$$
$$= C(N\partial^b M - M\partial^b N). \tag{A.73}$$

At last, we can calculate the consistency of secondary constraints in terms of the constraint algebra, those are

$$\frac{d}{dt}C(N) = \{C(N), H\}_{PB} \approx 0, \tag{A.74}$$

$$\frac{d}{dt}C(\vec{N}) = \{C(\vec{N}), H\}_{PB} \approx 0, \tag{A.75}$$

This means that there is no extra constraint. We should pay attention to the Poisson bracket, $\{C(N), C(M)\}_{PB}$, because it depends on the field, q_{ij}. Namely, the Poisson bracket among $C(N)'$s is not the Lie algebra in the exact sense. Contrary to that, other two, $\{C(\vec{N}), C(M)\}_{PB}$ and $\{C(\vec{N}), C(\vec{M})\}_{PB}$, are the Lie algebras precisely. Such a field dependence of the bracket causes the notorious problems when we quantize the theory. In addition, it is worthwhile to mention that secondary constraints, $C(x) \approx 0$ and $C^i(x) \approx 0$ are both local. Namely, these constraints should be imposed at each time and space.

B Regge action

In 1960, the Euclidean Einstein's gravity has been reformulated without coordinate by T. Regge [18]. In this theory, the Reimannian manifold is discretized by the flat-space simplices, and such a manifold is often called *simplicial manifold* or *Regge manifold.* The dynamical variable alternative to the metric in the Regge's formulation is the number of ways of gluing simplices, or the lengths of links. In this appendix, we try to derive the discretized action which we call *the Regge action* in d dimensions[15].

First, we define the simplex in each dimension. The 0-dimensional simplex is a point (node), the 1-dimensional one is a line (link), the 2-dimensional one is a triangle, the 3-dimensional one is a tetrahedron, the 4-dimensional one is a 4-simplex, $\cdots$, and the d-dimensional one is a d-simplex (Figure 34). Noting that the 4-simplex consists of the tetrahedra whose one of faces is replaced by

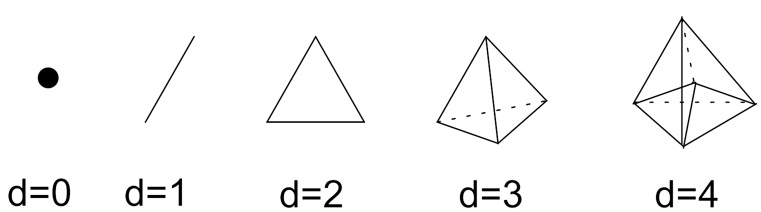


Figure 34: Simplices in several dimensions

the (boundary) tetrahedra. Similarly the 5-simplex consists of the tetrahedra whose one of faces is replaced by the (boundary) 4-simplex, and so on. In d dimensions, by gluing d-simplices as building blocks we can construct the Regge manifold.

[15]This appendix is based on [18, 27].

Where is the information as to the curvature embedded? To understand this, we take a look at the 2-dimensional case for simplicity. In 2 dimensions, the simplex is a flat triangle. It is obvious that the curvature singularity is localized at each node. In a similar manner, we can notice that in d dimensions the curvature singularity is localized at $(d-2)$-simplices. The curvature is originally determined by probing the infinitesimal variation of a vector under the rotation around the curvature, and the rotation is the concept pertained to the plane, the 2-dimensional surface. Thus the information of the curvature may exist in the $(d-2)$-dimensional object. We call such a $(d-2)$-simplex *a hinge*. Summing up above, we understand that the curvature is localized at the hinges.

Next, we would like to find out the quantity analogous to the Riemann tensor in the continuum theory. To do this, again we focus on the 2 dimensions, and introduce the deficit angle:

$$\delta_h \equiv 2\pi - \sum_i \theta_i, \tag{B.1}$$

where $\sum_i \theta_i$ is the sum of inner angles at some hinge. If we cut one link of the triangles attached to the hinge (the node), and push onto the flat plane, then we find the deficit of the angle, that is *the deficit angle*. For the schematical interpretation of the deficit angle, see Figure 35. We can extend

