# Langlands Program, Field Theory, and Mirror Symmetry

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**Abstract**

This article aims to give a self-contained introduction to the Langlands program in math and physics. Initially the Langlands correspondences are purely mathematical conjecture, however, recently there have been several related works from physics side. Giving some basic facts from mathematical side, we review those recent advances towards the Langlands program mostly from physical viewpoints, and see the physical geometric Langlands correspondence is indeed analogous to the picture we have in mathematics. Starting with $\mathcal{N} = 4$ super Yang-Mills theory on a four-dimensional manifold, we claim that the geometrical Langlands duality is physically understandable via mirror symmetries between (electric and magnetic) branes associated with the boundaries of the manifold. Main references are [26, 55].

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# Chapter 1

# introduction

The Langlands program, which originates from work by Robert Langlands [31], has a vast range of research activities mostly in mathematics today. In addition to the original Langlands correspondences between number theory and harmonic analysis, its geometric analogue has been considered on Riemann surfaces in several work by Drinfeld [7, 8] and Laffaorgue [32]. Even tough there are a lot of reasons to be fascinated by the Langlands program, here we would like to present a still mysterious correspondence between the elliptic curve (1.1) and the modular form (automorphic form) (1.2), which is firstly conjectured by Ramanujan and proved by Eichker in 1954. We first consider the following elliptic curve over the rational field

$$y^2 + y = x^3 - x, \tag{1.1}$$

and we define $a_p = p + 1 - \#\text{solutions of (1.1) of mod } p$, where $p$ is a prime. Now let us suddenly consider an form

$$q \prod_{n=1}^{\infty} (1 - q^n)^2 (1 - q^{11n})^2 = \sum_{n=1} b_n q^n. \tag{1.2}$$

Surprisingly, the equations $a_p = b_p$ hold for all prime $p$. We will reconsider this later in more precise way, however, this correspondence is yet an astonishing result. It is the Langlands program that connects these dualities in more general and modern ways from various areas of mathematics. In the classical Langlands program, the correspondence between elliptic curves and automorphic forms are all about the correspondence between the eigenvalues of Frobenius operator and Hecke operator.

By the way, one of the most important player in the Langlands program is the Langlands dual group ${}^LG$, where $G$ is a given reductive algebraic group. Though the Langlands program has nothing to do with physics at first sight, the dual group has been independently considered in physics as an extension of electric-magnetic duality [16, 37] and ${}^LG$ has been know as a GNO dual group among physicists. For a gauge theory with gauge group $G$ and a coupling constant $e$, there exists a dual theory with ${}^LG$ and $1/e$, which implies strong-weak duality or $S$-duality. In addition to the emergence of dual groups in physics, Kapustin and Witten showed in 2006 that there is a possibility that the Langlands correspondence is indeed accessible from physics via the $S$-duality in $\mathcal{N} = 4$ super Yang-Mills theory on a four-dimensional manifold and mirror symmetries. It is notable that the correspondence between Frobenius operators and Hecke operators in the Langlands program is mapped to the correspondence between Wilson operators and 't Hoof operators in physical point of view as shown by Kapustin [24]. Moreover Kapustin and Witten showed that the Hecke modification on a $G-$bundle indeed corresponds to inserting an 't Hooft operator. Hence we have clear picture to connect the Hecke operators and the 't Hooft operators.

The chart of this article is as shown in Figure 1. We will see how Ramanujan's findings are related to the Langlands conjecture in section 2.1, where several marvelous correspondences can be found. In section 2.2, we will roughly review how the Langlands conjecture can be interpreted from geometrical and categorical viewpoints. In physics sense, the correspondences between Frobenius and Hecke operator are interpreted from dualities between Wilson and 't Hooft operators via electric-magnetic duality. To reconsider the dual groups from physics side, we will review in section 3.1 classical electric-magnetic dualities for non-abelian gauge groups. In section 3.2 we consider $\mathcal{N} = 4$ super Yang-Mills theory (SYM theory) and show the supersymmetric conditions for bosonic fields result in the Hitchin moduli spaces when the four-manifold is reduced to two-dimension. One of the key feature is that SYM theories with gauge group $G$ and ${}^LG$ are in terms of $S$-dual in our case, and the Hitchin moduli spaces $\mathcal{M}_H({}^LG, C)$ and $\mathcal{M}_H(G, C)$ are believed to be mirror partners, which agree with the SYZ conjecture [46]. The mirror symmetries between $A$-brane and a $B$-brane bring about the geometric Langlands correspondence. We will show that there is a $B$-brane which is an eigenbrane of a Wilson operator and a $A$-brane which is an eigenbrane of an 't Hooft operator. The duality seen in Wilson and 't Hooft operators is one of most important in the Langlands program.

We give some examples for the 't Hooft/Hecke correspondence. In more concrete way, we will finally see a special $A$-brane is indeed connected to a $\mathcal{D}$-module.

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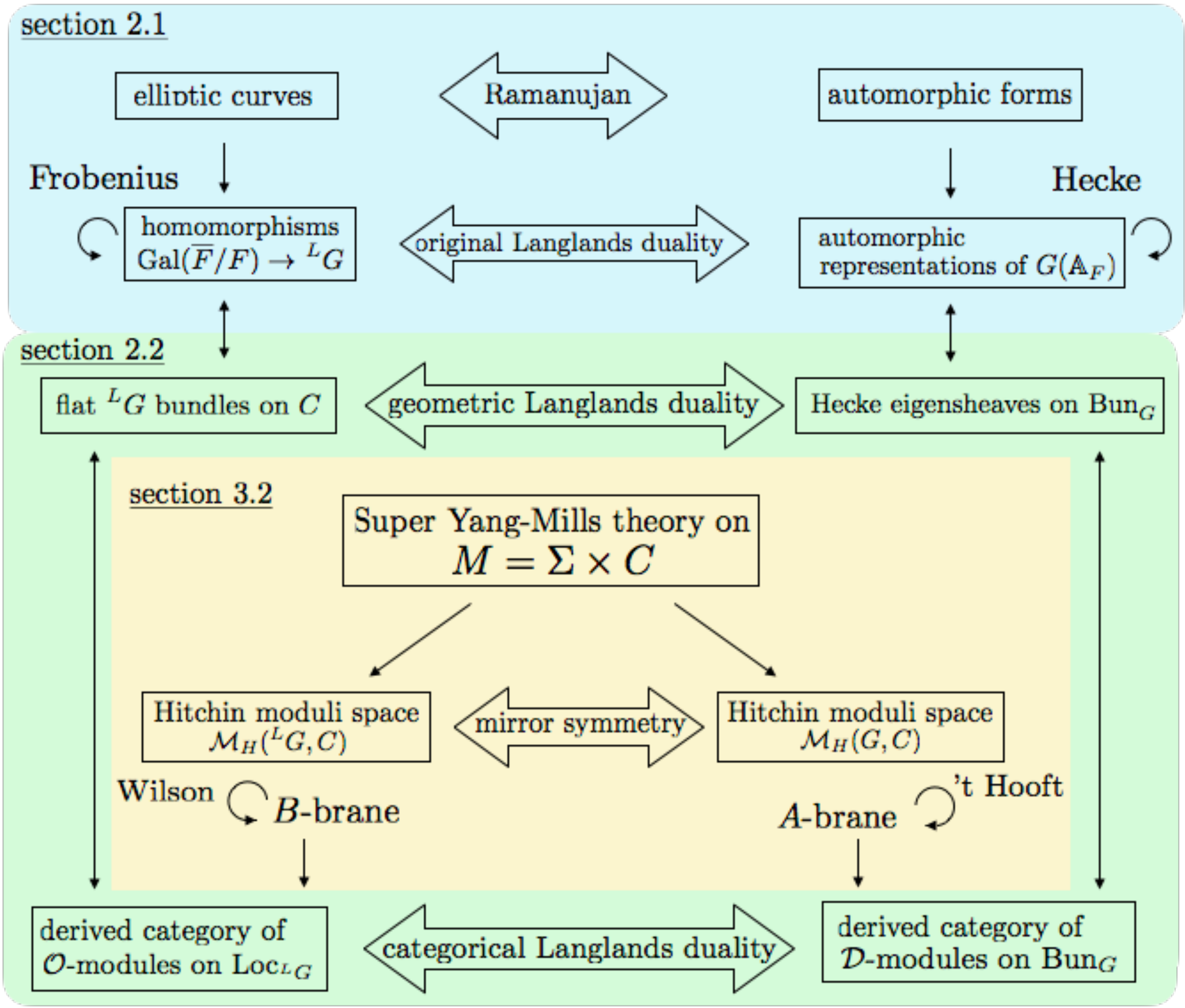



Figure 1.1: A rough image of the Langpands program. $G$ is a general reductive algebraic group (in super Yang-Mills theory, we choose it as a compact Lie group) and ${}^LG$ is its Langlands dual group. $F$ is a number field or the function field of a curve over $\mathbb{F}_q$. For the geometric Langlands correspondence, we take $F = k(C)$, the field of rational functions on a curve $C$ over a field $k$. We consider $\mathcal{N} = 4$ super Yang-Mills theory in a four-dimensional manifold $M = \Sigma \times C$, which is reduced to sigma models whose target spaces are $\mathcal{M}_H({}^LG, C)$ and $\mathcal{M}_H(G, C)$, where $C$ and $\Sigma$ are compact Riemann surfaces with some additional conditions. If $\Sigma$ has a boundary, we impose some boundary conditions on the sigma models, and there exists the corresponding branes. Hence one of physical interpretations of the Langlads correspondences is mirror symmetries between categories of $B$-brane and $A$-brane. Some of fundamental topics about mirror symmetry are wrapped up in appendix B.

# Chapter 2

# Langlands program in math

## 2.1 The Langlands conjecture over number fields

### 2.1.1 Preliminaries

In a classical sense, one of the main concepts of the Langlands conjecture, or Langlands program, is to relate an **automorphic form** and an **elliptic curve** by the use of the associated $L$-functions respectively. Hecke operators act on the automorphic form and Frobenius operators acts on an elliptic form in a natural way. Here those Frobenius operators generate a Galois group and hence it allows us to consider its Galois representations. On the other hand, when a special automorphic form is a simultaneous eigenform of the Hecke operators, its eigenvalues turn out to be the coefficients of the Fourier series of the original automorphic form. Very mysterious facts are that those Fourier coefficients can be related to the number of rational points of the corresponding elliptic curve. It would be natural to ask why those astonishing results arise and whether more general correspondences exist. Hence the Langlands conjecture over algebraic fields stated below.

Langlands conjecture

For any Galois representation $\sigma$, there exists an automorphic representation $\pi$ such that

$$L(s,\sigma) = L(s,\pi), \tag{2.1}$$

namely, for a given Hecke operator there is a corresponding Frobenius operator.

Our goal of this section is to provide precise and fundamental review of the Langlands program with a number of examples. The next series of subsections are equipped with self-contained explanations of terminologies and motivation to consider the Langlands conjecture.

In the series of following sections, we will recall some of fundamental concepts of related to elliptic curves and automorphic curves. There are a lot of celebrated references, for example [29, 39, 43, 44].

### 2.1.2 Automorphic forms and Hecke operators on $SL_2(\mathbb{Z})$

An automorphic form rerated to an elliptic curve is a holomorphic function $f$ over the upper half plane $H = \{z \in \mathbb{C} : \mathrm{Im} z > 0\} \cong SL_2(\mathbb{R})/SO(2)$. We see $G = SL_2(\mathbb{R})$ acts on $H$ naturally by

$$\gamma z = \frac{az+b}{cz+d} \quad \text{for} \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R}). \tag{2.2}$$

Indeed one can show that $\gamma z \in H$ straightforwardly. Automorphic forms are defined in terms of a discrete subgroup $\Gamma$ of $G$. We take $\Gamma = SL_2(\mathbb{Z})$, which is generated by

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{2.3}$$

When they act on $H$, we see by definition

$$Tz = z+1, \quad Sz = -\frac{1}{z}. \tag{2.4}$$

**Definition 2.1.1.**

A **holomorphic modular form** of weight $k$ is a holomorphic function $f : H \to \mathbb{C}$ such that

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z) \quad \text{for} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}). \tag{2.5}$$

and $f$ has a Fourier expansion

$$f(z) = \sum_{n=0} a(n,f)q^n, \quad q = e^{2\pi i z}. \tag{2.6}$$

Let $M_k(SL_2(\mathbb{Z}))$ denote the set of all holomorphic modular forms.

Note that the definition (2.6) make sense since a holomorphic function $f$ which obeys (2.5) is periodic $f(z+1) = f(z)$ regardless of weight $k$, which allows us to expand $f$ into a Fourier series

$$f(z) = \sum_{n\in\mathbb{Z}} a(n,f)q^n, \quad q = e^{2\pi i z}. \tag{2.7}$$

Now let us consider a situation in which $f \in M_k(SL_2(\mathbb{Z}))$ vanishes at infinity $z = i\infty$, namely the cusp of the fundamental domain of the action of $SL_2(\mathbb{Z})$ on the upper plane $H$. Such $f$ is called a holomorphic cusp form defied below.

**Definition 2.1.2.**

A **holomorphic cusp form** $f$ of weight $k$ is a holomorphic modular form of weight $k$ with $a(0,f) = 0$ at (2.6). We denote by $S_k(SL_2(\mathbb{Z}))$ the set of all holomorphic cusp forms.

Though we defined a holomorphic modular form with respect to $SL_2(\mathbb{Z})$, we can extend our definition to an automorphic form on $SL_2(\mathbb{R})$ as follows. Once given a holomorphic modular form $f \in M_k(SL_2(\mathbb{Z}))$, we define

$$\varphi_f(g) = (ci+d)^{-k} f(gi), \quad \text{for} \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R}). \tag{2.8}$$

Then $\varphi_f$ is $SL_2(\mathbb{Z})$ invariant, $\varphi_f(\gamma g) = \varphi_f(g)$, for any $g \in SL_2(\mathbb{R})$ since (2.5). Therefore we regard $\varphi_f$ as an automorphic form on $SL_2(\mathbb{R})$

$$\varphi_f : SL_2(\mathbb{Z}) \setminus SL_2(\mathbb{R}) \to \mathbb{C}. \tag{2.9}$$

Moreover if $f \in S_k(SL_2(\mathbb{Z}))$, then $\varphi_f \in L^2(SL_2(\mathbb{Z}) \setminus SL_2(\mathbb{R}))$. We will consider a more general pair $(G, \Gamma)$ and automorphic forms when we discuss automorphic representations.

**Definition 2.1.3.**

We define the **Hecke operators** $T_k(m) : M_k(SL_2(\mathbb{Z})) \to M_k(SL_2(\mathbb{Z}))$ for $1 \leq m \in \mathbb{Z}$ by

$$(T_k(m)f)(z) = m^{k-1} \sum_{ad=m} \sum_{b=0}^{d-1} d^{-k} f\left(\frac{az+b}{d}\right) \quad \text{for} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}). \tag{2.10}$$

Since $f$ has the Fourier series (2.72), the right hand side of the above definition (2.10) can be written in the form of

$$(T_k(m)f)(z) = \sum_{n=0}^{\infty} \left( \sum_{d|(m,n)} d^{k-1} a\left(\frac{mn}{d^2}, f\right) \right) q^n, \tag{2.11}$$

where $(m, n)$ is the greatest common divisor of $m$ and $n$, and $d$ runs all divisors of $(m, n)$. (For integers $a$ and $b$, we write $a|b$ if $a$ is a divisor of $b$.)

The Hecke operators are commutable

$$T_k(m)T_k(n) = T_k(n)T_k(m) = \sum_{d|(m,n)} d^{k-1} T_k\left(\frac{mn}{d^2}\right) \tag{2.12}$$

then the **Hecke ring**

$$\mathbb{T}_k = \mathbb{C}[T_k(m) : m = 1, 2, \cdots] \tag{2.13}$$

is an abelian $\mathbb{C}$-algebra.

If $f \in M_k(SL_2(\mathbb{Z}))$ is a eigenfunction of all $T_k(m)$ $(m = 1, 2, \cdots)$

$$T_k(m)f = \lambda(m, f)f, \tag{2.14}$$

then Hecke eigenvalues $\lambda(m, f)$ form the **Hasse-Weil $L$-function** of $f$

$$L(s, f) = \sum_{m=1}^{\infty} \lambda(m, f) m^{-s} \tag{2.15}$$

$$= \prod_{p:\text{prime}} (1 - \lambda(p, f)p^{-s} + p^{k-1-2s})^{-1}. \tag{2.16}$$

In particular, if $f=\sum_{n=1} a(n,f)q^n \in S_k(SL_2(\mathbb{Z}))$ is normalized $a(1,f)=1$ and is simultaneous eigenfunction of the Hecke operators $T_k(m)$, then the eigenvalues of $T_k(m)$ are the Fourier coefficients

$$T_k(m)f = a(m,f)f \tag{2.17}$$

since the coefficient of $q$ in $T_k(m)f$ is $a(m,f)$ from the equation (2.11). Moreover by virtue of the Hecke ring (2.12), we see that the Fourier coefficients satisfy

$$a(m,f)a(n,f) = \sum_{d|(m,n)} d^{k-1} a\left(\frac{mn}{d^2}, f\right). \tag{2.18}$$

### 2.1.3 Congruence subgroup of $SL_2(\mathbb{Z})$ and Hecke operators

For $(1 \leq)\ N \in \mathbb{Z}$, put

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : a \equiv d \equiv 1 \mathrm{mod}\ N,\ b \equiv c \equiv 0 \bmod N \right\}, \tag{2.19}$$

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : c \equiv 0 \bmod N \right\}, \tag{2.20}$$

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : a \equiv d \equiv 1 \mathrm{mod}\ N,\ c \equiv 0 \bmod N \right\}. \tag{2.21}$$

$\Gamma(N)$ is called the **principal congruence subgroup** of level$N$ in $\Gamma = SL_2(\mathbb{Z})$. A congruence subgroup $\Gamma$ is a subgroup of $SL_2(\mathbb{Z})$ such that $\Gamma(N) \subset \Gamma \subset SL_2(\mathbb{Z})$.

Now we consider cases $2 \leq N$. Let $\psi$ be an index of $(\mathbb{Z}/N\mathbb{Z})^\times$, the group of units of $\mathbb{Z}/N\mathbb{Z}$, and $f$ be a function on $H$. We define a function $f|_k g$ on $H$ by

$$(f|_k g)(z) = (\det g)^{k/2} f(g(z))(cz+d)^{-k}, \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_{2+}(\mathbb{R}), \tag{2.22}$$

where $GL_{2+}(\mathbb{R})$ is the set of matrices in $GL_2(\mathbb{R})$ whose determinant is positive.

We define the set of the holomorphic modular form and holomorphic cusp form of weight $k$ and level $N$, denoted by $M_k(\Gamma_0(N),\psi)$ and $S_k(\Gamma_0(N),\psi)$ respectively, as

$$M_k(\Gamma_0(N),\psi)=\left\{f\in M_k(\Gamma_1(N)) : f|_k\begin{pmatrix}a & b\\ c & d\end{pmatrix}=\psi(d)f,\ \text{for all}\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in\Gamma_0(N)\right\}, \tag{2.23}$$

$$S_k(\Gamma_0(N),\psi)=\left\{f\in S_k(\Gamma_1(N)) : f|_k\begin{pmatrix}a & b\\ c & d\end{pmatrix}=\psi(d)f,\ \text{for all}\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in\Gamma_0(N)\right\}, \tag{2.24}$$

where $\psi(d)=\psi(d \bmod N)$. They are general concepts of the holomorphic modular form and the holomorphic cusp form we defined in the previous subsection.