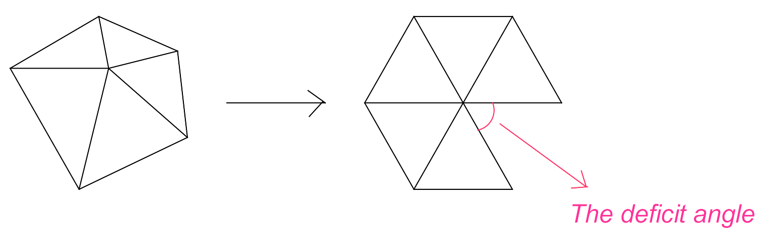


Figure 35: The deficit angle at 2 dimensions

this concept into the general case. In d dimensions, the deficit angle is defined by $\delta_h \equiv 2\pi - \sum$ (inner hyper-dihedral angles). What we need to do is defining the rotation around the hinge. The rotation is defined by designating the rotational axis so that the generator of the rotation in d dimensions can be defined by

$$S_{ab} \equiv \frac{1}{(d-2)!A_h}\epsilon_{abn_1\ldots n_{d-2}} l_{n_1} \ldots l_{n_{d-2}}, \tag{B.2}$$

where $\epsilon_{abn_1\ldots n_{d-2}}$ is the antisymmetric tensor with indices running from 1 to d, l_{n_i} is the link vector (the link having the direction), and A_h is the volume of the hinge. We have used the Einstein's convention of summation. For instance, in 2 dimensions, $S_{ab} = \epsilon_{ab}$. We also introduce the density of the hinge in the rotational plane as the 2-dimensional δ-function at the hinge:

$$\rho_h \equiv \delta^{(2)}(\text{hinge}). \tag{B.3}$$

Considering the infinitesimal closed path C and its inner area Γ, we can find the number of hinges in Γ:

$$N_\Gamma = \frac{1}{2!}\rho_h \Gamma_{ab} S_{ab}, \tag{B.4}$$

where Γ_{ab} is the area element. In this region we can take the continuum limit satisfying that ρ_h

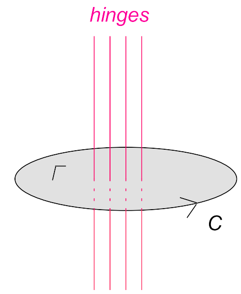


Figure 36: The path C around hinges

goes to infinity and δ_h goes to zero under the fixed $\rho_h\delta_h$. By this treatment the surface becomes smooth locally. Thus we can treat the rotation of the vector η_a along the closed path C. The variation of η_a under the infinitesimal rotation is

$$\delta\eta_a = (N_\Gamma\delta_h)S_{ab}\eta_b. \tag{B.5}$$

In general this quantity can be derived by using the Riemann tensor R_{abcd} such that

$$\delta\eta_a = \frac{1}{2!}\Gamma_{cd}R_{cdab}\eta_b. \tag{B.6}$$

Comparing the two, we can identify the Riemann tensor:

$$R_{abcd} = (\rho_h\delta_h)S_{ab}S_{cd} = \delta^{(2)}(\text{hinge})\delta_h S_{ab}S_{cd}. \tag{B.7}$$

Contracting indices, we get the Ricci scalar:

$$R = 2\delta^{(2)}(\text{hinge})\delta_h. \tag{B.8}$$

Remembering the Euclideanized Einstein-Hilbert action without boundary in d dimensions:

$$S_{EH} = -\frac{1}{4\pi G_N}\int d^d\xi\sqrt{g}R + \Lambda\int d^d\xi\sqrt{g}, \tag{B.9}$$

where G_N is the Newton's gravitational coupling constant, Λ is the bulk cosmological constant. Inserting $(B.8)$ into $(B.9)$ naively, we can assume the discretized action:

$$S_{\text{Regge}} = -\kappa\sum_h A_h\delta_h + \lambda\sum_i V_i, \tag{B.10}$$

where A_h is the $(d-2)$-dimensional volume of the hinge h, κ is the gravitational coupling constant, λ is the bulk cosmological constant, and V_i is the volume of the d-simplex labeled by i. We call this dicretized action *the Regge action*.

C Matrix model

G. 'tHooft pointed out the $U(N)$ gauge theory, especially for Yang-Mills theory, in $N \to \infty$ limit had the possibility to solve the quantum chromodynamics in the strong-coupled region [57]. Nowadays, contrary to the first motivation, the large-N gauge theory is used in different contexts, such as AdS/CFT correspondence in string theory. In this appendix, we would like to take advantage of the large-N theory in the 0-dimensional matrix model, which is closely related to the discretized 2-dimensional gravity. We will see that the vacuum amplitude with genus 0 (corresponding to the leading contribution in the large-N limit) can be calculated completely at all perturbative orders, that is nontrivial and surprising. Although we do not handle the higher genus contributions in this thesis, it is well known that $1/N$-expansion in the matrix model can be calculated to all orders by using the double scaling limit[16].

C.1 Large-N limit and 'tHooft coupling

At first, we would like to see what happens to the d-dimensional field theory in the large-N limit. To do that, we try to construct a good way to count the diagrams' orders with respect to the coupling constant. As our starting Lagrangian density, we take:

$$\mathcal{L} = \mathrm{tr}(\partial_I M \partial_I M) + \mathrm{tr}(M^2) + \sum_n g_n \mathrm{tr}(M^n), \tag{C.1}$$

where the index I runs from 1 to d. M is the $N \times N$ hermitian matrix, and g_n is the coupling constant of the n-point vertex. We can write down the propagator in the momentum space:

$$\langle M_{ij} M_{kl}(p) \rangle = \Delta(p) \delta_{il} \delta_{kj}, \tag{C.2}$$

where p denotes the momentum, and $\Delta(p)$ is the propagator. This propagator can be expressed by the double line, and each line has the different orientation because the low and the column of the matrix are related by hermiticity. For the index loop, we get

$$\delta_{ii} = N. \tag{C.3}$$

Now we can estimate the contribution of the coupling constants and the factor N of the amplitude, that is

$$g_3^{V_3} g_4^{V_4} \cdots N^F, \tag{C.4}$$

where V_n is the number of n-point vertices, and F means the number of index loop. It is convenient to introduce the total number of vertices $V = \sum_n V_n$, the number of propagators E, and the Euler character χ. In the following discussion, we restrict the diagram to the connected vacuum diagram, and in such connected diagrams we can apply the Euler's theorem:

$$V - E + F = \chi = 2 - 2h, \tag{C.5}$$

where h is the number of the holes of the surface or in other words, the genus[17]. We can also find the relation:

$$2E = \sum_n n V_n. \tag{C.6}$$

[16]This appendix is based on [25, 57]

[17]In this case, we have omitted the contribution of the boundaries and the cross caps. If we consider such contributions, the Euler character is changed. In general, we have $\chi = 2 - 2h - b - c$ where b and c are the number of boundaries and cross caps, respectively.

This is because there are n propagators at each n-point vertex, and each propagator has two vertices. Using this relation, the Euler's theorem can be rewritten by

$$V - E + F = \sum_n \left(1 - \frac{n}{2}\right) V_n + F = \chi. \tag{C.7}$$

Applying this relation to the amplitude, we get

$$g_3^{V_3} g_4^{V_4} \cdots N^F = (g_3 N^{(3/2-1)})^{V_3} (g_4 N^{(4/2-1)})^{V_4} \cdots N^\chi. \tag{C.8}$$

If the coupling constant g_n is proportianal to $N^{1-n/2}$, we can factor out the N dependence up to the over all factor N^χ. To see the N dependence clearly, we take

$$g_n \equiv N^{1-\frac{n}{2}} g, \tag{C.9}$$

where g is the new coupling constant, and we call it *the 'tHooft coupling*. If we take the large-N limit under this coupling constant fixed, the most leading contribution of the diagram is from $h = 0$. We call such a genus-zero diagram *the planar diagram*. Considering the planar diagrams only, the d-dimensional vacuum energy $E^{(d;N)}$ divided by N^2 is finite under the large-N limit:

$$\lim_{N\to\infty} \frac{1}{N^2} E^{(g;N)} \equiv E^{(d)}(g). \tag{C.10}$$

In the following discussion, we will calculate $E^{(d)}(g)$ including all diagrammatic contributions.

C.2 Matrix-model calculations

If we set each diagram is equivalent to unity except for the over all factor, the theory becomes 0-dimensional field theory which is called *the matrix model*. We take the partition function of the matrix model under the large-N limit as follows:

$$Z = \exp(-N^2 E^{(0)}) = \lim_{N\to\infty} \int d^{N^2} M \exp\left[-\left(\frac{1}{2}\mathrm{tr} M^2 + \frac{g}{N}\mathrm{tr} M^4\right)\right], \tag{C.11}$$

where

$$d^{N^2} M \equiv \prod_i dM_{ii} \prod_{i<j} d(\mathrm{Re}[M_{ij}]) d(\mathrm{Im}[M_{ij}]). \tag{C.12}$$