The **Hecke operators** $T_k(m)$ are defined for $f\in M_k(\Gamma_0(N),\psi)$ in a similar way that

$$(T_k(m)f)(z)=m^{k-1}\sum_{ad=m,(a,N)=1}\sum_{b=0}^{d-1}d^{-k}f\left(\frac{az+b}{d}\right)\quad\text{for}\quad\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in SL_2(\mathbb{Z}). \tag{2.25}$$

Let $f\in M_k(\Gamma_0(N),\psi),\ k>0$ be a simultaneous eigenfunction of the Hecke operators

$$T_k(m)f=\lambda(m,f)f. \tag{2.26}$$

Then the **Hasse-Weil $L$-function** is

$$L(s,f)=\sum_{m=1}^{\infty}\lambda(m,f)m^{-s} \tag{2.27}$$

$$=\prod_{p|N}(1-\lambda(p,f)p^{-s})^{-1}\prod_{p\nmid N}(1-\lambda(p,f)p^{-s}+\psi(p)p^{k-1-2s})^{-1}. \tag{2.28}$$

Moreover the eigenvalues $\lambda(m,f)$ coincide with the Fourier coefficients of $f(z)=\sum_{n=0}^{\infty}a(n,f)e^{2\pi inz}$, namely

$$a(n,f)=\lambda(m,f)a(1,f). \tag{2.29}$$

### 2.1.4 Remarks on history

The theory of automorphic forms we have been working on originates in the work of Ramanujan [41]. In 1916 Ramanujan considered a holomorphic cusp form of weight 12 and revel 1

$$\Delta(z)=q\prod_{n=1}^{\infty}(1-q^n)^{24} \tag{2.30}$$

and its Fourier series

$$\Delta(z) = \sum_{n=1}^{\infty} a(n,f)q^n = q - 24q^2 + 252q^3 - 1472q^4 + 4830q^5 - 6048q^6 - \cdots . \tag{2.31}$$

Here is a list of $a(n,\Delta)$ for $1 \leq n \leq 50$.

| $n$ | $a(n,\Delta)$ | $n$ | $a(n,\Delta)$ | $n$ | $a(n,\Delta)$ | $n$ | $a(n,\Delta)$ | $n$ | $a(n,\Delta)$ |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 11 | 534612 | 21 | -4219488 | 31 | -52843168 | 41 | 308120442 |
| 2 | -24 | 12 | -370944 | 22 | -12830688 | 32 | -196706304 | 42 | 101267712 |
| 3 | 252 | 13 | -577738 | 23 | 18643272 | 33 | 134722224 | 43 | -17125708 |
| 4 | -1472 | 14 | 401856 | 24 | 21288960 | 34 | 165742416 | 44 | -786948864 |
| 5 | 4830 | 15 | 1217160 | 25 | -25499225 | 35 | -80873520 | 45 | -548895690 |
| 6 | -6048 | 16 | 987136 | 26 | 13865712 | 36 | 167282496 | 46 | -447438528 |
| 7 | -16744 | 17 | -6905934 | 27 | -73279080 | 37 | -182213314 | 47 | 2687348496 |
| 8 | 84480 | 18 | 2727432 | 28 | 24647168 | 38 | -255874080 | 48 | 248758272 |
| 9 | -113643 | 19 | 10661420 | 29 | 128406630 | 39 | -145589976 | 49 | -1696965207 |
| 10 | -115920 | 20 | -7109760 | 30 | -29211840 | 40 | 408038400 | 50 | 611981400 |

Then he predicted that there are mysterious formulas

① Put $L(s,\Delta) = \sum_{n=1}^{\infty} a(n,\Delta)n^{-s}$, then we obtain the form (2.28)

$$L(s,\Delta) = \prod_{p:\text{prime}} (1 - a(p,f)p^{-s} + p^{11-2s})^{-1}, \tag{2.32}$$

which is equivalent to the following statements

$$\begin{cases} \text{(1a) } a(p^{k+1},\Delta) = a(p,\Delta)a(p^k,\Delta) - p^{11}a(p^{k-1},\Delta) \text{ for any prime } p \text{ and } k \geq 1. \\ \text{(1b) If } m \text{ and } n \text{ are co-prime, then } a(mn,\Delta) = a(m,\Delta)a(n,\Delta). \end{cases}$$

② For any prime $p$, there exists $0 \leq \theta_p \leq \pi$ such that

$$a(p,\Delta) = 2p^{\frac{11}{2}} \cos\theta_p. \tag{2.33}$$

Moreover he showed

③ For any prime $p$, $a(p,\Delta) = 1 + p^{11} \bmod 691$.

In the remaining of this subsection we show ① by following Mordell's argument [36]. He introduced the Mordell operator $T$

$$(T(p)\Delta)(z) = \sum_{n=1}^{\infty} \left( a(pn, \Delta) + p^{11} a\left(\frac{n}{p}, \Delta\right) \right) q^n \tag{2.34}$$

for a prime $p$, which is an example of the Hecke operator (2.11) from modern view point. We need to show that $f$ is indeed an eigenfunction of $T$. First of all, the function

$$F(z) = \frac{(T(p)\Delta)(z)}{\Delta(z)} \tag{2.35}$$

is constant for all $z$ since short calculations show

$$(T(p)\Delta)(z+1) = (T(p)\Delta)(z), \tag{2.36}$$

$$(T(p)\Delta)\left(\frac{1}{z}\right) = z^{12}(T(p)\Delta)(z), \tag{2.37}$$

which mean $T(p)\Delta$ transforms in the same way as $\Delta$ under $SL_2(\mathbb{Z})$.

Next we want to identify the constant as $a(p, \Delta)$. By the use of Fourier series expansions

$$\begin{aligned} F(z) &= \frac{(T(p)\Delta)(z)}{\Delta(z)} && (2.38) \\ &= \frac{a(p,\Delta)q + O(q^2)}{q - 24q^2 + O(q^3)} && (2.39) \\ &= \frac{a(p,\Delta) + O(q)}{1 - 24q + O(q^2)} && (2.40) \end{aligned}$$

we take the limit $q \to 0$ ($z \to i\infty$) then $F(z) \to a(p, \Delta)$. Then $F(z) = a(p, \Delta)$ follows since $F$ is constant everywhere.

$T(p)\Delta = a(p, \Delta)\Delta$ and the definition of the Mordell operator (2.34) imply

$$a(pn, \Delta) + p^{11} a\left(\frac{n}{p}, \Delta\right) = a(p, \Delta)a(n, \Delta), \quad \text{for } n = 1, 2, \cdots, p. \tag{2.41}$$

Putting $n = p^k$ in the above equation (2.41) gives ⓵ⓐ.

In order to show ⓵ⓑ it is enough to show $a(p^k m, \Delta) = a(p^k, \Delta)a(m, \Delta)$ if $p$ and $m$ are co-prime. We prove this by mathematical induction with respect to $k$. If $k = 0$, the statement is true. We assume the statement is true for $k = 0, 1, \cdots, l$. Again putting $n = p^l m$ in the equation (2.41) gives

$$\begin{aligned} a(p^{l+1}m, \Delta) &= a(p, \Delta)a(p^l m, \Delta) - p^{11} a\left(p^{l-1}m, \Delta\right) \\ &= a(p, \Delta)a(p^l, \Delta)a(m, \Delta) - p^{11}a(p^{l-1}, \Delta)a(m, \Delta) \\ &= a(p^{l+1}, \Delta)a(m, \Delta). \end{aligned} \tag{2.42}$$

Hence the statement is true for $k = l + 1$.

To finish the proof of ①, we derive the equation (2.32). ⓵ⓐ implies

$$1 = \sum_{k=0}^{\infty} (a(p^k, \Delta) - a(p, \Delta)a(p^{k-1}) + p^{11}a(p^{k-2}, \Delta))p^{-sk} \tag{2.43}$$

$$= \left( \sum_{k=0} a(p^k, \Delta)p^{-sk} \right) (1 - a(p, \Delta)p^{-s} + p^{11-2s}). \tag{2.44}$$

Then we have

$$\sum_{k=0}^{\infty} a(p^k, \Delta)p^{-sk} = (1 - a(p, \Delta)p^{-s} + p^{11-2s})^{-1}. \tag{2.45}$$

Now we use ⓵ⓑ in the above equation and obtain

$$\sum_{n=1}^{\infty} a(n, \Delta)n^{-s} = \prod_{p:\text{prime}} \left( \sum_{k=0}^{\infty} a(p^k, \Delta)p^{-ks} \right) \tag{2.46}$$

$$= \prod_{p:\text{prime}} (1 - a(p, \Delta)p^{-s} + p^{11-2s})^{-1}. \tag{2.47}$$

### 2.1.5 Automorphic representation on $GL_2(\mathbb{A}_{\mathbb{Q}})$

#### (a) $p$-adic number and adéle

Firstly we work on the simplest case. For a given prime $p \in \mathbb{Z}$, any nonzero rational number $a$ can be written as

$$a = p^m \frac{u}{v} \qquad m \in \mathbb{Z},\ p \text{ and each of } u,\ v \in \mathbb{Z} \text{ are relatively prime.} \tag{2.48}$$

Then we define the **$p$-adic valuation** $\text{ord}_p(a)$ of $a$ by $\text{ord}_p(a) = m$ and set $\text{ord}_p(0) = \infty$. We introduce the **$p$-adic absolute value** $|a|_p$ of $a \in \mathbb{Q}$ by

$$|a|_p = \begin{cases} p^{-\text{ord}_p(a)} & a \neq 0 \\ 0 & a = 0 \end{cases} \tag{2.49}$$

The $p$**-adic number field** $\mathbb{Q}_p$ is a completion of $\mathbb{Q}$ by the $p$**-adic metric**

$$d_p(a,b) = |a-b|_p \quad a,b \in \mathbb{Q}. \tag{2.50}$$

$\mathbb{Q}_p$ is a field of characteristic 0. Let

$$\mathbb{Z}_p = \{a \in \mathbb{Q}_p : \mathrm{ord}_p(a) \geq 0\} \tag{2.51}$$

be the ring of $p$-adic integers.

**Example 2.1.4.**

Consider the case $x = \frac{1}{2}$ for $p = 2$ and 3. For $p = 2$ one has $\mathrm{ord}_2(x) = -1$ hence $x$ is not 2-adic integer. On the contrary, for $p = 3$ one has $\mathrm{ord}_3(x) = 0$ hence $x$ is 3-adic integer.

In addition to the method we have used above, there are several ways to define $\mathbb{Z}_p$. A standard way is by means of **inverse limit** defined as follows.

**Definition 2.1.5.** Given a sequence consists of sets $X_n (n = 1, 2, \cdots)$ and maps $f_n : X_n \to X_{n+1} (n = 1, 2, \cdots)$

$$\cdots \xrightarrow{f_3} X_3 \xrightarrow{f_2} X_2 \xrightarrow{f_1} X_1, \tag{2.52}$$

we define the **inverse limit** of $X_n$ by

$$\varprojlim X_n = \left\{ (a_n)_{n\geq 1} \in \prod_{n=1}^{\infty} X_n : f_n(a_{n+1}) = a_n \text{ for all } n \geq 1 \right\}. \tag{2.53}$$

Indeed, one can show that for a prime $p$

$$\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^n\mathbb{Z}. \tag{2.54}$$

In accord with the terminology used for general algebraic fields, we include infinity $\infty$ to prime, which is called the **infinite place** associated with $\mathbb{Q}_\infty = \mathbb{R}$, while a finite prime $p$ is called the **finite place**. We denote the places by $v$.

The **adéle** ring $\mathbb{A}_\mathbb{Q}$ of $\mathbb{Q}$ is defined by the restricted direct product

$$\mathbb{A}_\mathbb{Q} = \prod_v{}' \mathbb{Q}_v = \mathbb{R} \times \prod_{p:\text{prime}} \mathbb{Q}_p \tag{2.55}$$

$$= \{(a_\infty, a_2, a_3, a_5, \cdots) : a_p \in \mathbb{Z}_p \text{ for all but finitely many } p\}. \tag{2.56}$$

The rational numbers are **embedded diagonally** into $\mathbb{A}_\mathbb{Q}$ naturally

$$\mathbb{Q} \ni x \mapsto (x, x, x, x, \cdots) \in \mathbb{A}_\mathbb{Q}. \tag{2.57}$$

We consider a group of $n \times n$ invertible matrices with entries in $\mathbb{A}_\mathbb{Q}$ denoted by $GL_n(\mathbb{A}_\mathbb{Q})$

$$GL_n(\mathbb{A}_\mathbb{Q}) = \left\{ (g_\infty, g_2, g_3, g_5, \dots) \in GL_n(\mathbb{R}) \times \prod_p GL_n(\mathbb{Q}_p) \right\} \tag{2.58}$$

such that $g_p \in GL_n(\mathbb{Z}_p)$ for all but finitely many $p$.

### (b) Automorphic representation

Let $G$ be a locally compact group and $\Gamma$ be its discrete subgroup. We define an action of $G$ on the Hilbert space $L^2(\Gamma \setminus G)$ with respect to a Haar measure on $G$ by

$$R : L^2(\Gamma \setminus G) \ni \psi \mapsto \psi_g \in L^2(\Gamma \setminus G), \tag{2.59}$$

where

$$\psi_g(x) = \psi(xg). \tag{2.60}$$

Then we obtain a representation of $G$

$$G \to \mathrm{Aut}_\mathbb{C}(L^2(\Gamma \setminus G)), \tag{2.61}$$

where $\mathrm{Aut}_\mathbb{C}(L^2(\Gamma \setminus G))$ is the set of isomorphisms of $L^2(\Gamma \setminus G)$ whose restrictions on $\mathbb{C}$ are the identity map $id_\mathbb{C}$. We call its subrepresentation **automorphic representation**. In particular, given an automorphic form $\varphi \in \mathrm{L}^2(\Gamma \setminus G)$, the natural action of $G$ on a subspace $V_\varphi = \{\varphi_g : g \in G\}$ gives the automorphic representation

$$G \to \mathrm{Aut}_\mathbb{C}(V_\varphi). \tag{2.62}$$

Let $K$ be a **number field**, namely a subfield of $\mathbb{C}$, and $v$ a **place** of $K$. An automorphic representation related to the Langlands program is an irreducible representation

$$\pi : GL_n(\mathbb{A}_K) \to \mathrm{Aut}_\mathbb{C}(L^2(GL_n(K) \setminus GL_n(\mathbb{A}_K))). \tag{2.63}$$

$\pi$ can be decomposed to a tensor product

$$\pi = \otimes_v \pi_v \tag{2.64}$$

of irreducible representations of the groups $GL_n(K_v)$. The **Hasse-Weil $L$-function** for the automorphic representation $\pi$ is an Euler product defined by

$$L(s,\pi) = \prod_{v\leq\infty} L(s,\pi_v) = \prod_{v\leq\infty} \det(1 - t(\pi_v)N(v)^{-s})^{-1}. \tag{2.65}$$

**(c) Relation with Fourier transformation**

We restrict ourself to the simplest example $G = \mathbb{R}$ and $\Gamma = \mathbb{Z}$ which amounts to the theory of Fourier series as follows. For $y \in \mathbb{Z}\setminus\mathbb{R}$ and $\psi \in L^2(\mathbb{Z}\setminus\mathbb{R})$ we define (2.60) by

$$(R(y)\psi)(x) = \psi(x+y), \tag{2.66}$$

then the Peter-Weyl theory allow us to decompose $R$ and $L^2(\mathbb{Z}\setminus\mathbb{R})$ irreducibly

$$R = \oplus_{n\in\mathbb{Z}}\chi_n, \tag{2.67}$$

$$L^2(\mathbb{Z}\setminus\mathbb{R}) = \oplus_{n\in\mathbb{Z}}L_n, \tag{2.68}$$

where the pairs $(\chi_n, L_n)$ are irreducible representations such that

$$\chi_n : \mathbb{Z}\setminus\mathbb{R} \to \mathbb{C}^\times,\ t \text{ mod}1 \to e^{2\pi i n t}, \tag{2.69}$$

and $L_n$ is a vector space with a basis $\{e^{2\pi i n t}\}_{n\in\mathbb{Z}}$. We define an operator $R(f) = \oplus_n\chi_n(f) : L^2(\mathbb{Z}\setminus\mathbb{R}) \to L^2(\mathbb{Z}\setminus\mathbb{R})$ for $f \in L^1(\mathbb{Z}\setminus\mathbb{R})$ by

$$\chi_{-n}(f) = \int_{\mathbb{Z}\setminus\mathbb{R}} f(t)\chi_{-n}(t) = \int_0^1 f(t)e^{-2\pi i n t}dt = \hat{f}(n), \tag{2.70}$$

which are the Fourier coefficients of $f$. Since the unitary dual group of $\widehat{\mathbb{Z}\setminus\mathbb{R}}$ is $\{\chi_n\}_{n\in\mathbb{Z}} \cong \mathbb{Z}$, the Fourier inverse transformation for $f$ is given by

$$f(t) = \int_{\mathbb{Z}} \chi_{-n}(f)\chi_{+n}(t)d\mu(\chi_n) \tag{2.71}$$

$$= \sum_{n\in\mathbb{Z}} \hat{f}(n)e^{2\pi i n t}, \tag{2.72}$$

where $d\mu$ is the Plancherel measure $d\mu(\chi_n) = 1$.

An essential formula related to the Langlands program is the **Poisson summation formula** for a function $g$ on $\mathbb{R}$

$$\sum_{n=-\infty}^{\infty} g(n) = \sum_{n=-\infty}^{\infty} \hat{g}(n), \tag{2.73}$$

where $\hat{g}$ is the Fourier transform

$$\hat{g}(y) = \int_{-\infty}^{\infty} g(x) e^{-2\pi i y x} dx. \tag{2.74}$$

To show the Poisson summation formula (2.73), we introduce

$$f(x) = \sum_{n=-\infty}^{\infty} g(x+n), \tag{2.75}$$

and then $f(x) = f(x+1)$ enable us to expand $f(x)$ into the Fourier series

$$f(x) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi i n x} \tag{2.76}$$

$$a_n = \int_0^1 f(x) e^{-2\pi i n x} dx = \int_{-\infty}^{\infty} g(x) e^{-2\pi i n x} dx = \hat{g}(n) \tag{2.77}$$

Set $x = 0$ in the above equations (2.75) and (2.76) leads to the Poisson summation formula (2.73).

### 2.1.6 Elliptic curves

A generic elliptic curve whose coefficients are integer is a curve of the form

$$E : y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6, \quad a_1, \cdots, a_6 \in \mathbb{Z}, \tag{2.78}$$

which dose not have any multiple root. This condition is equivalent to that **discriminant** $\Delta \neq 0$, which is defined as follows. Let

$$b_2 = a_1^2 + a_4, \tag{2.79}$$

$$b_4 = 2a_4 + a_1 a_3, \tag{2.80}$$

$$b_6 = a_3^2 + 4a_6, \tag{2.81}$$

$$b_8 = a_1^2 a_6 + 4a_2 a_6 - a_1 a_3 a_4 + a_2 a_3^2 - a_4^2 \tag{2.82}$$

We define the discriminant $\Delta$ of $E$ to be

$$\Delta = -b_2^2 b_8 - 8b_4^3 - 27b_6^2 + 9b_2 b_4 b_6. \tag{2.83}$$

Note that over $\mathbb{Q}$, when we change variables so that $Y = \frac{1}{2}(y - a_1 x - a_3),\ X = x$, the above elliptic curve (2.78) is transformed into

$$Y^2 = 4X^3 + b_2 X^2 + 2b_4 X + b_6. \tag{2.84}$$

An elliptic curve over $\mathbb{Q}$ can be transformed to the form (2.78) by proper choice of variables. We say that an elliptic curve $E$ over $\mathbb{Q}$ has **good reduction** at a prime $l$ if $E$ can be an elliptic curve over $\mathbb{F}_l$, namely $p$ does not divide the discriminant $\Delta$, otherwise $E$ has **bad reduction** at $l$. Note that any elliptic curve has good reduction for all but finitely many primes since the number of primes which divide $\Delta$ is finite.

Let $E$ has bad reduction at a prime $l$. We call $E$ has **multiplicative reduction** at $l$ if $l$ does not divide $\Delta$, otherwise we call $E$ has **additive reduction** there. We assume $l \neq 2$, then$E : y^2 = f(x)$ has multiplicative reduction is equivalent to $f(x)$ has a double root over $\mathbb{F}_l$, and $E$ has additive reduction is equivalent to $f(x)$ has a triple root.

Moreover if $E$ has multiplicative reduction at $l$ and the slopes of the tangent line at the node are in $\mathbb{F}_l$, then the reduction is said to be split multiplicative.

**Example 2.1.6.**

$$y^2 = x^3 - x = x(x-1)(x+1) \tag{2.85}$$

has good reduction except at $l = 2$.

Let $K$ be a field whose character is larger than 2. An **elliptic curve** $E$ over $K$ is an algebraic curve defined by

$$E: \quad y^2 = ax^3 + bx^2 + cx + d, \tag{2.86}$$

where $a, b, c, , d \in K,\ a \neq 0$ and the right hand side does not have any multiple root.

The elliptic curve $E$ can be transformed into

$$y^2 = 4x^3 - g_2 x - g_3, \quad g_2, g_3 \in \mathbb{Z} \tag{2.87}$$

over $\mathbb{C}$. The condition that $4x^3 - g_2 x - g_3$ is nonsingular is equivalent to a the fact that the discriminant $\Delta = g_2^3 - 27g_3^2 \neq 0$.

Let $E(K)$ be the set of $K$ rational points in $E$

$$E(K) = \{(x, y) \in K \times K : y^2 = ax^3 + bx^2 + cx + d\} \cup O, \tag{2.88}$$

where $O$ is the point at infinity. $E(K)$ has an additive group structure whose binary operation $+$ is defined by

1. $O$ is the unit element
2. If $P, Q \in E(K)$ and $P \neq Q \neq O$, let $R = (x, y)$ be the point of intersection of $E$ with the line joining $P$ and $Q$. Then $P + Q = (x, -y)$. If $P = Q \neq O$, then replace $R$ by the point of intersection with $E$ and the tangent at $P$.
3. The inverse of $P = (x, y)$ is $P^{-1} = (x, -y)$

**Theorem 2.1.7.** In general, a group $E(K)$ is a finitely generated abelian group, namely

$$E(K) \cong \mathbb{Z}^{\oplus r} \oplus E(K)_{\mathrm{tors}}, \tag{2.89}$$

where $E(K)_{\mathrm{tors}}$ is a certain finite group and called the **torsion subgroup** of $E(K)$. Especially, if $K = \mathbb{Q}$, then $E(K)_{\mathrm{tors}}$ is isomorphic to one of the following 15 groups:

$$\cdot\, \mathbb{Z}/m\mathbb{Z},\ 1 \leq m \leq 12,\ m \neq 11,$$

$$\cdot\, \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2m\mathbb{Z},\ 1 \leq m \leq 4.$$

**Example 2.1.8.** We consider an elliptic curve over $\mathbb{Q}$, $E : y^2 = x^3 + 2x + 3$ (Fig 2.1). If we choose $P = (-1, 0)$, and $Q = (3, -6)$, then $R = \left(\frac{1}{4}, -\frac{15}{8}\right)$ is the intersection point with $E$ and the line through $P$ and $Q$. Therefore we obtain $P+Q = \left(-\frac{1}{4}, -\frac{8}{15}\right)$. One can show that $2P = O$ and $nQ \neq O$ for all $n \in \mathbb{Z}$. Moreover it is know that $P$ and $Q$ generate $E(\mathbb{Q}) = \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

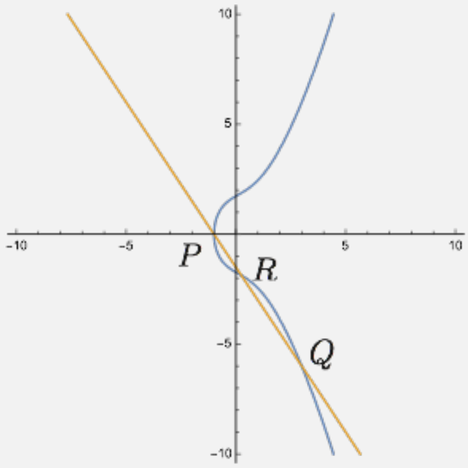



Figure 2.1: $E : y^2 = x^3 + 2x + 3$

Let $E$ be an elliptic curve over $\mathbb{Q}$. Recall that $E$ can be transformed into

$$y^2 = 4x^3 - g_2 x - g_3, \quad g_2, g_3 \in \mathbb{Z} \tag{2.90}$$

over $\mathbb{C}$. Let $\omega_1, \omega_2$ be a pair of complex numbers which are linearly independent on $\mathbb{R}$ and we denote $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ the lattice spanned by $\omega_1, \omega_2$. Topologically the quotient $\mathbb{C}/L$ is homeomorphic to a torus $S^1 \times S^1$.

we define the **Weierstraß $\wp$-function** by

$$\wp(z) = \frac{1}{z^2} + \sum_{(m,n)\neq(0,0)} \left( \frac{1}{(z - m\omega_1 - n\omega_2)^2} - \frac{1}{(m\omega_1 + n\omega_2)^2} \right), \tag{2.91}$$

The following theorem indicates that there is one to one correspondence between a point on $z \in \mathbb{C}/L$ and a rational point of $E(\mathbb{C})$ which is given explicitly by the Weierstraß $\wp$-function.

**Theorem 2.1.9.** For an elliptic curve $y^2 = 4x^3 - g_2x - g_3$, there exist $\omega_1, \omega_2 \in \mathbb{C}$ which are linearly independent over $\mathbb{R}$ and satisfy

$$g_2 = 60 \sum_{(m,n)\neq(0,0)} \frac{1}{(m\omega_1 + n\omega_2)^4}, \quad g_2 = 140 \sum_{(m,n)\neq(0,0)} \frac{1}{(m\omega_1 + n\omega_2)^6}. \tag{2.92}$$

Furthermore for the Weierstraß $\wp$-function we obtain an isomorphism of abelian group

$$\mathbb{C}/L \ni z \mapsto (\wp(z), \wp'(z)) \in E(\mathbb{C}), \tag{2.93}$$

where $\wp'(z) = \frac{d}{dz}\wp(z)$.

When a pair of two complex numbers $(\omega_1, \omega_2)$ which are linearly independent over $\mathbb{R}$ is given, we can order them so that $\mathrm{Im}(\omega_1/\omega_2) > 0$. Let $(\omega_1', \omega_2')$ be another pair of elements in $L$, then there exists a matrix $A$ such that

$$\begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} = A \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}, \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}). \tag{2.94}$$

The pair $(\omega_1', \omega_2')$ become basis of $L$ if and only if $\det A = \pm 1$. Moreover for $z = \omega_1/\omega_2,\ z' = \omega_1'/\omega_2'$, we see

$$\mathrm{Im}(z') = \mathrm{Im}\left(\frac{az+b}{cz+d}\right) = \frac{(ad-bc)\mathrm{Im}(z)}{|az+b|^2} \tag{2.95}$$

therefore $\mathrm{Im}(z') > 0$ if and only if $\det A > 0$. So $SL_2(\mathbb{Z})$ naturally acts on the set of basis $(\omega_1, \omega_2)$ for $L$ with $\mathrm{Im}(\omega_1/\omega_2) > 0$. Let $M = \{(\omega_1, \omega_2) : \mathrm{Im}(\omega_1/\omega_2) > 0\}$ and $\Lambda$ be the set of lattices in $\mathbb{C}$. We have proved that $SL_2(\mathbb{Z}) \setminus M \cong \Lambda$.

Since $\mathbb{C}^\times$ naturally acts on $M$ by scaler multiplication, $M/\mathbb{C}^\times$ can be identified with $H$ by the map

$$M/\mathbb{C}^\times \ni (\omega_1, \omega_2) \mapsto \omega_1/\omega_2 \in H. \tag{2.96}$$

Combing the above two identifications, we arrive at the following statement.

**Proposition 2.1.10.**

$$\Lambda/\mathbb{C}^\times \cong SL_2(\mathbb{Z}) \setminus M/\mathbb{C}^\times \cong SL_2(\mathbb{Z}) \setminus H. \tag{2.97}$$

### 2.1.7 Galois representation

Let $K$ denote a number field and $L$ a finite extension of $K$. We denote $\mathrm{Aut}_K(L)$ the set of automorphisms $\sigma$ over $K$ such that $\sigma|_K = \mathrm{id}_K$. $\mathrm{Aut}_K(L)$ has a natural group structure whose binary operation is given by composition of the automorphisms. Generally there is the inequality $\#(\mathrm{Aut}_K(L)) \leq [L:K]$. When the inequality is saturated we call $L$ is a **Galois extension** of $K$ and $\mathrm{Aut}_K(L)$ a **Galois group**, and we denote it by $\mathrm{Gal}(L/K)$.

**Example 2.1.11.** If $K = \mathbb{R}$ and $L = \mathbb{C}$, then $[\mathbb{C}:\mathbb{R}] = 2$ and $\mathrm{Aut}(\mathbb{C}/\mathbb{R}) = \{\mathrm{id}_{\mathbb{C}}, \sigma\}$, where $\sigma$ is complex conjugation. Hence $\mathbb{C}$ is a Galois extension of $\mathbb{R}$ and $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) = \{\mathrm{id}_{\mathbb{C}}, \sigma\} \cong \mathbb{Z}/2\mathbb{Z}$.

Let $E(\mathbb{C})_n$ be the subgroup of $\mathbb{C}/L$ defined by

$$E(\mathbb{C})_n = \{P \in \mathbb{C}/L : nP = 0\} \cong \mathbb{Z}/n\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}. \tag{2.98}$$

$P \in E(\mathbb{C})_n$ can be written in such a way that

$$P = \alpha e_1 + \beta e_2, \tag{2.99}$$

where $\alpha, \beta \in \mathbb{Z}/n\mathbb{Z}$ and $e_1, e_2$ are basis of $\mathbb{C}/L$. The action of $\sigma$ indeed maps $E(\mathbb{C})_n$ to itself since $n\sigma(P) = \sigma(nP) = 0$. Then when we put the image of $e_1, e_2$ with respect to $\sigma$ by

$$\sigma(e_1) = ae_1 + be_2, \quad \sigma(e_2) = ce_1 + de_2, \tag{2.100}$$

we succeed to construct a homomorphism

$$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\mathbb{Z}/n\mathbb{Z}) \tag{2.101}$$

$$\sigma \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{2.102}$$

For our purpose to construct a Galois representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ over $\mathbb{Z}_l$, where $l$ is a prime, we define the **Tate module** $T_l E(\mathbb{C})$ to be

$$T_l E(\mathbb{C}) = \varprojlim E(\mathbb{C})_{l^n}, \tag{2.103}$$

where the inverse limit of $E(\mathbb{C})_{l^n}$ is taken with respect to the map $E(\mathbb{C})_{l^{n+1}} \to E(\mathbb{C})_{l^n}$.

The Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on $T_l E(\mathbb{C})$ in a similar manner as it did on $E(\mathbb{C})_n$. Let $e_1, e_2$ be a set of basis of $T_l E(\mathbb{C})$ as $\mathbb{Z}_l$-module and the action of $\sigma$ be

$$\sigma(e_1) = ae_1 + be_2, \quad \sigma(e_2) = ce_1 + de_2, \tag{2.104}$$

where $a, b, c, d \in \mathbb{Z}_l$. Then we obtain a Galois representation

$$\rho_{E,l} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\mathbb{Z}_l) \tag{2.105}$$

$$\sigma \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{2.106}$$

We consider the case of $K = \mathbb{F}_p$, where $p$ is a prime, and $L = \mathbb{F}_q$, where $q = p^n$ ($n \in \mathbb{Z}$), be a finite field with $q$ elements. It is known that $\mathrm{Gal}(\mathbb{F}_q/\mathbb{F}_p) \cong \mathbb{Z}/n\mathbb{Z}$ is generated by **Frobenius automorphism**

$$\mathrm{Frob}_p : \mathbb{F}_q \ni x \mapsto x^p \in \mathbb{F}_q. \tag{2.107}$$

One reason the Frobenius automorphism is important for a Galois representation is as follows.

**Theorem 2.1.12.** Let $L$ be a finite Galois extension of $\mathbb{Q}$. For any conjugate class $c$ of $\mathrm{Gal}(L/\mathbb{Q})$ there exist infinite number of primes $p$ such that $p$ are unramified at $L$ and $c = \mathrm{Frob}_p$.

We define a Galois representation by a composite map

$$\rho_{E,\mathbb{Q}_l} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \xrightarrow{\rho_{E,l}} GL_2(\mathbb{Z}_l) \to GL_2(\mathbb{Q}_l), \tag{2.108}$$

which is known to be irreducible for any prime $l$.

The **Artin $L$-function** is defined by

$$L(s, \rho_{E,\mathbb{Q}_l}) = \prod_{p:\text{prime}} \det(1 - \rho_{E,\mathbb{Q}_l}(\mathrm{Frob}_p)p^{-s})^{-1}. \tag{2.109}$$

**Theorem 2.1.13.** Let $E$ be an elliptic curve over $\mathbb{Q}$ and $l, p$ ($l \neq p$) be primes at which $E$ has good reduction. Then we have

$$\det(\rho_p(\mathrm{Frob}_l)) = l. \tag{2.110}$$

Moreover if we put

$$\mathrm{Tr}(\rho_p(\mathrm{Frob}_l)) = a_l, \tag{2.111}$$

then $a_l$ is integer and

$$\#E(\mathbb{F}_l) = l + 1 - a_l, \tag{2.112}$$

where $\#E(\mathbb{F}_l)$ is the number of $\mathbb{F}_l$ rational points of $E$.

Hence the main part of the Artin $L$-function (2.109) where $E$ has good reduction can be written as

$$\prod_{\text{good } p} (1 - a_p p^{-s} + p^{1-2s})^{-1}. \tag{2.113}$$

For a bad prime $p$, let

$$a_p = \begin{cases} 0 & \text{additive reduction} \\ -1 & \text{non split multiplicative reduction} \\ 1 & \text{split multiplicative reduction} \end{cases} \tag{2.114}$$

and then the Artin $L$-function becomes

$$L(s, \rho_{E,\mathbb{Q}_l}) = \prod_{\text{bad } p} (1 - a_p p^{-s})^{-1} \prod_{\text{good } p} (1 - a_p p^{-s} + p^{1-2s})^{-1}. \tag{2.115}$$

Here we give a brief introduction to a mysterious result derived by Eichler. Our elliptic curve over $\mathbb{Q}$ is as follows:

$$E: \ y^2 + y = x^3 - x^2. \tag{2.116}$$

We can transform this elliptic curve to

$$Y^2 = X^3 - 4X^2 + 16 \tag{2.117}$$

under $Y = 8y - 4$ and $X = 4x$. This implies the elliptic curve (2.116) has bad reduction at 11 since

$$X^3 - 4X^2 + 16 \equiv (X + 1)^2 (X - 6) \mod 11. \tag{2.118}$$

$\mathbb{F}_l$ rational points $(x, y)$ of $E$ are

$$\begin{array}{rl} \mathbb{F}_2 & O,(0,0),(0,1),(1,0),(1,1) \\ \mathbb{F}_3 & O,(0,0),(0,2),(1,0),(1,2) \\ \mathbb{F}_5 & O,(0,0),(0,4),(1,0),(1,4) \\ \mathbb{F}_7 & O,(0,0),(0,6),(1,0),(1,6),(4,2),(4,4),(5,1),(5,5),(6,3) \\ \mathbb{F}_{11} & O,(0,0),(0,10),(1,0),(1,10),(5,3),(5,7),(7,5),(8,5),(10,4),(10,6) \end{array}$$

Then $a_l = l + 1 - \#E(\mathbb{F}_l)$ is listed in the following table.

| $l$ | 2 | 3 | 5 | 7 | $11\cdots$ |
|---|---|---|---|---|---|
| $a_l$ | -2 | -1 | 1 | -2 | $1\cdots$ |

Surprisingly the $a_l$ coincide with the coefficients $a(l)$ of the Fourier series of the following automorphic form of weight 2 and revel 11:

$$\begin{aligned} q\prod_{n=1}^{\infty}(1-q^n)^2(1-q^{11n})^2 =& \sum_{n=1}^{\infty} a(n)q^n \\ =& q - 2q^2 - q^3 + 2q^4 + q^5 + 2q^6 - 2q^7 - 2q^9 - 2q^{10} + q^{11} - 2q^{12} + 4q^{13}\cdots \end{aligned}$$

Recall that the Hasse-Weil $L$-function of this elliptic curve is

$$L(s, E) = \sum_{n=1}^{\infty} a(n)q^n \tag{2.119}$$

$$= (1 - a(11)11^{-s}) \prod_{p \neq 11}(1 - a(p)p^{-s} + p^{1-2s}). \tag{2.120}$$

Therefore the Artin $L$-function associated with the elliptic curve $y^2 + y = x^3 - x^2$ and the Hasse-Weil $L$-function associated with the automorphic form $q\prod_{n=1}^{\infty}(1-q^n)^2(1-q^{11n})^2$ coincide with each other.

There are some other cases for instance

$$\begin{aligned} y^2 = x^3 + 1 &\longleftrightarrow q\prod_{n=1}^{\infty}(1-q^{6n})^4 \\ y^2 = x^3 - x &\longleftrightarrow q\prod_{n=1}^{\infty}(1-q^{4n})^2(1-q^{8n})^2 \end{aligned}$$

More generally, for a given irreducible $n$-dimensional Galois representation

$$\sigma : \mathrm{Gal}(\overline{K}/K) \to GL_n(\mathbb{C}) \quad (\text{or } GL_n(\mathbb{Q}_l)), \tag{2.121}$$

the **Artin $L$-function** is defined by

$$L(s,\sigma) = \prod_{v\leq\infty} L(s,\sigma_v) = \prod_{v\leq\infty} \det(1-\sigma(\mathrm{Frob}_v)N(v)^{-s})^{-1}. \tag{2.122}$$

The Langlands conjecture is stated below.

**Langlands conjecture**

For any Galois representation $\sigma$, there exists an automorphic representation $\pi$ such that

$$L(s,\sigma) = L(s,\pi), \tag{2.123}$$

namely, **eigenvalues of Frobenius** correspond to **eigenvalues of Hecke operators**.

## 2.2 The geometric Langlands Program

### 2.2.1 Geometric interpretation of the Langlands program

In the previous section, we saw that the Langlands correspondence implied that $n$-dimensional representations of the Gaois group $\mathrm{Gal}(\overline{K}/K)$ could be described by automorphic representations of $GL_n(\mathbb{A}_K)$. In this section, we would like to explain the geometric analogue of them. The geometric Langlands correspondence has been constructed by [32, 5, 7, 8] and there is a huge number of surveys today. Some readable reviews are [9, 10, 11] and we recall some discussions given in [10].

**(a)Geometric counterparts of Galois representations**

We first give the geometric counterpart of the Galois group in brief. We will back to this topic in section 2.2.2. Let $C$ be a smooth projective connected algebraic curve over $\mathbb{C}$ and $F$ be the field of rational functions on $C$. We restrict ourself to the case where all coverings of $C$ are unramified. Let $G$ be a complex reductive algebraic group. Then a homomorphism $\mathrm{Gal}(\overline{F}/F) \to {}^LG$ is a homomorphism $\pi_1(C) \to {}^LG$, which can be understood in terms of a ${}^LG$ bundle $(\mathcal{E}, \nabla)$ with a flat connection on $C$, where $\mathcal{E}$ is a holomorphic ${}^LG$ bundle with a holomorphic connection $\nabla$ over $C$. Both of $\mathcal{E}$ and $\nabla$ originate from the $(0,1)$ and $(1,0)$ components of the given flat connection.

**(b) Geometric counterparts of automorphic representations**

Next, we consider the set of isomorphism classes of principal $G$-bundle on $C$, which we denote by $\mathrm{Bun}_G$. If $G = GL_1$, then $\mathrm{Bun}_G$ turns out to be the Picard variety Pic. However, $\mathrm{Bun}_G$ is not an algebraic variety for a general $G$, nevertheless the derived category, on which the Hecke functors act, of $\mathcal{D}$-modules on $\mathrm{Bun}_G$ is defined in general. The geometric analogue of an automorphic representation is believed to be a perversive sheaf [3, 22] on $\mathrm{Bun}_G$. According to the **Riemann-Hilbert correspondence** [27, 35], if $C$ is an algebraic variety, then the category of (holonomic) $\mathcal{D}$-modules with regular singularities on $C$ is equivalent to the category of perverse shaves on $C$. So we may pass the automorphic representations to the $\mathcal{D}$ modules. The $\mathcal{D}$ we consider for the geometric Langlands is so-called Hecke eigensheavs, which we roughly explain here.