Introducing the eigenvalue of M_{ij}, λ_i, we can get

$$Z = \lim_{N\to\infty} \int \prod_i d\lambda_i \prod_{i<j} (\lambda_i - \lambda_j)^2 \exp\left[-\left(\frac{1}{2}\sum_i \lambda_i^2 + \frac{g}{N}\sum_i \lambda_i^4\right)\right]. \tag{C.13}$$

To derive this relation, it is good to insert the unity [24]:

$$1 = \int \prod_i d\lambda_i dU \delta^{N^2}(UMU^\dagger - \Lambda) \Delta^2(\lambda), \tag{C.14}$$

where U is the unitary matrix which can diagonalize M, that is $\Lambda \equiv diag(\lambda_1, \cdots, \lambda_N)$, and $\Delta^2(\lambda)$ is the Faddeev-Popov determinant. In addition to that, we set

$$\int dU = 1, \tag{C.15}$$

and

$$M = U_0^\dagger \tilde{\Lambda} U_0. \tag{C.16}$$

Thus we get

$$\begin{aligned}
Z &= \lim_{N\to\infty} \int d^{N^2} M \int \prod_i d\lambda_i dU \delta^{N^2}(UMU^\dagger - \Lambda)\Delta^2(\lambda) \exp\left[-\left(\frac{1}{2}\mathrm{tr}M^2 + \frac{g}{N}\mathrm{tr}M^4\right)\right] \\
&= \lim_{N\to\infty} \int \prod_i d\lambda_i dU \Delta^2(\lambda) \exp\left[-\left(\frac{1}{2}\sum_i \lambda_i^2 + \frac{g}{N}\sum_i \lambda_i^4\right)\right] \\
&= \lim_{N\to\infty} \int \prod_i d\lambda_i \Delta^2(\lambda) \exp\left[-\left(\frac{1}{2}\sum_i \lambda_i^2 + \frac{g}{N}\sum_i \lambda_i^4\right)\right].
\end{aligned} \tag{C.17}$$

Next, we need to determine $\Delta^2(\lambda)$. Considering $U = (1+T)U_0$ for small T, we get

$$\begin{aligned}
UMU^\dagger &= (1+T)U_0(U_0^\dagger \tilde{\Lambda} U_0)U_0^\dagger(1-T) \\
&= \tilde{\Lambda} + T\tilde{\Lambda} - \tilde{\Lambda}T + O(T^2).
\end{aligned}$$

This implies

$$\delta^{N^2}(UMU^\dagger - \Lambda) = \delta^N\left(\delta_{ij}(\tilde{\lambda} - \lambda)_i\right)\delta^{N^2-N}\left(T_{ij}(\tilde{\lambda_j} - \tilde{\lambda_i})\right). \tag{C.18}$$

Using this equation, we get

$$\begin{aligned}
1 &= \int \prod_i d\lambda_i dU \delta^{N^2}(UMU^\dagger - \Lambda)\Delta^2(\lambda) \\
&= \int \prod_i d\lambda_i dT \delta^N\left(\delta_{ij}(\tilde{\lambda} - \lambda)_i\right)\delta^{N^2-N}\left(T_{ij}(\tilde{\lambda_j} - \tilde{\lambda_i})\right)\Delta^2(\lambda) \\
&= \int dT \delta^{N^2-N}\left(T_{ij}(\tilde{\lambda_j} - \tilde{\lambda_i})\right)\Delta^2(\tilde{\lambda}) \\
&= \Delta^2(\tilde{\lambda}) \int \prod_{j>i} dT(\tilde{\lambda_j} - \tilde{\lambda_i})^{-2}\delta^{N^2-N}\left(T_{ij}\right).
\end{aligned}$$

Finally, we get

$$\Delta^2(\tilde{\lambda}) = \prod_{j>i}(\tilde{\lambda_j} - \tilde{\lambda_i})^2. \tag{C.19}$$