Let $V$ be a finite dimensional representation of ${}^LG$ and $x$ be a point in $C$. The Hecke functors are labeled by $(x, V)$. We consider the simplest case $G = GL_1$. We denote by $\mathcal{L}$ a line bundle $\mathcal{L}$ on $C$ and by $\mathcal{L}(x)$ the line bundle whose sections are sections of $\mathcal{L}$ which have a pole of order 1 at $x \in C$. We consider the map $h_x : \mathrm{Pic} \to \mathrm{Pic}$, which sends $\mathcal{L}$ to $\mathcal{L}(x)$. Regarding $1 \in \mathbb{Z}$ as the set of one-dimensional representations of ${}^LG = GL_1$, the Hecke functor $H_{1,x}$ satisfies $H_{1,x}(\mathcal{F}) = h_x^*(\mathcal{F})$, where $\mathcal{F}$ is a $\mathcal{D}$-module on $\mathrm{Bun}_{GL_1}$. Recall that automorphic functions in the Langlands program are eigenfunctions of the Hecke operators (2.14). The geometric analogues of them are some sheaves, called Hecke eigensheavs, on $\mathrm{Bun}_G$. In our case, the Hecke eigensheaf is a certain $\mathcal{D}$-module $\mathcal{F}$ on $\mathrm{Bun}_G$ such that

$$\begin{aligned} H_V(\mathcal{F}) &\simeq V_{\mathcal{E}} \boxtimes \mathcal{F} \\ H_{V,x}(\mathcal{F}) &\simeq V \otimes \mathcal{F}, \ \ x \in C, \end{aligned} \tag{2.124}$$

where $(\mathcal{E}, \nabla)$ is a flat ${}^LG$ bundle on $C$ and $V_{\mathcal{E}} = \mathcal{E} \times_{{}^LG} V$ is the flat vector bundle on $C$ associated to $\mathcal{E}$ and $V$.

In summary, the geometric Langlands correspondence roughly claims that for each holomorphic ${}^LG$-bundle $\mathcal{E}$ with a holomorphic connection $\nabla$ on a complex algebraic curve $C$, there is a corresponding $\mathcal{D}$-module $\mathcal{F}_{\mathcal{E}}$ on $\mathrm{Bun}_G$. Here $\mathcal{F}_{\mathcal{E}}$ is a Hecke eigensheaf with eigenvalue $\mathcal{E}$.

### 2.2.2 Flat bundles and local systems

In this part, we consider the left side of the Langlands program in the figure 1. We will mostly take $G$ as a Lie group if there is no remark.

Let $\mathcal{V}$ be the sheaf of smooth vector fields on a real manifold $M$, $\mathcal{O}_M$ be the sheaf of smooth functions on $M$, and $E$ be a complex vector bundle on $M$. We denote by $\mathrm{End}(E)$ the bundle of endomorphisms of $E$. Then a connection $\nabla$ is a map

$$\nabla : \mathcal{V} \to \mathrm{End}(E) \tag{2.125}$$

which assigns an endomorphism $\nabla_v$ for a smooth vector field $v$ defined on an open set $U \subset M$ such that

1. $\nabla_{v+w} = \nabla_v + \nabla_w$.

2. $\nabla_{fv} = f\nabla_v$ for $f \in \mathcal{O}_X$.

3. $\nabla_v(f\phi) = f\nabla_v(\phi) + \phi \cdot v(f)$ for $f \in \mathcal{O}_M$ and $\phi \in \mathrm{End}(E)$.

The first and second conditions are linearity of $\nabla_v$ and the third condition is the Leibniz rule.

Let $U$ be a sufficiently small open subset of $M$ on which $E$ is trivial. We choose coordinates $x_1, \cdots, x_n$ associated with $U$. A connection is given by the form

$$\nabla_{x_i} = \partial_{x_i} + A_i(x), \tag{2.126}$$

where $A_i,\ i = 1, \cdots, n$, are $m \times m$ matrix-valued smooth functions on $U$. We call a connection is **flat** if the curvature vanish $F_{ij} = 0$, where

$$\begin{aligned} F_{ij} &:= [\nabla_{x_i}, \nabla_{x_j}] \\ &= \nabla_{x_i} A_j - \nabla_{x_j} A_i + [A_i, A_j]. \end{aligned} \tag{2.127}$$

With respect to a section $s : M \to E$ and a connection $\nabla$, we consider the differential equation

$$\nabla_v s(x) = 0, \tag{2.128}$$

where $v$ is a smooth vector field defined on a small open subset $U$. According to the theorem of existence and uniqueness of linear differential equations, for a given point $x_0 \in M$ and a given vector

$V$ in the fiber $E_{x_0}$, there is an unique local solution $s_V$ of (2.135) such that $s_V(x_0) = V$. Each $s_V$ for $V \in E_{x_0}$ is a section of $E$ over $U$, hence it gives the natural trivialization of $E$ over $U$.

When we consider a loop in $M$ with the base point $x_0$, we have a homomorphism $\pi_1(M, x_0) \to \mathrm{Aut}E_{x_0}$. If we identify $E_{x_0}$ with $\mathbb{C}^m$, then the homomorphism becomes $\pi_1(M, x_0) \to GL_n(\mathbb{C})$, which is called a **holonomy homomorphism** of a flat bundle $E$. More strongly the equivalence class of isomorphisms of **flat** $G$-bundles on a manifold $M$ is equivalent to the conjugacy class of homomorphisms $\pi_1(M) \to G$.

**Theorem 2.2.1.** Let $M$ be a smooth manifold and $G$ be a Lie group. Then with respect to a **flat** $G$-bundle on $M$, the correspondence which assigns its holonomy isomorphism $\pi_1(M) \to G$ is one-to-one.

This gives us a suitable reason why we work on the space of flat ${}^LG$-bundles for the geometric Langlands program. Recall that a Galois representation $\mathrm{Gal}(\overline{F}/F) \to^L G$ is replaced by a homomorphism $\pi_1(C) \to^L G$ in section 2.2.1. More precisely we will work on holomorphic ${}^LG$-bundles on a complex algebraic variety $C$, and we will add some conditions on the above theorem. In order to consider this, we first recall some definitions of those concepts.

Let $E$ be a holomorphic vector bundle on a complex variety $C$. For an open subset $U \subset C$ we choose its holomorphic coordinates $z_1 \cdots, z_n$. The holomorphic sections of $E$ are sections which are annihilated by all $\overline{\partial}$-operators $\nabla_{\overline{z}_i}$, where $\overline{z}_1, \cdots, \overline{z}_n$ are anti-holomorphic coordinates associated with $U$. We introduce a holomorphic structure on $E$ by defining a connection $\nabla$ in such a way that

$$\begin{aligned} \nabla_{z_i} &= \partial_{z_i} + A_i(z) \\ \nabla_{\overline{z}_i} &= \partial_{\overline{z}_i}, \end{aligned} \tag{2.129}$$

where $A_i(z),\ i = 1, \cdots, n$, are matrix-valued holomorphic functions of $z_j,\ j = 1, \cdots, n$. The flatness conditions are defined as before: $F_{ij} = [\nabla_{z_i}, \nabla_{z_j}] = 0$.

From now on we consider a holomorphic $G$-bundle $\mathcal{P}$ on $C$. Let $V$ be a finite dimensional vector space and $\rho : G \to GL(V)$ be a representation of $G$. A flat connection on $\mathcal{P}$ can be constructed from the **associated vector bundle**:

$$V_{\mathcal{P}} = \mathcal{P} \times_G V = (P \times V)/G, \tag{2.130}$$

where the action of $G$ on $\mathcal{P} \times V$ is defined by

$$g : (u, y) \mapsto (ug, \rho(g)^{-1}y) \ \text{ for } g \in G,\ (u, y) \in \mathcal{P} \times V \tag{2.131}$$

and we identify $(u, y) \sim (ug, \rho(g)^{-1}y)$. If $\{g_{ij}\}$ is a translation function of $\mathcal{P}$, then the translation function of $V_{\mathcal{P}}$ is given by $\{\rho(g_{ij})\}$.

We choose a sufficiently small open set $U$ so that $\mathcal{P}$ is trivial there. Then a flat connection on $\mathcal{P}$ is given by operators

$$\nabla_{z_i} = \partial_{z_i} + a_i(z), \tag{2.132}$$

where $a_i$'s are $\mathfrak{g}$-valued holomorphic functions on $U$. A gauge transformation is given by

$$\nabla_{z_i} \mapsto \partial_{z_i} + g a_i(z) g^{-1} - (\partial_{z_i} g) g^{-1}, \tag{2.133}$$

where $g$ is a translation function, which is holomorphic on $U$ by definition. This method to endow the principle bundle $\mathcal{P}$ with a flat connection is known as the Tannakian formalism. Now the theorem we saw for a smooth real manifold $M$ is changed to as follows.

**Theorem 2.2.2.** Let $C$ be a compact complex algebraic variety and $G$ be a complex algebraic group. Then with respect to a flat $G$-bundle on $C$, the correspondence which assigns its holonomy isomorphism $\pi_1(C) \to G$ is one-to-one.

If $C$ is non-compact, generally a pole of higher order associated with the connection produces many more gauge equivalence classes of flat $G$-bundle than the number of homomorphisms $\pi_1(C) \to G$.

We denote $\mathrm{Loc}_G$ by the set of gauge equivalence class of $G$-bundles on $C$, which is called the moduli stack of local systems on $C$. We refer to a point of $\mathrm{Loc}_G$ as a **local system** on $C$. In summary, a local system is a pair $(\mathcal{P}, \nabla)$ of a holomorphic $G$-bundle $\mathcal{P}$ and a holomorphic connection $\nabla$, which brings the corresponding flat connection on $\mathcal{P}$. Note that the connections are allow to have a pole in of arbitrary order.

### 2.2.3 $\mathcal{D}$-module

We first describe fundamental concepts of $\mathcal{D}$-module with an elementary model. Let $X$ be a open set of $\mathbb{C}^n$ and $\mathcal{O}$ be the commutative ring of all holomorphic functions on $X$. We denote by $D$ the set of all linear differential operators whose coefficients are holomorphic functions on $X$. Namely, an element of $D$ has the form

$$\sum_{i_1,\cdots,i_n}^{\infty} f_{i_1\cdots i_n}\left(\frac{\partial}{\partial x_1}\right)^{i_1}\cdots\left(\frac{\partial}{\partial x_n}\right)^{i_n}, \quad f_{i_1\cdots i_n}\in\mathcal{O}, \tag{2.134}$$

where $x_1,\cdots,x_n$ are coordinates of $X$ and $i_1,\cdots,i_n$ are non-negative integers. Note that $D$ is a non-commutative ring in general and $\mathcal{O}$ is a (left) $D$-module under the action of $D$ to $\mathcal{O}$ as differential operators. To see more concrete way, we choose a differential operator $P\in D$ and consider a differential equation with respect to $\Phi\in\mathcal{O}$:

$$P\Phi=0. \tag{2.135}$$

What is the set of solutions of this equation? In order to answer this question we define a new $D$-module by $\Delta=D\cdot\Phi=D/I$, where $I=\{P\in\mathcal{D}:P\Phi=0\}$.

**Example 2.2.3.** Let $X=\mathbb{C}$ and $D=\mathbb{C}[x,\partial_x]$.

1. If $\Phi=1$, then $\Delta=\mathbb{C}[x]$.
2. if $\Phi=1/x$, which is annihilated by $\partial_x x$. Hence this case results in $\Delta=\mathbb{C}[x,1/x]$.
3. If $\Phi=\delta(0)$, the delta function supported at 0, then we have $\Delta=\mathbb{C}[x,\partial_x]/(x)=\mathbb{C}[\partial_x]$.

Then the set $\mathrm{Hom}_D(\Delta,\mathcal{O})$ of all $D$-homomorphisms from $\Delta$ to $\mathcal{O}$ gives the answer. The reason is as follows: We observe the isomorphism

$$\begin{aligned}\mathrm{Hom}_D(\Delta,\mathcal{O})&\simeq\{\phi\in\mathrm{Hom}_D(D,\mathcal{O}):\phi(P)=0\}\\&\simeq\{f\in\mathcal{O}:Pf=0\}.\end{aligned} \tag{2.136}$$

The last identification obeys from the fact $\mathrm{Hom}_D(D,\mathcal{O})\simeq\mathcal{O}$ ($\phi\mapsto\phi(1)$) and the formula $Pf=P\phi(1)=\phi(P1)=\phi(P)=0$. Hence we obtain an alternative perspective toward the space of solutions of the differential equation (2.135). This expression $\mathrm{Hom}_D(\Delta,\mathcal{O})$ has advantage since it is independent of any detail of the expression $P\Phi=0$. Importantly, this is true for the system of general linear differential operators $P_{ij}\in D$. In the sense of category, these result in to consider an opposite functor $\mathrm{Hom}_D(-,\mathcal{O})$ from the category $M(D)$ of $D$ modules to the category $M(\mathbb{C})$ of $\mathbb{C}$-modules.

So far we have considered the simplest example where $X$ is a open set of $\mathbb{C}^n$ and holomorphic functions are defined on $X$ globally. In principle, we can generalize $X$ to any complex manifold or any algebraic variety over a field $k$, and the holomorphic functions can be locally defined on $X$. In such a case, we work on the sheaf $\mathcal{O}_X$ of holomorphic functions on $X$, instead of $\mathcal{O}$, and the sheaf $\mathcal{D}_X$ of the linear differential operators on $X$. Also, instead of $D$-module on $X$, we generally consider the sheaf of $\mathcal{D}_X$-modules. Of course, the space of the solutions associated to differential operators should be sheafified.

To describe more concretely, let $X$ be a $n$-dimensional smooth algebraic variety over $\mathbb{C}$. We choose an open set $U \subset X$ with coordinates $x_1 \cdots, x_n$. The sheaf $\mathcal{V}_U$ of vector fields on $U$ is defined by

$$\mathcal{V}_U = \bigoplus_{i=1}^{n} \mathcal{O}_U \partial_{x_i}, \tag{2.137}$$

where $\partial_{x_i}$'s are operators on $U$ such that $[\partial_{x_i}, x_j] = \delta_{ij}$. Moreover $\mathcal{D}_U = \mathcal{D}_X|_U$ is

$$\mathcal{D}_U = \bigoplus_{i_1 \cdots, i_n} \mathcal{O}_U \partial_1^{i_i} \cdots \partial_n^{i_n}, \tag{2.138}$$

where $i_1 \cdots, i_n$ are non-negative integers. So this definition is a natural extension of (2.134). We say $M$ is a $\mathcal{D}_X$ module if $M$ is a sheaf on $X$ and for any open set $U \subset X$, $M(U)$ is a $\mathcal{D}_X(U)$-module and its structure commutes with the restriction map of the sheaf. In other wards, $M$ is a $\mathcal{O}_X$-module with a flat connection $\nabla$, namely $\nabla$ defines the action $\nabla_v : M \to M$ for $v \in \mathcal{V}_X$ which obeys the same properties given in section 2.2.2 just replaced $\mathrm{End}(E)$ with $M$.

### 2.2.4 Schubert variety

Now let us move on to the right side of the figure 1. For $G = GL_n$ case, the Hecke operators are parametrized by points in the Grassmannian manifold $\mathrm{Gr}(r, n)$, which is a union of Schubert varieties. In order to state those meanings, we recall some fundamental topics of them.

Let $V$ be a $n$-dimensional vector space over $\mathbb{C}$. A **flag manifold** $Fl(V)$ consists of all sequences of vector subspaces of $V$

$$0 = V_0 \subset V_1 \subset \cdots \subset V_{n-1} \subset V_n = V, \ \ \dim_{\mathbb{C}} V_i = i. \tag{2.139}$$

The manifold structure of $Fl(V)$ results from that fact that $GL(V)$ acts on $Fl(V)$ transitively by

$$g(V_0 \subset V_1 \subset \cdots \subset V_{n-1} \subset V_n) := (g(V_0) \subset g(V_1) \subset \cdots \subset g(V_{n-1}) \subset g(V_n)), \tag{2.140}$$

where $g \in GL(V)$. Then $Fl(V)$ can be identified with element of $GL(V)$ modulo $\tilde{B}(V)$, where $\tilde{B}(V)$ is the subgroup of $GL(V)$ which fixes a flag $V_\bullet = (V_0 \subset V_1 \subset \cdots \subset V_{n-1} \subset V_n)$. After all, we have $Fl(V) \cong GL(V)/B(V)$.

Now we choose $V = \mathbb{C}^n$ and write $Fl(\mathbb{C}^n) = Fl_n$. In this case $\tilde{B}(\mathbb{C}^n) = \tilde{B}_n(\mathbb{C})$ is the group of all upper triangular $n \times n$ matrices with coefficients in $\mathbb{C}$. Hence $\dim_{\mathbb{C}} Fl_n = \frac{1}{2}n(n-1)$. By using the unitary group $U_n$, $Fl_n$ can be written as $Fl_n \cong GL_n(\mathbb{C})/\tilde{B}_n(\mathbb{C}) \cong U_n/U(1)^n$. Consequently, $Fl_n$ is compact since $U_n$ is so. Let $S_n$ be the symmetric group on $n$ letters. Then each elements in $w \in S_n$ has a matrix representation $A_w \in GL_n(\mathbb{C})$, which is constructed by putting $A_w e_i = e_{w(i)}$, where $e_1, \cdots, e_n$ are the standard basis of $\mathbb{C}^n$. Since $A_w$ is an element of $GL_n(\mathbb{C})$, it gives a corresponding point $p_w$ in $Fl_n$. More precisely, $p_w$ corresponds to flags $V_i = \langle e_{w(1)}, \cdots, e_{w(i)} \rangle$, $i = 1, \cdots, n$.

The following proposition plays a fundamental role when we introduce Schubert varieties.

**Proposition 2.2.4.** The set of fixed points in $Fl_n$ under the action of torus $T = U(1)^n$ corresponds to $\{p_w : w \in S_n\}$.

The above statement says that if a flag $V_\bullet = (V_0 \subset V_1 \subset \cdots \subset V_{n-1} \subset V_n)$ satisfies $t(V_\bullet) = V_\bullet$ for any $t \in T$, then generators of each $V_i = \langle v_1, \cdots, v_i \rangle$ can be written as a liner combination of basis $v_1, \cdots, v_n$ of $\mathbb{C}^n$ in such a way that

$$v_i = \sum_{j=1}^{i} a_{ij} e_{w(j)}, \quad a_{ij} \in \mathbb{C}, \; a_{ii} \neq 0, \tag{2.141}$$

where $w$ is an elements of $S_n$. Used mathematical induction for the equation (2.141), the proposition 2.2.4 will be shown.

For our convenience, let us put $B = SL_n(\mathbb{C}) \cap \tilde{B}$. Then $Fl_n$ can be written as $Fl_n \cong SL_n(\mathbb{C})/B$. With respects to a torus fixed point $p_w$ of $Fl_n$, the $B$-orbit $X_w^\circ = Bp_w$ is called a **Schubert cell**. We write $X_w$ for the closure of $X_w^\circ$ in $Fl_n$ and call it a **Schubert variety**. In general $X_w$ has singularities.

Schubert varieties are also accessible via Grassmann manifolds. An approachable review will be found in [34]. Let us see how to achieve there for our purpose of geometrical Langlands program. Let $V$ be a vector space over $\mathbb{C}$. A **Grassmann manifold** $G(r,V)$ is defined by the set of all $r$-(complex) dimensional vector subspace of $V$. $GL(V)$ acts transitively on $r$-dimensional subspaces of $V$, hence if $H$ is the stabilizer subgroup of $GL(V)$ with respect to an element $U \in GL(V)$, then it is possible to identify $G(r,V)$ as $GL(V)/H$, which gives the manifold structure of $G(r,V)$. We now choose $V = \mathbb{C}^n$ and simply write $G(r,\mathbb{C}^n) = G(r,n)$.

Recall a flag $V_\bullet = (V_0 \subset V_1 \subset \cdots \subset V_{n-1} \subset V_n)$ consists of $r$-dimensional vector spaces $V_r$. Therefore there is a surjection

$$\begin{aligned} \rho_r : Fl_n &\to G(r,n) \\ V_\bullet &\mapsto V_r. \end{aligned} \tag{2.142}$$

We choose a set of non-negative numbers $\lambda = (\lambda_1, \cdots, \lambda_r)$ so that $1 \leq \lambda_1 < \lambda_2 < \cdots < \lambda_r \leq n$. Then, Schubert cells and varieties of Grassmann manifold are defined by

$$X^\circ_\lambda(V_\bullet) = \{U \in G(r,n) : \dim_{\mathbb{C}}(U \cap V_{\lambda_i}) = i\} \tag{2.143}$$

$$X_\lambda(V_\bullet) = \{U \in G(r,n) : \dim_{\mathbb{C}}(U \cap V_{\lambda_i}) \geq i\}. \tag{2.144}$$

### 2.2.5 More on Hecke operators

Here we explain some important properties of Hecke operators in more algebraic way. We first consider $G = GL_n$ case. We write $\mathrm{Bun}_n$ for the moduli stack of $GL_n$-bundles on a smooth projective connected curve $C$ over a field $k$. Let $(\mathcal{M}, \mathcal{M}')$ be a pair of points in $\mathrm{Bun}_n$ such that $\mathcal{M}' \subset \mathcal{M}$. We pic an arbitral point $x$ up form $C$ and denote $\mathcal{O}_x$ by the skyscraper sheaf supported at $x$. The **Hecke correspondence** (or modification) $Hecke_r$ is the moduli space of $(\mathcal{M}, \mathcal{M}', x, \beta)$, where $x$ is a point in $C$ and $\beta : \mathcal{M}' \hookrightarrow \mathcal{M}$ is an embedding of the sheaves of sections such that $\mathcal{M}/\mathcal{M}'$ is supported at $x$ and isomorphic to the direct sum $\mathcal{O}_x^{\oplus r}$ of $r$ copies of $\mathcal{O}_x$. We define the correspondence $Hekce_{r,x}$ over $\mathrm{Bun}_n \times \mathrm{Bun}_n$ by

$$\begin{aligned} \mathrm{supp}(x, \mathcal{M}, \mathcal{M}') &= x \\ Hecke_{r,x} &= \mathrm{supp}^{-1}(x). \end{aligned} \tag{2.145}$$

Hence we obtain the diagram

$$
\begin{tikzcd}
 & Hecke_{r,x} \arrow[dl, "h^{\leftarrow}"'] \arrow[dr, "h^{\rightarrow}"] & \\
\mathrm{Bun}_n & & \mathrm{Bun}_n,
\end{tikzcd}
\tag{2.146}
$$

where $h^{\leftarrow}(x, \mathcal{M}, \mathcal{M}') = \mathcal{M}$ and $h^{\rightarrow}(x, \mathcal{M}, \mathcal{M}') = \mathcal{M}'$. Hence the Hecke functor $H_{r,x}$ is written as

$$H_{r,x}(\mathcal{F}) = h^{\rightarrow}_{x*} h^{*\leftarrow}_{x}(\mathcal{F}), \tag{2.147}$$

where $\mathcal{F}$ is a $\mathcal{D}$-module on $\mathrm{Bun}_n$.

For a simple case $(r, n) = (1, 2)$, the points in the fiber of $Hecke_{1,x}$ in the left $\mathrm{Bun}_2$, the sheaf of sections of a rank 2 vector bundle on $C$, is all locally free sheaves $\mathcal{M}'$ such that $\mathcal{M}/\mathcal{M}' = \mathcal{O}_x$. Each $\mathcal{M}'$ corresponds to a line bundle $\mathcal{L}_x$ in the dual space $\mathcal{M}^*_x$, and a section $s$ of $\mathcal{M}'$ is a section of $\mathcal{M}$ such that $\langle v, s(x)\rangle = 0$ for all non-zero $v \in \mathcal{L}_x$. Then $Hecke_{1,x}$ is a $\mathbb{P}^1$-fibration over left $\mathrm{Bun}_2$. In the same way, it is a $\mathbb{P}^1$-fibration over right $\mathrm{Bun}_2$.

Generally, $Hecke_{r,x}$ is a fibration over left and right $\mathrm{Bun}_n$s whose fiber is the Grassmannian $\mathrm{Gr}(r, n)$.

For a generic reductive group $G$, the space of Hecke correspondences is parametrized by the **affine Grassmannian**. Let $R$ be a ring. We define the **ring of formal power series** $R[[t]]$ by

$$R[[t]] = \left\{ \sum_{n=0}^{\infty} a_n t^n : a_n \in R \right\} \tag{2.148}$$

and **ring of formal Laurient series** $R((t))$ by

$$R((t)) = \left\{ \sum_{n=m}^{\infty} a_n t^n : m \in \mathbb{Z}, a_n \in R \right\}. \tag{2.149}$$

If $R$ is a field, then $R((t))$ become a field. For $R = \mathbb{C}$, the **loop group** $G((t))$ is defined by $G((t)) = G(\mathbb{C}((t)))$. We define $G[[t]]$ in the similar way. The affine Grassmannian is given by $\mathrm{Gr}_G = G[[t]]/G((t))$. It is known that there is a bijection between $\mathrm{Bun}_G$ and $G_{\text{out}}\backslash G((t))/G[[t]]$, where $G_{\text{out}} = \{C\backslash x \to G\}$ is the group of algebraic maps. It is known that the space of (spherical) Hecke modifications is Schubert cell in the affine Grassmannian.

The physical counterparts of the Hecke operators are 't Hooft operators as we will see in the later section. We tell the result first. Let $G$ be a compact reductive real Lie group. An 't Hooft operator is classified by the conjugacy class of a homomorphism $\rho : U(1) \to G$, which is analytically continued to a homomorphism $\rho_{\mathbb{C}} : \mathbb{C}^{\times} \to G_{\mathbb{C}}$, where $G_{\mathbb{C}}$ is the complexfication of $G$. Then $\rho_{\mathbb{C}}$ defines a point on $\mathrm{Gr}_G$ and the orbit, which is the Schubert cell, of this point under the action of $G((t))$ depends only on the conjugacy class of $\rho$.

# Chapter 3

# Langlands program in physics

## 3.1 The early days of the Langlands programs in physics

The electric-magnetic duality is a key to understand the Langlands dual group. Classically it say Maxwell's equations are invariant under exchanging the electric field and the magnetic field. Furthermore, from $U(1)$ gauge theory, it results in Dirac's quantization condition. In viewing these charges as coupling constants of U(1) gauge theory and the dual ${}^LU(1)$ theory respectively, we arrive at the picture of S-duality (for introduction, see [47]). And this $S$-duality is the most elementary perspective of modern $S$-duality with supersymmetries. One trivial but important fact is that the dual of $U(1)$ is again $U(1)$, which implies that the magnetic charge is scaler. However if we consider a gauge theory whose gauge group is a general compact real Lie group $G$, the "magnetic" charge is $\mathfrak{g}$ valued in general (strictly speaking, it takes values in the maximal torus of $\mathfrak{g}$) and the dual theory has the gauge group ${}^LG$. Hence we obtain more nontrivial duality of gauge theories. In below, we will consider how the Langlands dual group emerges and what the dual theory looks like.

In 1974, 't Hooft [49] and Polyakov [48] constructed frameworks to consider a monopole with charge $e$ in Yagn-Mills-Higgs model with SO(3) gauge group. They made assumptions, which is called the **'t Hooft-Polyakov anzats** today, on the form of the Higgs field $\Phi$ and the gauge potential $W$, and showed that the magnetic field $F_{ij}$ is written as a product of a Higgs filed $\Phi$ and field strength $G_{ij}$ in such a way that $F_{ij} = \Phi \cdot G_{ij}$, and at the region sufficiently away from the monopole, the magnetic field becomes $F_{ij} = \epsilon_{ijk}\frac{1}{e}\frac{x_k}{|x|^3}$. This shows that 't Hooft-Polyakov monopole is indeed the Dirac charge up to sign. In the following section, we extend the gauge group to a more general one and consider monopoles. Then it turns out that the corresponding monopole charge $Q$ is also quantized. While these series of discussions we encounter the **GNO dual groups** which are examples the **Langlands dual groups**.

In this section we review the paper of Goddard, Nuyts and Olive [16]. We work on the $3+1$-dimensional Yang-Mills-Higgs system with a $n$-dimensional compact and connected gauge group $G$. Let $\mathfrak{g}$ be the Lie algebra of $G$, and we denote by $\{\tau_a\}_{a=1}^n$ the a of the basis with structure coefficients $f_{abc}$, namely any $\tau_a$ and $\tau_b$ satisfy $[\tau_a, \tau_b] = f_{abc}\tau_c$.

We work with Lagrangian density $\mathcal{L}$ given by

$$\mathcal{L} = -\frac{1}{4}G^a_{\mu\nu}G^{a\mu\nu} + \frac{1}{2}|D_\mu\Phi|^2 - V(\Phi), \tag{3.1}$$

where $\mu, \nu = 0, \cdots, 3$ and

$$D_\mu\Phi = \partial_\mu\Phi - eW_\mu\Phi, \tag{3.2}$$

$$G_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu - e[W_\mu, W_\nu], \tag{3.3}$$

where $W_\mu = W^a_\mu\tau_a$ are the $\mathfrak{g} = \{\tau_a\}_{a=1}^n$ valued potentials and $V(\Phi)$ is an $G$ invariant Higgs potential. In this system, the energy density $\mathcal{H}$ corresponding to the given Lagrangian density is

$$\mathcal{H} = \frac{1}{2}|\dot{W}_i|^2 + \frac{1}{2}|\dot{\Phi}_i|^2 + \frac{1}{4}G^a_{ij}G^{aij} + \frac{1}{2}|D_i\Phi|^2 + V(\Phi), \tag{3.4}$$

where a dot indicates the time derivative. Since the energy is the integral over $\mathbb{R}^3$ of the density $\mathcal{H}$, especially $D_\mu\Phi = 0$ and $V(\Phi) = 0$ as $|x| \to \infty$. So let $M_0$ denote the manifold of vaccua, namely $M_0 = \{\Phi : V(\Phi) = 0\}$. Since $V$ is $G$ invariant, so is $M_0$. Suppose $G$ acts on $M_0$ transitively, namely ${}^\forall\Phi, \Phi' \in M_0, {}^\exists g \in G$ such that $\Phi = g\Phi'$. We put an asymptotic 2-dimensional sphere $S^2_\infty$ whose radius is sufficiently large so that $\Phi \in M_0$. Then the condition $V(\Phi) = 0$ as $|x| \to \infty$ means that $\Phi$ is a continuous map from $S^2_\infty$ to $M_0$. Hence we would be allowed to categorize $\Phi$ by $\pi_2(M_0)$. Let us fix a base point $x_0$ in $S^2_\infty$ and assume $\Phi(x_0) = \phi_0 \in M_0$. The $G$**-orbit** $G\phi_0$ is a submanifold $G\phi_0 = \{g\phi_0 : g \in G\} \subset M_0$ and the **isotropy subgroup** $I$ is defined by $I = \{g \in G : g\phi_0 = \phi_0\}$. Hence we have homeomorphic $G/I \cong G\phi_0 \cong M_0$ since $G$ acts on $M_0$ transitively. Assume $M_0 \cong G/I$

is connected and consider the homotopy exact sequence as the fiber bundle $(G, G/I, I)$

$$\cdots \to \pi_2(G) \to \pi_2(G/I) \to \pi_1(I) \to \pi_1(G) \to \cdots . \tag{3.5}$$

Note that $\pi_2(G) = 1$ for any connected and compact Lie group $G$. Then we obtain isomorphic

$$\pi_2(G/I) \cong \ker(\pi_1(I) \to \pi_1(G)). \tag{3.6}$$

Therefore our problems to classify $\Phi$ by $\pi_2(G/I)$ result in to study loops in $I$ which is null-homotopic in $G$.

Now we consider a monopole in this model. If the gauge group $G$ is SO(3), this results in a 't Hooft-Polyakov monopole as we mentioned.

We construct a loop in $I$ and show the given monopole charge $Q(x)$ determines the element of $\pi_1(I)$ in an appropriate manner. Now we invite a magnetic monopole with magnetic charge $Q(x)$ which takes value in Lie algebra $\mathfrak{i}$. We assume that the field strength take the form

$$G_{ij} = \epsilon_{ijk} \frac{x_k}{|x|^3} \frac{Q(x)}{4\pi}, \ \ i, j = 1, 2, 3, \tag{3.7}$$

at any site sufficiently distant from the central region of the monopole in a appropriate Lorentz frame.

**Claim 3.1.1.**

$$D_i Q(x) = 0 \ \text{ on } S^2_\infty \tag{3.8}$$

**Derivation.** The Bianchi identity

$$\frac{1}{2} \epsilon^{\mu\nu\rho\lambda} D_\nu G_{\rho\lambda} = 0 \tag{3.9}$$

brings about

$$x_i D_i Q(x) = 0. \tag{3.10}$$

And the equation of motion

$$\partial_\mu G^{a\mu\nu} - e f^{abc} W^b_\mu G^{c\mu\nu} + e\Phi \tau_a D^\nu \Phi = 0 \tag{3.11}$$

gives $D_\mu G^{\mu\nu} = 0$ since $D^\nu \Phi = 0$ on the sphere, where $D_\mu G^{\mu\nu} = \partial_\mu G^{\mu\nu} - e[W_\mu, G^{\mu\nu}]$. Then we obtain

$$\epsilon_{ijk} x_j D_k Q(x) = 0. \tag{3.12}$$

Consequently the assertion follows. $\square$

Some loops in $I$ are obtained by the following way. Firstly, we divide $S^2_\infty$ in to two open hemispheres $\Sigma_\pm$ such that $S^2_\infty = \Sigma_+ \cup \Sigma_-$ and $\Sigma_+ \cap \Sigma_-$ is a small band around the equator. Since $\Sigma_\pm$ are contractible, there are local gauge transformations $g_\pm : \Sigma_\pm \to G$ such that

$$\Phi(x) = g_\pm(x)\phi_0, \quad x \in \Sigma_\pm. \tag{3.13}$$

This gives us on $\Sigma_+ \cap \Sigma_-$

$$g_+(x)\phi_0 = g_-(x)\phi_0. \tag{3.14}$$

Thefore there we obtain $g_-^{-1}(x)g_+(x)\phi_0 = \phi_0$, and this shows that $g_-^{-1}(x)g_+(x) \in I$, which is what we have sought. Moreover, this loop $h(x) = g(x)_-^{-1}g_+(x)$ is trivial in $G$, because for all $h(x)$ there exists continuous paths to connect between $g_\pm$ and the identity element of $G$ since $G$ is path-connected. Let $g(\tau, x)_\pm \in G,\ \tau \in [0,1]$ be such a continuous path which satisfies $g(0,x)_\pm = 1$ and $g(1,x)_\pm = g(x)_\pm$. Then the operation to glue $g_\pm(\tau, x)$ into $g_-^{-1}(\tau, x)g_+(\tau, x)$ is also continuous since $G$ is Lie group, and $h(\tau, x) = g(\tau, x)_-^{-1}g_+(\tau, x)$ connects 1 and $h(x)$ in $G$. Therefore we have the following Higgs configuration

$$\Phi(\tau, x) = \begin{cases} g(\tau, x)_+\phi_0 & x \in \Sigma_+ \\ g(\tau, x)_-\phi_0 & x \in \Sigma_-. \end{cases} \tag{3.15}$$

On the other hand, if $h(x) \in \ker(\pi_1(I) \to \pi_1(G))$ is given, then choose a continuous path $g(\tau, x) \in G$ so that $g(0, x) = 1$ and $g(1, x) = h(x)$. Then we arrive at

$$\Phi(\tau, x) = g(\tau, x)\phi_0. \tag{3.16}$$

Clearly, the set of the equivalent classes of $h(x)$ forms a subgroup of $\pi_1(I)$ and is indeed $\ker(\pi_1(H) \to \pi_1(G))$.

Let us introduce parameters $(s, t) \in [0,1] \times [0,1]$ and coordinates $x_i(s,t)$ for $S^2_\infty \in \mathbb{R}^3$. In order to consider a loop $h(x(s,t))$, we fix $x(s,t)$ at $P \in S^2_\infty$ whenever $s$ or $t = 0$ or 1.

**Claim 3.1.2.** For a fixed $s$, $\Phi(s,t)$ is given by

$$\Phi(s,t) = P\exp\left(e\int_0^t dt' W_i(s,t')\frac{\partial x_i}{\partial t'}\right)\phi_0, \tag{3.17}$$

where $\Phi(s,t)$ and $W_i(s,t')$ are abbreviation of $\Phi(x(s,t))$ and $W_i(x(s,t'))$ respectively. Hence

$$h(s) = P\exp\left(e\int_0^1 dt W_i(s,t)\frac{\partial x_i}{\partial t}\right) \tag{3.18}$$

defines a loop in $I$.

**Derivation.** Since $D_i\Phi = 0$ on $S^2_\infty$, $D_{\frac{d}{dt}}\Phi = \frac{\partial x_i}{\partial t}D_i\Phi = 0$. Then

$$0 = \frac{\partial x_i}{\partial t}D_i\Phi = \frac{\partial \Phi}{\partial t'} - e\frac{\partial x_i}{\partial t'}W_i\Phi. \tag{3.19}$$

Note that $W$ is non-Abelian, by which we need the path-ordering operator $P$. The integral with respect to $t'$ over [0,t] gives the solution $\Phi(s,t)$. Since $\frac{\partial x_i}{\partial t} = 0$ at $s = 0, 1$, $h(0) = h(1)$ follows. Moreover $\Phi(s,1) = \phi_0 \in I$, since $x(s,1) = P$. Then $h(s)$ is a loop in $I$. $\square$

In order to show that the monopole charge is quantized, let us calculate the development of $h(s)$.

**Claim 3.1.3.** $h(s) = \exp\left[-\frac{e}{4\pi}Q\int_0^s ds'\int_0^1 dt\epsilon_{ijk}\frac{\partial x_i}{\partial t}\frac{\partial x_j}{\partial s'}\right]h(0)$.