In fact, the $1/N$-expansion means $\hbar$-expansion so that in the large-N limit, we can use the saddle-point method to solve $E^{(0)}(g)$. To carry out our plan, we rewrite the partition function [25]:

$$\begin{aligned}
Z &= \lim_{N\to\infty} \int \prod_i d\lambda_i \prod_{i<j}(\lambda_i - \lambda_j)^2 \exp\left[-\left(\frac{1}{2}\sum_i \lambda_i^2 + \frac{g}{N}\sum_i \lambda_i^4\right)\right] \\
&= \lim_{N\to\infty} \int \prod_i d\lambda_i \exp\left[\sum_{i<j}\log|\lambda_i - \lambda_j| + \sum_{j<i}\log|\lambda_i - \lambda_j| - \left(\frac{1}{2}\sum_i \lambda_i^2 + \frac{g}{N}\sum_i \lambda_i^4\right)\right] \\
&= \lim_{N\to\infty} \int \prod_i d\lambda_i \exp\left[\sum_{i\neq j}\log|\lambda_i - \lambda_j| - \left(\frac{1}{2}\sum_i \lambda_i^2 + \frac{g}{N}\sum_i \lambda_i^4\right)\right] \\
&\equiv \lim_{N\to\infty} \int \prod_i d\lambda_i \exp(-S).
\end{aligned} \tag{C.20}$$

If we take the leading one configuration of λ_i in the large-N limit, we get

$$E^{(0)}(g) = \lim_{N\to\infty} \frac{1}{N^2} S. \tag{C.21}$$

Considering the saddle-point equation, $\frac{dS}{d\lambda_i} = 0$, the equation of motion can be derived as follows:

$$\frac{1}{2}\lambda_i + \frac{2g}{N}\lambda_i^3 = \sum_{j\neq i} \frac{1}{\lambda_i - \lambda_j}. \tag{C.22}$$

One can notice the quite interesting thing, that is that because of the Coulomb like potential, the eigenvalues repel each other. We put this aside for now. To calculate $E^{(0)}(g)$, we define λ_i as

$$\lambda_i \equiv \sqrt{N}\lambda(i/N). \tag{C.23}$$

In the large-N limit, it is natural to take

$$\lim_{N\to\infty} \sum_i \frac{1}{N}\lambda\left(\frac{i}{N}\right) = \int_0^1 dx \lambda(x). \tag{C.24}$$

By this expression, $E^{(0)}(g)$ can be rewritten by

$$E^{(0)}(g) = \int_0^1 dx \left(\frac{1}{2}\lambda(x)^2 + g\lambda(x)^4\right) - \int_0^1 dx \int_0^1 dy \log|\lambda(x) - \lambda(y)|. \tag{C.25}$$

Similarly, the equation of motion obeys

$$\frac{1}{2}\lambda(x) - 2g\lambda(x)^3 = P\int_0^1 \frac{dy}{\lambda(x) - \lambda(y)}, \tag{C.26}$$

where $P\int$ means the principal part of the integral. Now we introduce the density of the eigenvalues, $u(\lambda)$ given by

$$u(\lambda) \equiv \frac{dx}{d\lambda}. \tag{C.27}$$

Letting $u(\lambda)$ be the density, we should impose that $u(\lambda)$ is positive, even, and normalized in the following form:

$$\int_{-2a}^{2a} d\lambda u(\lambda) = 1. \tag{C.28}$$

Taking λ as the variable, the equation of motion is given by

$$\frac{1}{2}\lambda - 2g\lambda^3 = P\int_{-2a}^{2a} \frac{d\mu u(\mu)}{\lambda - \mu}, \tag{C.29}$$

where of course $u(\lambda)$ vanishes outside the support $(-2a, 2a)$. To evaluate this equation, we introduce an analytic function:

$$F(\lambda) = \int_{-2a}^{2a} \frac{d\mu u(\mu)}{\lambda - \mu}, \tag{C.30}$$

where λ is complex outside the support. $F(\lambda)$ given by this form satisfies the good properties which we have discussed in this thesis . In particular, the value of λ near the interval, for small ϵ we can get

$$F(\lambda \pm i\epsilon) = \int_{C_\pm} d\mu \frac{u(\mu)}{\lambda \pm i\epsilon - \mu}$$

$$
\begin{aligned}
&= P\int_{-2a}^{2a}\frac{d\mu u(\mu)}{\lambda-\mu} \mp i\pi \mathrm{Res}[F(\mu)]|_{\mu=\lambda} \\
&= \frac{1}{2}\lambda + 2g\lambda^3 \mp i\pi u(\lambda),
\end{aligned} \tag{C.31}
$$

where the path C_+ is the upper half semi circle, and C_- is the lower half one. The solution can be given by

$$
F(\lambda) = \frac{1}{2}\lambda + 2g\lambda^3 - \left(\frac{1}{2} + 4ga^2 + 2g\lambda^2\right)\sqrt{\lambda^2 - 4a^2}, \tag{C.32}
$$

with

$$
12ga^4 + a^2 - 1 = 0. \tag{C.33}
$$

In the region $|\lambda| \le 2a$, we find

$$
u(\lambda) = \frac{1}{\pi}\left(\frac{1}{2} + 4ga^2 + 2g\lambda^2\right)\sqrt{4a^2 - \lambda^2}. \tag{C.34}
$$

This expression must satisfy $\int d\lambda\ u = 1$ so that

$$
\begin{aligned}
1 &= \int_{-2a}^{2a} d\lambda \frac{1}{\pi}\left(\frac{1}{2} + 4ga^2 + 2g\lambda^2\right)\sqrt{4a^2 - \lambda^2} \\
&= \int_{\pi}^{0} d\lambda \frac{1}{\pi}\left(\frac{1}{2} + 4ga^2 + 8a^2 g\cos^2(\theta)\right)(2a\sin(\theta))(-2a\sin(\theta)) \\
&= a^2 + 8ga^4 + 4ga^4.
\end{aligned} \tag{C.35}
$$

This implies $12ga^4 + a^2 - 1 = 0$ which must be satisfied so that $u(\lambda)$ is well-defined in $|\lambda| \le 2a$ as the density. Rewriting $E^{(0)}(g)$ again in terms of the integration of λ, we get

$$
E^{(0)}(g) = \int_{-2a}^{2a} d\lambda u(\lambda)\left(\frac{1}{2}\lambda^2 + g\lambda^4\right) - \int_{-2a}^{2a} d\lambda \int_{-2a}^{2a} d\mu u(\lambda)u(\mu)\log|\lambda - \mu| \tag{C.36}
$$

Remembering the equation of motion $(C.29)$, and integrating over it from 0 to λ with respect to $\tilde{\lambda}$, that is

$$
\int_0^{\lambda} d\tilde{\lambda}\left(\frac{1}{2}\tilde{\lambda} + 2g\tilde{\lambda}^3\right) = \int_0^{\lambda} d\tilde{\lambda} P\int_{-2a}^{2a} d\mu \frac{u(\mu)}{\tilde{\lambda} - \mu}.
$$

This integral can be evaluated as follows:

$$
\begin{aligned}
(\mathrm{LHS}) &= \frac{\lambda^2}{4} + \frac{g}{2}\lambda^4, \\
(\mathrm{RHS}) &= P\int_{-2a}^{2a} d\mu u(\mu)(\log|\lambda - \mu| - \log|\mu|).
\end{aligned}
$$

Multiplying $u(\lambda)$ and then integrating over the both sides with respect to λ from $-2a$ to $2a$, we get

$$
P\int_0^{2a} d\lambda u(\lambda)\left(\frac{1}{2}\lambda^2 + g\lambda^4\right) = P\int_{-2a}^{2a} d\lambda P\int_{-2a}^{2a} d\mu u(\lambda)u(\mu)(\log|\lambda - \mu| - \log|\mu|). \tag{C.37}
$$

Using this relation, we get

$$
E^{(0)}(g) = \int_0^{2a} d\lambda u(\lambda)\left(\frac{1}{2}\lambda^2 + g\lambda^4 - 2\log|\lambda|\right). \tag{C.38}
$$

References

[1] H. Kodama, T. Yoneya, H. Kawai, et.al., "Quantum Gravity," Mathematical Science Additional volume, October issue, 2009 (in Japanese).

[2] P. Horava, "Membranes at Quantum Criticality," hep-th/0812.4287.

[3] P. Horava, "Quantum Gravity at a Lifshitz Point," hep-th/0901.3775.

[4] P. Horava, "Spectral Dimension of the Universe in Quantum Gravity at a Lifshitz Point," hep-th/0902.3657.

[5] E. M. Lifshitz, "On the Theory of Second-Order Phase Transitions I and II," Zh. Eksp. Teor. Fiz. **11** (1941) 255 and 269.

[6] P. M. Chaikin and T. C. Lubensky, "Principles of Condensed Matter Physics," Cambridge U.P. (1995).

[7] R. M. Hornreich, M. Luban, and S. Shtrikman, "Critical Behavior at the Onset of $\vec{\mathbf{k}}$-Space Instability on the λ Line," Phys. Rev. Lett. **35** (1975) 1678.