**Derivation.** We defined $g$ by

$$g(s,t) = P\exp\left(e\int_0^t dt' W_i(s,t')\frac{\partial x_i}{\partial t'}\right). \tag{3.20}$$

With respect to $\Phi(s,t) = g(s,t)\phi_0$, $D_t\Phi = D_{\frac{d}{dt}}\Phi = 0$ leads to $D_t g(s,t) = 0$. Then $D_t g f = g\partial_t f$ for all $f$, and hence $g^{-1}D_t = \partial_t g^{-1}$ follows. Therefore we obtain

$$\partial_t(g^{-1}D_s g) = g^{-1}D_t D_s g \tag{3.21}$$

$$= g^{-1}[D_t, D_s]g \tag{3.22}$$

$$= g^{-1}\frac{\partial x_i}{\partial t}\frac{\partial x_j}{\partial s}[D_i, D_j]g \tag{3.23}$$

$$= g^{-1}\frac{\partial x_i}{\partial t}\frac{\partial x_j}{\partial s}(-eG_{ij})g. \tag{3.24}$$

The integral of this form over $t \in [0,1]$ and the fact $g^{-1}(s,t)D_s g(s,t)|_{t=0} = h^{-1}(s)\partial_s h(s)$ give

$$h^{-1}(s)\partial_s h(s) = -e\int_0^1 dt\frac{\partial x_i}{\partial t}\frac{\partial x_j}{\partial s}g^{-1}(s,t)G_{ij}g(s,t) \tag{3.25}$$

$$= -e\int_0^1 dt\frac{\partial x_i}{\partial t}\frac{\partial x_j}{\partial s}g^{-1}(s,t)\epsilon_{ijk}\frac{x_k}{|x|^3}\frac{Q(x(s,t))}{4\pi}g(s,t) \tag{3.26}$$

$$= -\,e\int_0^1 dt\frac{\partial x_i}{\partial t}\frac{\partial x_j}{\partial s}g^{-1}(0,0)\epsilon_{ijk}\frac{x_k}{|x|^3}\frac{Q(x(0,0))}{4\pi}g(0,0) \tag{3.27}$$

$$= -\,e\frac{Q}{4\pi}\int_0^1 dt\epsilon_{ijk}\frac{\partial x_i}{\partial t}\frac{\partial x_j}{\partial s}\frac{x_k}{|x|^3} \tag{3.28}$$

We used the fact $D_i Q(x(s,t)) = 0$ to obtain

$$g^{-1}(s,t)Q(x(s,t))g(s,t) = g^{-1}(0,0)Q(x(0,0))g(0,0). \tag{3.29}$$

The integration of this form over $s$ yields

$$h(s) = \exp\left[-e\frac{Q}{4\pi}\int_0^s ds'\int_0^1 dt\epsilon_{ijk}\frac{\partial x_i}{\partial t}\frac{\partial x_j}{\partial s'}\frac{x_k}{|x|^3}\right]h(0), \tag{3.30}$$

where $\exp : \mathfrak{i} \to I$ is the exponential map, which corresponds to construct the Chevalley group. Especially,

$$h(1) = \exp\left[-e\frac{Q}{4\pi}\int_0^1 ds\int_0^1 dt\epsilon_{ijk}\frac{\partial x_i}{\partial t}\frac{\partial x_j}{\partial s}\frac{x_k}{|x|^3}\right]h(0). \quad \square \tag{3.31}$$

$\square$

Next we evaluate the above integral. Since the integral is invariant under rescaling $x$, we are allowed to set the radius $|x|$ of $S^2_\infty$ to 1. Furthermore, $\epsilon_{ijk}\frac{\partial x_i}{\partial t}\frac{\partial x_j}{\partial s}x_k$ is the pul back via the embedding $S^2 \ni (s,t) \to x_i(s,t) \in \mathbb{R}^3$ of the form $\omega = \frac{1}{2}\epsilon_{ijk}x_k dx_i \wedge dx_j$. Then the above integral becomes

$$h(1) = \exp\left[-e\frac{Q}{4\pi}\int_{S^2}\omega\right]h(0) \tag{3.32}$$

$$= \exp\left[-e\frac{Q}{4\pi}\int_{D^3}d\omega\right]h(0),\ \partial D^3 = S^2 \tag{3.33}$$

$$= \exp[-eQ]h(0). \tag{3.34}$$

To derive the last equation we used the fact that $d\omega = \frac{1}{2}\epsilon_{ijk}dx_i \wedge dx_j \wedge dx_k$ is indeed the 3 times of the standard volume form in $\mathbb{R}^3$.

Recall $h(1) = h(0)$, then we arrive at the following pregnant result.

**Claim 3.1.4.** $\exp[-eQ] = 1 \in I$.

We shall investigate the structure of the group which produces the above simple formula. Since $H$ is connected and compact Lie group, the Lie algebra $\mathfrak{i}$ of $I$ has Cartan decomposition with the associated set $\Delta$ of roots

$$\mathfrak{i} = \mathfrak{t} \oplus \sum_{\alpha \in \Delta} \mathfrak{i}_\alpha, \tag{3.35}$$

where $\mathfrak{t}$ is the **Cartan subalgebra**, which is equivalent to the Lie algebra $\mathfrak{t}$ of the **maximal torus** $T$,

$$\mathfrak{t} = \{X \in \mathfrak{i} : [X, X'] = 0, ^\forall X' \in \mathfrak{i}\}, \tag{3.36}$$

and $\mathfrak{i}_\alpha$ is defined by

$$\mathfrak{i}_\alpha = \{X \in \mathfrak{i} : [I, X] = \alpha(X)I, ^\forall I \in \mathfrak{i}\}. \tag{3.37}$$

We focus on $\mathfrak{t}$. Let the rank of $\mathfrak{h}$ be $l$. That is to say, we take $\{T_1, \cdots, T_l\}$ as the basis of $\mathfrak{t}$. And we denote by $e_\alpha$ the generator of $\mathfrak{i}_\alpha$. Then they satisfy the following properties:

1. $\mathfrak{i} = \cup_{s \in I} \mathrm{Ad}(s)\mathfrak{t}$, namely

$$^\forall X \in \mathfrak{i},\ ^\forall Y \in \mathfrak{t},\ ^\exists s \in I \text{ such that } X = sYs^{-1}. \tag{3.38}$$

2. $^\exists s \in I$ such that $s\mathfrak{t}s^{-1} = \mathfrak{t}$.

3. $[T_i, e_\alpha] = \alpha_i e_\alpha$.

4. The center of $H$ is contained in $T$, i.e. $Z(H) \subset T$.

The above properties 1. and 2. allow us to write $-eQ \in \mathfrak{i}$ as

$$-eQ = s^{-1} \left( \sum_{i=1}^{l} \beta_i T_i \right) s. \tag{3.39}$$

Then the quantization condition $\exp[-eQ] = 1$ becomes

$$\exp \left( \sum_{i=1}^{l} \beta_i T_i \right) = 1. \tag{3.40}$$

This shows that the exponential map is $\exp : \mathfrak{t} \to T$. Alternatively, the map $\exp : \mathfrak{t} \to T$ is surjective, and there exists a non negative integer $k(\leq l)$ such that

$$\exp^{-1}(1) = \sum_{i=1}^{k} \mathbb{Z} T_i \tag{3.41}$$

$$= X_*(T) \tag{3.42}$$

where $\{T_i\}$ is an appropriate choice of the basis of $\mathfrak{t}$, and $X_*(T)$ is the **coweight lattice** of homeomorphism $U(1) \to T$. Then we find

$$T \cong \mathfrak{t}/X_*(T). \tag{3.43}$$

Moreover using the property 3. ($[T_i, e_\alpha] = \alpha_i e_\alpha$), we obtain

$$1 = \exp\left(\sum_{i=1}^{l} \beta_i T_i\right) = \exp\left(\sum_{i=1}^{l} \beta_i \alpha_i\right). \tag{3.44}$$

Therefore we arrive at $\beta \cdot \alpha \in 2\pi\mathbb{Z}$.

A homeomorphism $\chi : T \to U(1)$ is called a **character** of $T$

$$\chi_w\left(e^X\right) = e^{iw(X)}, \quad X \in \mathfrak{t}, \tag{3.45}$$

where $w$ is a **weight**. The set of weights spans a lattice $X^*(T)$, which is called **weight lattice** of homeomorphism $\chi : T \to U(1)$. So our quantization condition $\exp[-eQ] = 1$ means that there exists a compact connected semisimple Lie group ${}^L I$ whose weight lattice is $X_*(T)$. Therefore with respect the given root data $(X^*(T), \Delta, X_*(T), \Delta^\vee)$ for $I$, we obtain its dual root data $(X_*(T), \Delta^\vee, X^*(T), \Delta)$ for ${}^L I$. So $\beta_i$ are coweights of $I$ and, at the same time, are weights of ${}^L I$. We call ${}^L I$ **GNO dual group** of $I$ which is an example of the **Langlands dual groups**. In appendix A, we explain general Langlands dual groups of connected reductive groups.

## 3.2 $\mathcal{N} = 4$ SYM from a geometric Langlands point of view

### 3.2.1 Overview

A physical approach to the Langlands program was firstly proposed by Kapustin and Witten [26]. Their method based on $\mathcal{N} = 4$ super Yang-Mils theory in four dimensions. We review it in this section. Here is a flow chart of our discussion.

### 3.2.2 Review of $\mathcal{N} = 4$ SYM

#### (a) Action

Our signatures of the metric $g_{IJ}$ of ten-dimensional Minkowski space $\mathbb{R}^{1,9}$ or Euclidean space $\mathbb{R}^{10}$ are in accord with those of used in [26], namely Lorentz signature $- + + \cdots +$ or Euclidean signatures $+ + + \cdots +$. The gamma matrices $\Gamma_I$ $(I = 0, \cdots, 9)$ obey the Clifford algebra $\{\Gamma_I, \Gamma_J\} = 2g_{IJ}$. The chirality operator $\Gamma$ is as defined in appendix C,

$$\Gamma_E = (-i)^5 \Gamma_0 \cdots \Gamma_9 \tag{3.46}$$

for the Euclidean signature, and

$$\Gamma_L = (-i)^6 \Gamma_0 \cdots \Gamma_9 \tag{3.47}$$

for the Lorentz signature. We use label $E$ or $L$ to indicate Euclidean or Lorentz signature. In this paragraph we work with the Lorentz signature.

The $\Gamma$ classifies the chirality of spinors. Let $\mathcal{S}^+$ and $\mathcal{S}^-$ be the set of positive and negative chirality spinors respectively. A number of detailed properties about the gamma matrices and spin representations $\mathcal{S}^\pm$ are given in appendix C.

We first consider $\mathcal{N} = 1$ super Yang-Mils theory in ten-dimensions. Let $A$ be a gauge field, which is a connection on a $G$-bundle $E$. We prefer to choose $A$ anti-hermitian then the curvature $F$ is $F = dA + A \wedge A$. We denote by $\lambda$ a fermion field, namely $\lambda$ is a section of $\mathcal{S}^+ \otimes \mathrm{ad}(E)$. Our action is given by

$$I_{10} = \frac{1}{e'^2} \int d^{10}x \mathrm{Tr} \left( \frac{1}{2} F_{IJ} F^{IJ} - i\bar{\lambda} \Gamma^I D_I \lambda \right) \tag{3.48}$$

A constant bosonic Majorana-Weyl spinor $\epsilon$ that obeys

$$\Gamma_L \epsilon = -i\epsilon \tag{3.49}$$

generates supersymmetry. The action (3.48) is invariant under the supersymmetry transformation

$$\delta_S A_I = i\bar{\epsilon}\Gamma_I \lambda, \tag{3.50}$$

$$\delta_S \lambda = \frac{1}{2}\Gamma^{IJ} F_{IJ} \epsilon, \tag{3.51}$$

and the conserved supercurrent which generates the above supersymmetries is

$$J^I = \frac{1}{2} \mathrm{Tr} \Gamma^{JK} F_{JK} \Gamma^I \lambda. \tag{3.52}$$

**(b) Dimensional reduction from ten-dimensions to four-dimensions**

We simply take all the fields to be independent of the coordinates $x_4, \cdots, x_9$ to reduce to four dimensions. For example, the four components $A_I$ $(I = 0, \cdots, 3)$ are the four dimensional gauge field $A_\mu$ $(\mu = 0, \cdots, 3)$, and the others become four dimensional scaler fields $\phi_i = A_{i+4}$ $(i = 0, \cdots, 5)$. Hence the field strength $F_{IJ} = \partial_I A_J - \partial_J A_I + [A_I, A_J]$ become

$$F_{IJ} = \begin{cases} \partial_\mu A_\nu - \partial_\mu A_\nu + [A_\mu, A_\nu] & 0 \leq I = \mu,\ J = \nu \leq 3 \\ [\phi_i, \phi_j] & 4 \leq I = i+4,\ J = j+4 \leq 9 \\ D_\mu \phi_j = \partial_\mu \phi_j + [A_\mu, \phi_j] & 0 \leq I = \mu \leq 3,\ 4 \leq J = j+4 \leq 9. \end{cases} \tag{3.53}$$

The dimensional reduction produces as $Spin(6)$ symmetries in four dimensions and the $Spin(6)$ group acts on the scaler fields $\phi_i$ naturally.

The bosonic part of the action of the four dimensional $\mathcal{N} = 4$ supersymmetric Yang-Mils theory consists of $S_4$ is reduced from $I_{10}$ to

$$S_4 = \frac{1}{e^2} \int d^4x \mathrm{Tr} \left( \frac{1}{2} \sum_{\mu,\nu=0}^{3} F_{\mu\nu} F^{\mu\nu} + \sum_{\mu=0}^{3} \sum_{i=1}^{5} D_\mu \phi_i D^\mu \phi^i + \frac{1}{2} \sum_{i,j=1}^{5} [\phi_i, \phi_j]^2 \right). \tag{3.54}$$

Moreover there is an additional term, known as a topological term of the theory, given by

$$S_\theta = \frac{\theta}{8\pi^2} \int \mathrm{Tr}(F \wedge F) = \frac{\theta}{16\pi^2} \int d^4x \mathrm{Tr}(F_{\mu\nu} (*F)^{\mu\nu}), \tag{3.55}$$

where $*F$ is the Hodge dual of $F$ and $(*F)_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}$. Together with $S_4$ and $S_\theta$, we have two real parameters $e$ and $\theta$. We combine them into the complex coupling parameter

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{e^2}. \tag{3.56}$$

We show that $\tau$ is identified with the coupling constant of our theory. When we put

$$G_{\mu\nu} = F_{\mu\nu} + i(*F)_{\mu\nu}, \tag{3.57}$$

then we obtain

$$G_{\mu\nu} G^{\mu\nu} = 2F_{\mu\nu} F^{\mu\nu} + 2i F_{\mu\nu} (*F)^{\mu\nu}. \tag{3.58}$$

The gauge field part of $S_4$ and $S_\theta$ become

$$\int d^4x \mathrm{Tr} \left( \frac{1}{2e^2} F_{\mu\nu} F^{\mu\nu} + \frac{\theta}{16\pi^2} F_{\mu\nu} (*F)^{\mu\nu} \right) = \frac{1}{16\pi} \int d^4x \mathrm{Tr}(\tau G_{\mu\nu} G^{\mu\nu}). \tag{3.59}$$

So we regard $\tau$ as the coupling constant of 4 dim $\mathcal{N}=4$ SYM theory.

Note that $\tau$ is defined on the upper half plane $H$ and recall the group $SL_2(\mathbb{Z})$, which $S$ and $T$ (defined by (2.4)) generate, naturally acts on $H$ and $\tau$ as

$$S : \tau \mapsto -\frac{1}{\tau}, \tag{3.60}$$

$$T : \tau \mapsto \tau + 1. \tag{3.61}$$

If $\theta = 0$ the $S$ transformation results in the classical electric-magnetic duality $e \mapsto 1/e$ and as we saw in the previous section 3.1, the gauge group $G$ transforms to its Langlands dual group ${}^L G$ in the dual theory. If $G$ is simply-laced, the $S$-**duality** conjecture asserts that the gauge theories of $(G, \tau)$ and $({}^L G, -1/\tau)$ are equivalent. If $G$ is not simply-laced, we modify the coupling constant $\tau$ and $S$-transformation as

$$\tilde{\tau} = \frac{1}{|\alpha_{\mathrm{long}}|^2}\tau, \tag{3.62}$$

$$S : \tilde{\tau} \to -\frac{1}{n_{\mathfrak{g}}\tilde{\tau}}, \tag{3.63}$$

$$S = \begin{pmatrix} 0 & 1/\sqrt{n_{\mathfrak{g}}} \\ -\sqrt{n_{\mathfrak{g}}} & 0 \end{pmatrix}, \tag{3.64}$$

where $n_{\mathfrak{g}} = |\alpha_{\mathrm{long}}|^2/|\alpha_{\mathrm{short}}|^2$, and $\alpha_{\mathrm{long}}$ and $\alpha_{\mathrm{long}}$ are the long root and short root of $\mathfrak{g}$ respectively. Consulting the Dynkin diagrams (see appendix A), we notice

$$n_{\mathfrak{g}} = \begin{cases} 1 & \text{for } A_n,\ D_n \text{ and } E_n \\ 2 & \text{for } B_n,\ C_n \text{ and } F_4 \\ 3 & \text{for } G_2 \end{cases} . \tag{3.65}$$

So if $n_{\mathfrak{g}} \neq 1$, the $S$-transformation is not a generator of $SL_2(\mathbb{Z})$, instead $S^2, STS$ and $T$ generate a subgroup $\Gamma_0(n_{\mathfrak{g}})$ of $SL_2(\mathbb{Z})$ (see (2.20) for its definition). More detailed discussions for generic $S$-duality for $\mathcal{N}=4$ super Yang-Mills theories can be found in [2, 6].

### (c) Transformation of the supersymmetries

From now on we will work with the Euclidean signature. The Euclidean ten-dimensional SYM theory has the standard $Spin(10)$ symmetry in the presence of fermionic fields and it decomposed to $Spin(4) \times Spin(6)$ symmetry after we reduced to the dimensional reduction. So the chirality condition $\Gamma_E \epsilon = \epsilon$ in ten-dimension becomes

$$\hat{\Gamma}\Gamma' \epsilon = \epsilon, \tag{3.66}$$

where $\hat{\Gamma} = \Gamma_0 \cdots \Gamma_3$ classifies $Spin(4)$ chirality and $\Gamma' = \Gamma_4 \cdots \Gamma_9$ classifies $Spin(6)$ chirality. Both of $\hat{\Gamma}$ and $\Gamma'$ have the same eigenvalues $\pm 1$. Recall the isomorphisms $Spin(4) \cong SL(2, \mathbb{C})$ and $Spin(6) \cong SU(4)$. The two spin representations of $Spin(4)$ corresponds to $(\mathbf{2}, \mathbf{1})$ and $(\mathbf{1}, \mathbf{2})$, one of which is complex conjugate of the other. We choose sigs so that $\hat{\Gamma}$ acts as $+1$ and $-1$ on $(\mathbf{2}, \mathbf{1})$ and $(\mathbf{1}, \mathbf{2})$ respectively.

Similarly the spin representations of $Spin(6)$ are $\mathbf{4}$ and $\overline{\mathbf{4}}$ of $SU(4)_R$, and we prefer signs so that $\Gamma'$ acts $+1$ and $-1$ on $\overline{\mathbf{4}}$ and $\mathbf{4}$ respectively.

In summary, the four-dimensional supersymmetries transform as

$$(\mathbf{2}, \mathbf{1}, \overline{\mathbf{4}}) \oplus (\mathbf{1}, \mathbf{2}, \mathbf{4}) \tag{3.67}$$

under $Spin(4) \times Spin(6) \cong SL(2, \mathbb{C}) \times SU(4)_R$. The fermion fields transform in the same way.

**(d) Topological twisting**

We successfully obtained four-dimensional supersymmetric Yang-Mils theory on flat space, however when we try to transplant this theory to a general four manifold, we need a trick, called **twisting**, otherwise all the supersymmetries are destined to die. The procedure of twisting is to replace the symmetry group $Spin(4)$ of $\mathbb{R}^4$ to a subgroup $Spin'(4)$ of $Spin(4) \times Spin(6)$, which is isomorphic to $Spin(4)$ and acts on $\mathbb{R}^4$, but acts on the gauge theory in a different way. In the following section, we will mainly use Euclidean signature. By the Wick rotation, we have $\Gamma_L = i\Gamma_E$, so $\Gamma_E \epsilon = -i\epsilon$.

We pick a homomorphism $\varkappa : Spin(4) \to Spin(6)$ and set

$$Spin'(4) = (1 \times \varkappa)(Spin(4)) \subset Spin(4) \times Spin(6). \tag{3.68}$$

Basically we have three different ways to perform twisting since there are isomorphisms $Spin(6) \cong SU(4)$ and $Spin(4) \cong SU(2) \times SU(2)$, which we refer to as $SU(4)_R$ and $SU(2)_l \times SU(2)_r$ respectively.