[8] P. H. Damgaad and H. Huffel, "STOCHASTIC QUANTIZATION," Phys. Rep. 152, Nos. 5 & 6 (1987) 227-398.

[9] R. Dijkgraaf, D. Orlando and S. Reffert, "Relating Field Theory via Stochastic Quantization," hep-th/0903.0732.

[10] D. Orlando and S. Reffert, "On the renormalizability of Horava-Lifshitz-type Gravities," hep-th/0905.0301

[11] M. Henneaux, A. Kleinschmidt, G. L. Gomez, "A dynamical inconsistency of Horava gravity," hep-th/0912.0399

[12] M. Li and Y. Pang, "A Trouble with Horava-Lifshitz Gravity," hep-th/0905.2751.

[13] J. Distler and H. Kawai, "WHO'S AFRAID OF JOSEPH LIOUVILLE ?," Nucl. Phys. B321 (1989) 509.

[14] F. David, "CONFORMAL FIELD THEORIES COUPLED TO 2-d GRAVITY IN THE CONFORMAL GAUGE," Mod. Phys. Lett. A3 (1988) 1651.

[15] A. M. Polyakov, "QUANTUM GEOMETRY OF BOSONIC STRINGS," Phys. Lett. B103 (1981) 207.

[16] Y. Nakayama, "Liouville Field Theory –A decade after the revolution," hep-th/0402009

[17] J. Polchinski, "String theory. Vol. 1: An introduction to the bosonic string," Cambridge U.P. (1998)

[18] T. Regge, "General Relativity without Coordinates," Nuovo Cim. A 19 (1961) 558-571.

[19] J. B. Hartle and S. W. Hawking, “Wave function of the universe,” Phys. Rev. D vol. 28, No. 12 (1983)2960-2975.

[20] J. Ambjorn, B. Durhuus and T. Jonsson, “Quantum Geometry,” Cambridge U.P. (1997)

[21] J. Ambjorn, J. Jurkiewicz and R. Loll, “Quantum gravity as sum over spacetimes,” gr-qc/0906.3947.

[22] J. Ambjorn, J. Jurkiewicz and Y. Watabiki, “Dynamical Triangulations, a Gateway to Quantum Gravity ?,” hep-th/9503108.

[23] J. Ambjorn and J. Jurkiewicz, “Mutiloop correlators for two-dimensional quantum gravity,” Phys. Lett. B vol. 251, No. 4 (1990)517-524.

[24] P. Di Francesco, P. Ginsparg and J. Zinn-Justin, “2D Gravity and Random Matrices,” hep-th/9306153.

[25] E. Brezin, C. Itzykson, G. Parisi and J. B. Zuber, “Planar Diagrams,” Commun. math. Phys. 59, 35-51 (1978).

[26] D. Bessis, C. Itzykson and J. B. Zuber, “Quantum Field Theory Techniques in Graphical Enumeration,” Adv. Appl. Math. 1 (1980) 109-157.

[27] T. Matsubara, “3-dimensional quantum gravity in the discrete space-time,” Soryushiron-kenkyu, 85, (1992), No.6, 136-235.

[28] J. Ambjorn, B. Durhuus and J. Frohlich, “Diseases of Triangulated Random Surface Models, and Possible Cures,” Nucl. Phys. B **257** (1985) 433.
J. Ambjorn, B. Durhuus, J. Frohlich and P. Orland, “The Appearance of Critical Dimensions in Regulated String Theories,” Nucl. Phys. B **270** (1986) 457. F. David, “Planar Diagrams, Two-Dimensional Lattice Gravity and Surface Models,” Nucl. Phys. B **257** (1985) 45.
A. Billoire and F. David, “Microcanonical Simulations Of Randomly Triangulated Planar Random Surfaces,” Phys. Lett. B **168** (1986) 279.

V. A. Kazakov, A. A. Migdal and I. K. Kostov, “Critical Properties of Randomly Triangulated Planar Random Surfaces,” Phys. Lett. B **157** (1985) 295.

D. V. Boulatov, V. A. Kazakov, I. K. Kostov and A. A. Migdal, “Analytical and Numerical Study of the Model of Dynamically Triangulated Random Surfaces,” Nucl. Phys. B **275** (1986) 641.

[29] J. Ambjorn and R. Loll, “Non-perturbative Lorentzian Quantum Gravity, Causality and Topology Change,” hep-th/9805108.