The $\mathbf{4}$ representation of $SU(4)_R$ can be realized as representations of $SU(2)_l \times SU(2)_r$ in the following ways:

$$\text{(i) } (\mathbf{2},\mathbf{1}) \oplus (\mathbf{1},\mathbf{1}) \oplus (\mathbf{1},\mathbf{1}) \quad \text{(ii) } (\mathbf{2},\mathbf{1}) \oplus (\mathbf{2},\mathbf{1}) \quad \text{(iii) } (\mathbf{2},\mathbf{1}) \oplus (\mathbf{1},\mathbf{2}). \tag{3.69}$$

Here we make some short comments on the three twists. (i) is the Donaldson-Witten twist [51] and this corresponds to regard $\mathcal{N}=4$ as $\mathcal{N}=2$ with matter. (ii) is the Vafa-Witten twist [50]. $S$-duality conjecture was tested by this twisting method. In this article, we prefer to choose the last case (iii) so as to rerate to the geometrical Langlands program and we call it the **GL twist**. This choice of $\varkappa$ amounts to embedding $SU(2)_l \times SU(2)_r$ in $SU(4)_R$ in such a way that

$$\varkappa : SU(2)_l \times SU(2)_r \mapsto \begin{pmatrix} SU(2)_l & 0 \\ 0 & SU(2)_r \end{pmatrix} \in SU(4)_R. \tag{3.70}$$

This embedding commutes with an additional $U(1)$ group generated by $\mathcal{K}$:

$$\mathscr{K} = -i\mathcal{K} = \begin{pmatrix} 1_2 & 0 \\ 0 & -1_2 \end{pmatrix}. \tag{3.71}$$

We will classify charges of fields by $\mathscr{K}$. Therefore the $\mathbf{4}$ of $SU(4)_R \cong Spin(6)$ transforms under $SU(2)_l \times SU(2)_r \times U(1)$ as $(\mathbf{2},\mathbf{1})^1 \oplus (\mathbf{1},\mathbf{2})^{-1}$.

In addition to the above method of twisting, there is an alternative description in terms of $SO$ group. Let $\gamma_i$ $(i = 1, \cdots, 6)$ be the gamma matrices in six dimensions, which are $8 \times 8$ matrices. We regard Weyl spinors $\psi, \chi$ in four-dimensions as eight-dimensional spinors by identifying $\psi = \binom{\psi}{0}$ and $\chi = \binom{\chi}{0}$, then a vector in six-dimensions wan be written in such a way that

$$^t\psi C \gamma_i \chi = (C\overline{\sigma}_i)_{\alpha\beta}\psi^\alpha \chi^\beta \quad i = 1, \cdots, 6,\ \alpha, \beta = 1, \cdots, 4, \tag{3.72}$$

where $C$ is the charge conjugation matrix and we used the **chiral representation** of $\gamma_i$, namely

$$\gamma_i = \begin{pmatrix} 0 & \sigma_i \\ \overline{\sigma}_i & 0 \end{pmatrix}, \tag{3.73}$$

where $\sigma_i$ is four-dimensional gamma matrices. So the fundamental, six-dimensional vector representations $\mathbf{6}$ of $SO(6)$ is the same as the Weyl representation $\wedge^2\mathbf{4}$ of $Spin(6)$. Hence

$$\begin{aligned} \mathbf{6} &= \wedge^2((\mathbf{2},\mathbf{1})^1 \oplus (\mathbf{1},\mathbf{2})^{-1}) \\ &= (\mathbf{2},\mathbf{2})^0 \oplus (\mathbf{1},\mathbf{1})^2 \oplus (\mathbf{1},\mathbf{1})^{-2}, \end{aligned} \tag{3.74}$$

where $(\mathbf{2},\mathbf{2})^{\mathbf{0}}$ can be identified with the vector representation $\mathbf{4}$ of $SO(4)$. Therefore the embedding corresponds to the homeomorphism

$$SO(4)\times U(1)\cong SO(4)\times SO(2)\rightarrow \begin{pmatrix} SO(4) & 0 \\ 0 & SO(2) \end{pmatrix}\in SO(6). \tag{3.75}$$

That is to say, the four $\phi_0,\cdots,\phi_3$ of the six spin zero fields $\phi_0,\cdots,\phi_5$ of our theory, which transform as $\mathbf{6}$ of $SO(6)$, rotate as $\mathbf{4}$ of $SO(4)$ and the rest $\phi_4,\phi_5$ are the $SO(4)$ scalers transformed by $SO(2)$. So the scalers $\sigma=\frac{1}{\sqrt{2}}(\phi_4-i\phi_5)$ and $\overline{\sigma}=\frac{1}{\sqrt{2}}(\phi_4+i\phi_5)$ have charges $\mathscr{K}=2$ and $\mathscr{K}=-2$ respectively.

In summary, the bosonic fields of our theory are gauge fields $A=A_\mu dx^\mu$, which is an adjoint valued one-form, another adjoint valued one-form $\phi=\phi_\mu dx^\mu$, and the complex scalers $\sigma,\overline{\sigma}$.

It is necessary to make a survey of how the sypersymmetries transform under $Spin'(4)\times U(1)$. Recall the four-dimensional original supersymmetries transform under $Spin(4)\times Spin(6)$ as (3.67)

$$(\mathbf{2},\mathbf{1},\overline{\mathbf{4}})\oplus(\mathbf{1},\mathbf{2},\mathbf{4}). \tag{3.76}$$

So the supersymmetries which transform as $(\mathbf{2},\mathbf{1})$ of $Spin(4)$ now transform as

$$(\mathbf{2},\mathbf{1})^{\mathbf{0}}\otimes((\mathbf{2},\mathbf{1})^{-\mathbf{1}}\oplus(\mathbf{1},\mathbf{2})^{\mathbf{1}})=(\mathbf{1},\mathbf{1})^{-\mathbf{1}}\oplus(\mathbf{3},\mathbf{1})^{-\mathbf{1}}\oplus(\mathbf{2},\mathbf{2})^{\mathbf{1}} \tag{3.77}$$

under $Spin'(4)\times U(1)$.

Similarly, the supersymmetries which transform as $(\mathbf{2},\mathbf{1})$ of $Spin(4)$ now transform as

$$(\mathbf{1},\mathbf{2})^{\mathbf{0}}\otimes((\mathbf{2},\mathbf{1})^{\mathbf{1}}\oplus(\mathbf{1},\mathbf{2})^{-\mathbf{1}})=(\mathbf{1},\mathbf{1})^{-\mathbf{1}}\oplus(\mathbf{1},\mathbf{3})^{-\mathbf{1}}\oplus(\mathbf{2},\mathbf{2})^{\mathbf{1}} \tag{3.78}$$

under $Spin'(4)\times U(1)$.

The representation $(\mathbf{3},\mathbf{1})\oplus(\mathbf{1},\mathbf{3})$ implies there are anti-symmetric two forms since the Lie algebra of $SU(2)\times SU(2)$ is the same as that of $SO(4)$ and $(\mathbf{3},\mathbf{1})\oplus(\mathbf{1},\mathbf{3})$ is the adjoint representation of $SO(4)$. So we call the anti-symmetric two form $\chi^+$ of $(\mathbf{3},\mathbf{1})^{-\mathbf{1}}$ self-dual two form and another anti-symmetric two form $\chi^-$ of $(\mathbf{1},\mathbf{3})^{-\mathbf{1}}$ anti-self-dual two form.

In summary, our fermion fields are as listed in the following table.

| | | | |
|---|---|---|---|
| $\mathscr{K}=1$ | $\psi,\tilde{\psi}$ | two one forms | $(\mathbf{2},\mathbf{2})^{\mathbf{1}}$ |
| $\mathscr{K}=-1$ | $\chi^+$ | self-dual two form | $(\mathbf{3},\mathbf{1})^{-\mathbf{1}}$ |
| | $\chi^-$ | anti-self-dual two form | $(\mathbf{1},\mathbf{3})^{-\mathbf{1}}$ |
| | $\eta,\tilde{\eta}$ | two zero forms | $(\mathbf{1},\mathbf{1})^{-\mathbf{1}}$ |

**(e) The SUSY generators in four dimensions**

The generators of $Spin'(4)$ are $\Gamma_{\mu\nu} + \Gamma_{\mu+4,\nu+4}$ $(\mu, \nu = 0, \cdots, 3)$ so the supercharges preserved by twisting are those obey

$$(\Gamma_{\mu\nu} + \Gamma_{\mu+4,\nu+4})\epsilon = 0. \tag{3.79}$$

Note this equation is equivalent to

$$\Gamma_{\mu,\mu+4}\epsilon = \Gamma_{\nu,\nu+4}\epsilon \tag{3.80}$$

for all $\mu, \nu = 0, 1, 2, 3$.

The $Spin'(4)$ invariant spinor $\epsilon$ can be decomposed to a linear combination $\epsilon = u\epsilon_l + v\epsilon_r$, where $\epsilon_l$ and $\epsilon_r$ are told apart by $\hat{\Gamma}_E = \Gamma_0\Gamma_1\Gamma_2\Gamma_3$ in such a way that

$$\hat{\Gamma}_E\epsilon_l = -\epsilon_l, \tag{3.81}$$

$$\hat{\Gamma}_E\epsilon_r = \epsilon_r. \tag{3.82}$$

Let $N = \frac{1}{4}\sum_{\mu=0}^{3}\Gamma_{\mu+4}\Gamma_\mu$ then $\epsilon_l$ and $\epsilon_r$ can be exchanged by $N$, namely

$$\epsilon_r = N\epsilon_l, \tag{3.83}$$

$$\epsilon_l = -N\epsilon_r. \tag{3.84}$$

Since $\epsilon_l$ and $\epsilon_r$ satisfy (3.80), we can show that

$$\begin{aligned}\epsilon_r &= N\epsilon_l \\ &= \Gamma_{\mu+4,\mu}\epsilon_l.\end{aligned} \tag{3.85}$$

Hence $\epsilon_l$ and $\epsilon_r$ obey

$$\Gamma_{\mu+4}\epsilon_l = -\Gamma_\mu\epsilon_r,\ \Gamma_{\mu+4}\epsilon_r = \Gamma_\mu\epsilon_l. \tag{3.86}$$

It would be convenient to use $\Gamma_{8\pm i9} = \frac{1}{\sqrt{2}}(\Gamma_8 \pm i\Gamma_9)$, then we have

$$\bar{\epsilon}_l\Gamma_{8+i9}\epsilon_r = \bar{\epsilon}_r\Gamma_{8+i9}\epsilon_l = 0, \tag{3.87}$$

$$\Gamma_{8-i9}\epsilon = 0. \tag{3.88}$$

The first equations follow by the chirality condition. We normalize $\epsilon_l$ and $\epsilon_r$ up to sign so that

$$\bar{\epsilon}_l\Gamma_{8+i9}\epsilon_l = \bar{\epsilon}_r\Gamma_{8+i9}\epsilon_r = 1. \tag{3.89}$$

The spinor $\epsilon = u\epsilon_l + v\epsilon_r$ is parametrized by $\mathbb{CP}^1$ since we are not interested in any overall constant scaling of $\epsilon$. Let $Q_l$ and $Q_r$ be the supercharges generated by $\epsilon_l$ and $\epsilon_r$. Let $\delta_T = \epsilon_l\delta_r + \epsilon_l\delta_r$ be a susy transformation operator generated by $\epsilon_l$ and $\epsilon_r$. Then any field $\Phi$ transforms

$$\delta_T\Phi = [Q, \Phi\}, \quad \delta_l\Phi = [Q_l, \Phi\}, \quad \delta_r\Phi = [Q_r, \Phi\}, \tag{3.90}$$

where

$$[A, B\} = AB - (-1)^{|A||B|}BA, \tag{3.91}$$

$$|A| = \begin{cases} +1 & A \text{ is Grassmann odd} \\ 0 & A \text{ is Grassmann even.} \end{cases} \tag{3.92}$$

**(f) The SUSY transformations**

In order to specify the super symmetry transformations of the bosonid and the fermionic fields, we expand the ten-dimensional spinor $\lambda$ in terms of fermionic fields in four dimensions:

$$\lambda = \left(\eta + \sum_\mu \Gamma^\mu\psi_\mu\Gamma_{8+i9} + \sum_{\mu<\nu}\Gamma^{\mu\nu}\chi^+_{\mu\nu}\right)\epsilon_l \tag{3.93}$$

$$+ \left(\tilde{\eta} + \sum_\mu \Gamma^\mu\tilde{\psi}_\mu\Gamma_{8+i9} + \sum_{\mu<\nu}\Gamma^{\mu\nu}\chi^-_{\mu\nu}\right)\epsilon_r, \tag{3.94}$$

where $\psi$ and $\tilde{\psi}$ are one forms, and $\chi^\pm$ are the selfdual and anti-selfdual parts of a two form $\chi$. Note that

$$\Gamma^{\mu\nu}\chi_{\mu\nu}\epsilon_l = \Gamma^{\mu\nu}\chi^+_{\mu\nu}\epsilon_l, \tag{3.95}$$

$$\Gamma^{\mu\nu}\chi_{\mu\nu}\epsilon_r = \Gamma^{\mu\nu}\chi^-_{\mu\nu}\epsilon_l \tag{3.96}$$

since $\Gamma_{\mu\nu}\epsilon_l$ is self dual and $\Gamma_{\mu\nu}\epsilon_r$ is anti-self dual.

Now we perform the susy transformation with respect to the bosonic and fermionic fields. The transformation laws $\delta_S A_I = i\bar{\epsilon}\Gamma_I\lambda$ (3.50) of bosonic fields lead to

$$\delta_T A_\mu = iu\psi_\mu + iv\tilde{\psi}_\mu, \tag{3.97}$$

$$\delta_T\phi_\mu = iv\psi_\mu - iu\tilde{\psi}_\mu, \tag{3.98}$$

$$\delta_T\sigma = 0, \tag{3.99}$$

$$\delta_T\overline{\sigma} = iu\eta + iv\tilde{\eta}. \tag{3.100}$$

The equation (3.99) means the charge of $\sigma$ is $\mathscr{K}=2$, and then there is no field with $\mathscr{K}=3$.

Similarly, the transformation rules $\delta_S\lambda = \frac{1}{2}\Gamma^{IJ}F_{IJ}\epsilon$ (3.51) of the ferimonic sectors with $\mathscr{K}=-1$ lead

$$\delta_T\chi^+ = u(F-\phi\wedge\phi)^+ + v(D\phi)^+, \tag{3.101}$$

$$\delta_T\chi^- = v(F-\phi\wedge\phi)^- - u(D\phi)^-, \tag{3.102}$$

$$\delta_T\eta = vD^*\phi + u[\overline{\sigma},\sigma], \tag{3.103}$$

$$\delta_T\tilde{\eta} = -uD^*\phi + v[\overline{\sigma},\sigma], \tag{3.104}$$

where $D^*\phi = \star D\star\phi = D_\mu\phi^\mu$, with the Hodge star $\star$.

Those of $\mathscr{K}=1$ transform as

$$\delta_T\psi = uD\sigma + v[\phi,\sigma], \tag{3.105}$$

$$\delta_T\tilde{\psi} = vD\sigma - u[\phi,\sigma]. \tag{3.106}$$

The nilpotency of the supersymmetric transformations is summarized in the form of

$$\delta_T^2\Phi = -i(u^2+v^2)\mathcal{L}_\sigma(\Phi), \tag{3.107}$$

where we denote by $\Phi$ a field. If $\Phi$ is gauge invariant, then $\mathcal{L}_\sigma(\Phi)=0$ follows.

For bosonic fields, it is straightforward to show

$$\delta_T^2 A = -i(u^2+v^2)(-D\sigma) \tag{3.108}$$

$$\delta_T^2\phi = -i(u^2+v^2)[\sigma,\phi] \tag{3.109}$$

$$\delta_T^2\psi = -i(u^2+v^2)[\sigma,\psi] \tag{3.110}$$

$$\delta_T^2\tilde{\psi} = -i(u^2+v^2)[\sigma,\psi] \tag{3.111}$$

$$\delta_T^2\sigma = 0. \tag{3.112}$$

However when one naively calculate similarly for fermion fields, it is false. To cure the problems we need to introduce an auxiliary field $P$ in the transformation laws

$$\delta_T \overline{\sigma} = iu\eta + iv\tilde{\eta} \tag{3.113}$$

$$\delta_T \eta = vP + u[\overline{\sigma}, \sigma] \tag{3.114}$$

$$\delta_T \tilde{\eta} = -uP + v[\overline{\sigma}, \sigma] \tag{3.115}$$

$$\delta_T \eta = vD^*\phi + u[\overline{\sigma}, \sigma] \tag{3.116}$$

$$\delta_T P = -iv[\sigma, \eta] + iu[\sigma, \tilde{\eta}]. \tag{3.117}$$

And then under those transformations, we obtain the explicit form $\mathcal{L}_\sigma(\Phi) = [\sigma, \Phi]$ for a fermion filed $\Phi$. If necessary, imposing $P = D^*\phi$ makes those transformations consistent with the original ones.

After all, if $\Phi$ is a gauge invariant field, either it is bosonic or fermionic, $\Phi$ indeed obeys $\delta_T^2\Phi = 0$ since we can choose a gauge $\sigma = 0$.

### 3.2.3 The "Topological" Lagrangian

In this section we mention the Lagrangian which posses topological symmetry for all $t$ and reduces when $M$ is flat to the Lagrangian of the underlying $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. Here we just write results.

$$V = \frac{2}{e^2}\int_M d^4x\sqrt{g}\mathrm{Tr}\left(-\frac{1}{2}\eta\tilde{\eta} - i\overline{\sigma}D^*\phi\right). \tag{3.118}$$

$$S_0 = \delta_l\delta_r V \tag{3.119}$$

$$\begin{aligned} S_0 = \frac{2}{e^2}\int_M d^4x\sqrt{g}\mathrm{Tr}\Big(&\frac{1}{2}P^2 - PD^*\phi + \frac{1}{2}[\overline{\sigma}, \sigma] - D_\mu\overline{\sigma}D^\mu\sigma - [\phi_\mu, \sigma][\phi^\mu, \overline{\sigma}] \\ &+ i\tilde{\eta}D_\mu\tilde{\psi}^\mu + i\eta D_\mu\psi^\mu - i\tilde{\eta}[\psi_\mu, \phi^\mu] + \imath\eta[\tilde{\psi}, \phi^\mu] \\ &- \frac{i}{2}[\sigma, \tilde{\eta}]\tilde{\eta} - \frac{i}{2}[\sigma, \eta]\eta + i[\overline{\sigma}, \psi_\mu]\psi^\mu + i[\overline{\sigma}, \tilde{\psi}_\mu]\tilde{\psi}_\mu\Big). \end{aligned} \tag{3.120}$$

The Euler-Lagrange equation for $P$ is $P = D^*\phi$.

$$
\begin{aligned}
S_1 = \frac{1}{e^2}\int_M d^4x\sqrt{g}\Big(&\frac{1}{2}F_{\mu\nu}F^{\mu\nu} + D_\mu\phi_\nu D^\mu\phi^\nu + R_{\mu\nu}\phi^\mu\phi^\nu + \frac{1}{2}[\phi_\mu,\phi_\nu]^2 - (D^*\phi)^2 \\
&- 2\chi^+_{\mu\nu}\Big(iD\psi + i[\tilde{\psi},\phi]\Big)^{\mu\nu} - 2\chi^-_{\mu\nu}\Big(iD\tilde{\psi} - i[\psi,\phi]\Big)^{\mu\nu} \\
&- i\chi^+_{\mu\nu}[\sigma,\chi^{+\mu\nu}] - i\chi^-_{\mu\nu}[\sigma,\chi^{-\mu\nu}]\Big) \\
&+ \frac{t-t^{-1}}{e^2(t+t^{-1})}\int_M \mathrm{Tr}F\wedge F.
\end{aligned} \tag{3.121}
$$

$$
S_2 = -\left(\frac{t-t^{-1}}{e^2(t+t^{-1})} - i\frac{\theta}{8\pi^2}\right)\int_M \mathrm{Tr}F\wedge F. \tag{3.122}
$$

We define our action by

$$
\begin{aligned}
S &= S_0 + S_1 + S_2 \\
&= \delta_T V + \frac{i\Psi}{4\pi}\int_M \mathrm{Tr}F\wedge F.
\end{aligned} \tag{3.123}
$$

The $t$-dependent terms in $S_1$ and $S_2$ vanish in total $S$. This action $S$ is "topological" in the sense that any supersymmetric correlation function does not depend on a given metric. To see this let $\mathcal{O}_1,\cdots,\mathcal{O}_n$ be an operators preserving supersymmetry, namely $\delta_T\mathcal{O}_i = 0$. We call such operators **observables**. The correlation function is defined by

$$
\langle\mathcal{O}_1\cdots\mathcal{O}_n\rangle = \int[D\Xi]\mathcal{O}_1\cdots\mathcal{O}_n e^{-S[\Xi]}, \tag{3.124}
$$

where $\Xi$ are the fields of the theory and $[D\Xi]$ is a path-integral measure invariant under supersymmetric transformations. Varying the correlator with respect to the metric $g$ on M, we find

$$
\begin{aligned}
\frac{\delta}{\delta g^{\mu\nu}}\langle\mathcal{O}_1\cdots\mathcal{O}_n\rangle &= \int[D\Xi]\mathcal{O}_1\cdots\mathcal{O}_n\frac{\delta}{\delta g^{\mu\nu}}(-S[\Xi])e^{-S[\Xi]} \\
&= \int[D\Xi]2\sqrt{g}\mathcal{O}_1\cdots\mathcal{O}_n T_{\mu\nu}e^{-S[\Xi]} \\
&= \int[D\Xi]2\sqrt{g}\mathcal{O}_1\cdots\mathcal{O}_n\delta_T b_{\mu\nu}e^{-S[\Xi]} \\
&= \int[D\Xi]\delta_T\left(2\sqrt{g}\mathcal{O}_1\cdots\mathcal{O}_n b_{\mu\nu}e^{-S[\Xi]}\right) \\
&= 0.
\end{aligned} \tag{3.125}
$$

Hence the correlation functions are independent of the metric. There are a few comments on the above derivation. In the second and third equation, we wrote

$$-2\sqrt{g}T_{\mu\nu} := \frac{\delta S}{\delta g^{\mu\nu}} = \delta_T\left(\frac{\delta V}{\delta g^{\mu\nu}}\right) = \delta_T(-2\sqrt{g}b_{\mu\nu}) \tag{3.126}$$

with respect to the action $S = \delta_T V +$ "topological term".