[30] W. Westra, “Toplogy Change and the Emergence of Geometry in Two Dimensional Causal Quantum Gravity,” hep-th/ 0810.0771.

[31] S. Zohren, “Analytic Results in 2D Causal Dynamical Triangulations: A Review,” hep-th/0609177.

[32] J. Ambjorn, J. Correia, C. Kristjansen and R. Loll, “On the relation between Euclidean and Lorentzian 2D quantum gravity,” hep-th/991226.

[33] F. Markopoulou, L. Smolin, “Gauge fixing in causal dynamical triangulations,” Nucl. Phys. B 739 (2006) 120-130.

[34] R. Nakayama, “2D Quantum Gravity in the Proper-Time Gauge,” hep-th/9312158.

[35] J. Ambjorn, R. Loll, W. Westra and S. Zohren, “Putting a cap on causality violations in CDT,” gr-qc/0709.2784.

[36] G. T. Horowitz, “Topology change in classical and quantum gravity,” Class. Quantum Grav. 8 (1991) 587-601.

[37] J. Louko and R. D. Sorkin, “Complex actions in two-dimensional topology change,” gr-qc/9511023.

[38] S. Jain and S. D. Mathur, “World-Sheet Geometry and Baby Universes in 2-D Quantum Gravity,” hep-th/9204017.

[39] H. Kawai. N. Kawamoto, T. Mogami and Y. Watabiki, “Transfer Matrix Formalism for Two-Dimensional Quantum Gravity and Fractal Structures of Space-time,” hep-th/9302133.

[40] N. Ishibashi, H. Kawai, “String Field Theory of Noncritical Strings,” hep-th/9307045.

[41] J. Ambjorn, Y. Watabiki, W. Westra and S. Zohren, “A String Field Theory based on Causal Dynamical Triangulations,” hep-th/0802.0719.

[42] J. Ambjorn, R. Loll, Y. Watabiki, W. Westra and S. Zohren, “A new continuum limit of matrix models,” hep-th/0810.2408.

[43] J. Ambjorn, R. Loll, Y. Watabiki, W. Westra and S. Zohren, “A Matrix Model for 2D Quantum Gravity defined by Causal Dynamical Triangulations,” hep-th/0804.0252.

[44] J. Ambjorn, R. Loll, Y. Watabiki, W. Westra and S. Zohren, “New aspects of two-dimensional quantum gravity,” hep-th/0911.4208.

[45] J. Ambjorn, R. Loll, Y. Watabiki, W. Westra and S. Zohren, “Proper time is stochastic time in 2d quantum gravity,” hep-th/0911.4211.

[46] J. Ambjorn, J. Jurkiewics and R. Loll, “Dynamically Triangulating Lorentzian Quantum Gravity,” hep-th/0105267.

[47] R. Loll, “The emergence of Spacetime or Quantum Gravity on Your Desktop,” gr-qc/0711.0273.

[48] J. F. Wheater and J. Correia, “The spectral dimension of non-generic branched polymers,” Nucl. Phys. B 73 (1999) 783-785.

[49] D. Benedetti and J. Henson, “Spectral geometry as a probe of quantum spacetime,” hep-th/0911.0401.

[50] D. V. Vassilevich, “Heat kernel expansion: User’s manual,” Phys. Rep. 388 (2003) 279-360.

[51] J. Ambjorn, J. Jurkiewics and R. Loll, “Spectral Dimension of the Universe,” hep-th/0505113.

[52] J. Ambjorn, J. Jurkiewics, R. Loll and G. Vernizzi, “3D Lorentzian Quantum Gravity from the asymmetric ABAB matrix model,” hep-th/0311072

[53] R. L. Arnowitt, S. Deser and C. W. Misner, “Canonical variables for general relativity,” Phys. Rev. **117**, 1595 (1960); R. L. Arnowitt, S. Deser and C. W. Misner, “The Dynamics of general relativity,” arXiv:gr-qc/0405109.

[54] J. D. Romano, “GEOMETRODYNAMICS VS. CONNECTION DYNAMICS,” gr-qc/9303032.

[55] J. Baez and J. P. Muniain, “GAUGE FIELDS, KNOTS ADN GRAVITY,” World Scientific (1994)

[56] R. M. Wald, “General Relativity,” Chicago U.P. (1984)

[57] G.’tHooft, “A PLANAR DIAGRAM THEORY FOR STRONG INTERACTION,” Nucl. Phys. B 72 (1974) 461-473.