To verify the last equation is held, we reparametrize $\varXi$ and consider the trivial equation

$$\int [D\varXi']\Omega' = \int [D\varXi]\Omega, \tag{3.127}$$

where $\Omega$ is an operator. Suppose $\varXi'$ is obtained by the variation $\varXi' = \varXi + \delta_T\varXi$ and expand $\Omega' = \Omega + \delta_T\Omega$, then we obtain

$$\begin{aligned} 0 &= \int [D\varXi']\Omega' - \int [D\varXi]\Omega \\ &= \int [D\varXi]\delta_T\Omega. \end{aligned} \tag{3.128}$$

We used the initial assumption that the measure preserves the supersymmetry $[D\varXi] = [D\varXi']$.

### 3.2.4 Hitchin moduli space

#### (a) Canonical parameter

$$S = \delta V + \frac{i\Psi}{4\pi}\int_M \mathrm{Tr}F\wedge F \tag{3.129}$$

where $V$ is a gauge invariant function and $\Psi$, called the **canonical parameter**, is given by

$$\Psi = \mathrm{Re}(\tau) + i\mathrm{Im}(\tau)\frac{t^2-1}{t^2+1}. \tag{3.130}$$

It is straightforward to show

$$S : \Psi \mapsto -\frac{1}{\Psi}, \tag{3.131}$$

$$T : \Psi \mapsto \Psi + 1. \tag{3.132}$$

**(b) Hitchin equations**

We consider the case where the fields are invariant under supersymmetry. Since this condition does not depend on the overall scaling, it is parametrized by $t = v/u \in \mathbb{C} \cup \{\infty\}$. Especially vanishing of the right hand sides of (3.101) $\sim$ (3.106) gives

$$(F - \phi \wedge \phi + tD\phi)^+ = 0, \tag{3.133}$$

$$(F - \phi \wedge \phi - t^{-1}D\phi)^- = 0, \tag{3.134}$$

$$D^*\phi + \frac{1}{t}[\overline{\sigma}, \sigma] = 0, \tag{3.135}$$

$$D^*\phi - t[\overline{\sigma}, \sigma] = 0, \tag{3.136}$$

$$D\sigma + t[\sigma, \phi] = 0, \tag{3.137}$$

$$D\sigma - \frac{1}{t}[\sigma, \phi] = 0. \tag{3.138}$$

We combine (3.133) and (3.134) into the equation

$$F + \frac{t - t^{-1}}{2}D\phi + \frac{t + t^{-1}}{2} \star D\phi - \phi \wedge \phi = 0. \tag{3.139}$$

The most important value of $t$ is $t = \pm i$ and $t = \pm 1$.

For $t = \pm i$, let $\mathcal{A} = A + i\phi$ and $\bar{\mathcal{A}} = A - i\phi$ be the $G_{\mathbb{C}}$ valued connections. Then the equation (3.139) is equivalent to that curvature $\mathcal{F}_{\mathcal{A}} = d\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$ vanish $\mathcal{F}_{\mathcal{A}} = 0$. Especially $t = i$ and $\mathcal{A} = A + i\phi$ lead

$$\mathcal{F}_{\mathcal{A}} = 0, \tag{3.140}$$

$$D_A^*\phi + i[\sigma, \overline{\sigma}] = 0, \tag{3.141}$$

$$D_{\mathcal{A}}\sigma = D_A\sigma + [A + i\phi, \sigma] = 0. \tag{3.142}$$

Readers should be careful with the different notations $D_A$ and $D_{\mathcal{A}}$.

Now we take our four-manifold $M$ to be a product $M = W \times \mathbb{R}$, $W = C \times I$, where $C$ is a compact Riemann surface and $I$ is a compact subspace of $\mathbb{R}$. We denote by $y$ a coordinate on $I$ and take $y = 0$ is an end of $I$. We write $\sigma = \frac{1}{\sqrt{2}}(X_1 + iX_2)$ and $\phi = \phi_y dy,\ \phi_y = X_3$, where $X_i (i = 1, 2, 3)$ take values in the real adjoint bundle $\mathrm{ad}(E)$. Suppose $A = 0$, then the equations (3.140)$\sim$(3.142) result in the **Nahm equations**

$$\frac{dX_i}{dy} + \epsilon_{ijk}[X_j, X_k] = 0 \tag{3.143}$$

as mentioned in [54].

For $t = 1$, we take $M = W \times \mathbb{R}$, where $W$ is a three-manifold, and denote by $s$ a coordinate of $\mathbb{R}$ so that $\phi = \phi_s ds$. Then the equation (3.139) results in

$$F = - \star D\phi \tag{3.144}$$

since $ds \wedge ds = 0$, and this is nothing but the **Bogomoln'yi equation**, which will play a significant role when we discuss 't Hooft operator in section 3.2.6.

Now we take our four-manifold $M$ as a product of two Riemannn surfaces $C$ and $\Sigma$, where we assume $C$ is compact, has genus $g > 1$ and sufficiently smaller than $\Sigma$ so that we can reduce the four-dimensional gauge theory on $M = C \times \Sigma$ to an effective theory on $\Sigma$. The generic configurations of the effective theory on $\Sigma$ turns out to be those that satisfy

$$F - \phi \wedge \phi = 0, \quad D\phi = D^* \phi = 0, \tag{3.145}$$

$$D\sigma = [\phi, \sigma] = [\overline{\sigma}, \sigma] = 0, \tag{3.146}$$

where $\phi$ and $A$ are pull-backs from $C$. The equations in (3.145) are known as **Hitchin equations**, and the space of solutions up to gauge transformations is a complex and noncompact manifold of dimension $2(g-1)\mathrm{dim}G$, called the **Hitchin moduli space**, which we denote $\mathcal{M}_H$ or $\mathcal{M}_H(G, C)$. Eventually the four-dimensional twisted $\mathcal{N} = 4$ SYM theory reduced to a non-linear sigma model of maps $\Phi : \Sigma \to \mathcal{M}_H(G, C)$. What our assumption that $\Sigma$ is sufficiently larger than $C$ amount to is this, that there exist non-constant but slowly varying maps $\Phi : \Sigma \to \mathcal{M}_H(G, C)$.

Here we give some important properties of the Hitchin moduli space:

1. non-compact
2. connected if $G$ is simply connected
3. completely integrable Hamiltonian system
4. hyper Kähler manifold

We rationalize 2 from the fact that the connected components of $\mathcal{M}_H(G, C)$ is labeled by elements of $H^2(C, \pi_1(G)) \cong \pi_1(G)$.

#### (c) The Hyper Kähler structure of Hitchin moduli space

Our goal of this paragraph is to show a guideline to prove the following theorem. Some of basic concepts about Käher and hyper Kähler manifolds are summarized in appendix B.1.

**Theorem 3.2.1.** the Hitchin moduli space $\mathcal{M}_H(G,C)$ has hyper Kähler structure.

From now, we work on a local complex coordinate $z$ on $C$. Then the gauge fields and scaler fields are written as

$$A = A_z dz + A_{\overline{z}} d\overline{z}, \tag{3.147}$$

$$\phi = \phi_z dz + \phi_{\overline{z}} d\overline{z}. \tag{3.148}$$

Let $\mathcal{W}$ be the space of all $A$ and $\phi$. We regard $\mathcal{W}$ as an infinite dimensional affine space, which has a metric defined by

$$ds^2 = -\frac{1}{4\pi}\int_C |d^2z| \mathrm{Tr}(\delta A_z \otimes \delta A_{\overline{z}} + \delta A_{\overline{z}} \otimes \delta A_z + \delta\phi_z \otimes \delta_{\overline{z}} + \delta\phi_{\overline{z}} \otimes \delta\phi_z), \quad |d^2z| = idzd\overline{z}, \tag{3.149}$$

where we write $\delta$ for the exterior derivative on $\mathcal{M}_H(G,C)$. We introduce three complex complex structures $I, J$ and $K$ on $\mathcal{W}$. Basically we define the actions of those operators by their transposed matrices as follows. We define $I$ by

$${}^tI(\delta A_z) = -i\delta A_z, \tag{3.150}$$

$${}^tI(\delta A_{\overline{z}}) = i\delta A_{\overline{z}}, \tag{3.151}$$

$${}^tI(\delta\phi_z) = i\delta\phi_z, \tag{3.152}$$

$${}^tI(\delta\phi_{\overline{z}}) = -i\delta\phi_{\overline{z}}, \tag{3.153}$$

which satisfy $I^2 = ({}^tI)^2 = -1$. $\delta A_{\overline{z}}$ and $\delta\phi_z$ are the linear holomorphic functions in $I$.

Likewise we define $J$ by

$${}^tJ(\delta A_z) = -\delta\phi_z, \tag{3.154}$$

$${}^tJ(\delta A_{\overline{z}}) = -\delta\phi_{\overline{z}}, \tag{3.155}$$

$${}^tJ(\delta\phi_z) = \delta A_z, \tag{3.156}$$

$${}^tJ(\delta\phi_{\overline{z}}) = \delta A_{\overline{z}}, \tag{3.157}$$

which satisfy $J^2 = ({}^tJ)^2 = -1$. $\mathcal{A}_z = A_z + i\phi_z$ and $\mathcal{A}_{\overline{z}} = A_{\overline{z}} + i\phi_{\overline{z}}$ are the linear holomorphic functions in $J$.

Finally we define $K$ by

$$ {}^tK(\delta A_z) = i\delta\phi_z, \tag{3.158} $$

$$ {}^tK(\delta A_{\overline{z}}) = -i\delta\phi_{\overline{z}}, \tag{3.159} $$

$$ {}^tK(\delta\phi_z) = i\delta A_z, \tag{3.160} $$

$$ {}^tK(\delta\phi_{\overline{z}}) = -i\delta A_{\overline{z}}, \tag{3.161} $$

which satisfy $K^2 = ({}^tK)^2 = -1$. $A_z + \phi_z$ and $A_{\overline{z}} - \phi_{\overline{z}}$ are the linear holomorphic functions in $K$.

These linear operators $I, J$ and $K$ obey ${}^tK{}^tJ{}^tI = -1$. Taking the transpose, we get $IJK = -1$. Moreover

$$ ds^2 ={}^tI \otimes {}^tI(ds^2) ={}^tJ \otimes {}^tJ(ds^2) ={}^tK \otimes {}^tK(ds^2). \tag{3.162} $$

Therefore, referring the definition B.1.21, we have checked $\mathcal{W}$ is a hyper Kähler manifold with complex structures $I, J, K$.

It is convenient to introduce the complex structure $I_w$, which is parametrized by $w \in \mathbb{CP}^1$, defined by

$$ I_w = \frac{I - w\overline{w}}{I + w\overline{w}} I + \frac{i(w - \overline{w})}{I + w\overline{w}} J + \frac{w + \overline{w}}{I + w\overline{w}} K. \tag{3.163} $$

The linear holomorphic functions in $I_w$ are $A_z + w^{-1}\phi_z$ and $A_{\overline{z}} - w\phi_{\overline{z}}$. For instance $I_1 = K$ and $I_{-i} = J$. We set $I_0 = I$ and $I_\infty = -I$.

By the definition B.1.10 of the fundamental 2-forms, it is straightforward to find $\omega_I$, $\omega_J$ and $\omega_K$ associated with the three complex structures $I, J$ and $K$.

$$ \begin{aligned} \omega_I &={}^tI \otimes 1(ds^2) \\ &= -\frac{i}{2\pi}\int_C |d^2z| \mathrm{Tr}(\delta A_{\overline{z}} \wedge \delta A_z - \delta\phi_{\overline{z}} \wedge \delta\phi_z) \\ &= -\frac{1}{4\pi}\int_C \mathrm{Tr}(\delta A \wedge \delta A - \delta\phi \wedge \delta\phi), \end{aligned} \tag{3.164} $$

$$ \begin{aligned} \omega_J &={}^tJ \otimes 1(ds^2) \\ &= \frac{1}{2\pi}\int_C |d^2z| \mathrm{Tr}(\delta\phi_{\overline{z}} \wedge \delta A_z + \delta\phi_z \wedge \delta A_{\overline{z}}), \end{aligned} \tag{3.165} $$

$$
\begin{aligned}
\omega_K &= {}^t K \otimes 1(ds^2) \\
&= \frac{i}{2\pi} \int_C |d^2 z| \mathrm{Tr}(\delta\phi_{\overline{z}} \wedge \delta A_z - \delta\phi_z \wedge \delta A_{\overline{z}}) \\
&= \frac{1}{2\pi} \int_C \mathrm{Tr}\ \delta\phi \wedge \delta A.
\end{aligned}
\tag{3.166}
$$

There is one more step to show that the Hichin moduli space $\mathcal{M}_H(G, C)$ is indeed a hyper Kähler manifold. Let $\epsilon$ generate an infinitesimal gauge transformation and $V(\epsilon)$ be the corresponding vector field on $\mathcal{W}$, which acts by $\delta A = -D\epsilon,\ \delta\phi = [\epsilon, \phi]$. The appropriate moment maps $\mu_I, \mu_J$ and $\mu_K$ turn out to be

$$
\mu_I = -\frac{1}{4\pi} \int_C \mathrm{Tr}\ \epsilon(F - \phi \wedge \phi), \tag{3.167}
$$

$$
\mu_J = -\frac{1}{4\pi} \int_C |d^2 z| \mathrm{Tr}\ \epsilon(D_z \phi_{\overline{z}} + D_{\overline{z}} \phi_z), \tag{3.168}
$$

$$
\mu_K = -\frac{1}{4\pi} \int_C |d^2 z| \mathrm{Tr}\ \epsilon(D_z \phi_{\overline{z}} - D_{\overline{z}} \phi_z). \tag{3.169}
$$

Indeed, one can check they are the right momentum maps as follows. We perform some calculations for (3.168). The starting point is make a small change for $\phi_z$ and $A_z$ so that $\phi_{\overline{z}} \to \phi_{\overline{z}} + \delta\phi_{\overline{z}}$ and $A_z \to A_z + \delta A_z$. Then $D_z \phi_{\overline{z}} = \partial_z \phi_{\overline{z}} + [A_z, \phi_{\overline{z}}]$ is transformed into

$$
D_z \phi_{\overline{z}} \to D_z \delta\phi_{\overline{z}} + [\delta A_z, \phi_{\overline{z}}] + [\delta A_z, \delta\phi_{\overline{z}}] + D_z \phi_z \tag{3.170}
$$

Therefore we obtain $\delta D_z \phi_{\overline{z}} = D_z \delta\phi_{\overline{z}} + [\delta A_z, \phi_{\overline{z}}]$. We define the first term of (3.168) as

$$
\mu_1 = -\frac{1}{4\pi} \int_C |d^2 z| \mathrm{Tr}\ \epsilon D_z \phi_{\overline{z}}. \tag{3.171}
$$

Then

$$
\begin{aligned}
\delta\mu_1 &= -\frac{1}{4\pi} \int_C |d^2 z| \mathrm{Tr}\ \epsilon \delta D_z \phi_{\overline{z}} \\
&= -\frac{1}{4\pi} \int_C |d^2 z| \mathrm{Tr}\ \epsilon(D_z \delta\phi_{\overline{z}} + [\delta A_z, \phi_{\overline{z}}]) \\
&= -\frac{1}{4\pi} \int_C |d^2 z| \mathrm{Tr}(-D_z \epsilon \wedge \delta\phi_{\overline{z}} + \epsilon[\delta A_z, \phi_{\overline{z}}]) \\
&= \frac{1}{4\pi} \int_C |d^2 z| \mathrm{Tr}(D_z \epsilon \wedge \delta\phi_{\overline{z}} + [\epsilon, \phi_{\overline{z}}] \wedge \delta A_z) \\
&= \frac{1}{4\pi} \int_C |d^2 z| \mathrm{Tr}(-\delta A_z \wedge \delta\phi_{\overline{z}} + \delta\phi_{\overline{z}} \wedge \delta A_z) \\
&= \frac{1}{2\pi} \int_C |d^2 z| \mathrm{Tr}(\delta\phi_{\overline{z}} \wedge \delta A_z)
\end{aligned}
\tag{3.172}
$$

The second term $\mu_2$ of (3.168) is derived likewise and it is actually $\delta\mu_2 = \frac{1}{2\pi}\int_C |d^2z| \mathrm{Tr}(\delta\phi_z \wedge \delta A_{\overline{z}})$. As a consequence, we finished to show $\delta\mu = \delta\mu_1 + \delta\mu_2 = \iota_V \omega_J$.

As mentioned in the theorem B.1.28, the quotient $(\mu_I^{-1}, \mu_J^{-1}, \mu_K^{-1})(0)/H$, where $H$ is a Lie group acts on $\mathcal{W}$ with the moment maps, is a hyper Kähler manifold. And the space $(\mu_I^{-1}, \mu_J^{-1}, \mu_K^{-1})(0)$ consists of those $(A, \phi)$ which obey the Hitchin equations (3.145)

$$F - \phi \wedge \phi = 0, \quad D_z \phi_{\overline{z}} = D_{\overline{z}} \phi_z = 0. \tag{3.173}$$

So the Hitchin moduli space $\mathcal{M}_H(G, C)$ is indeed a hyper Kähler manifold.

### (d) The Hitchin fibration

Our main ingredient in this section is a nonlinear sigma model of the map $\Psi : \Sigma \to \mathcal{M}_H(G, C)$. What we have shown in the previous subsection is $\mathcal{M}_H(G, C)$ has a hyper Kähler structure. What we are interested in next is the space called the **Hitchin fibration** on which we will see that A and B type D-branes exist.

In [19], N. Hitchin introduced a **Higgs $G$-bundle** on $X$. Firstly we describe it in its complex structure $I$, in which we regard $A_{\overline{z}}$ and $\phi_z$ as holomorphic variables. A Higgs $G$-bundle on $C$ is a pair $(E, \varphi)$, where $E$ is a algebraic $G$-bundle on $C$ and $\varphi = \phi_z dz \in H^0(C, \mathfrak{g}_E \otimes K_C)$ is a Higgs field on it, where $\mathfrak{g}_E = E \times_G \mathfrak{g}$ is the adjoint vector bundle and $K_C$ is the canonical bundle of $E$.

In complex structure $I$, the Hitchin moduli space $\mathcal{M}_H(G, C)$ is understood as the moduli space of **semistable** Higgs $G$-bundles on $C$.

For any Higgs bundle $(E, \varphi)$ and an invariant polynomial $P$ of degree $d$ on the Lie algebra $\mathfrak{g}$, the Hitchin fibration is constructed via a gauge invariant map $\pi$, which is surjective [19, 20],

$$\pi : \mathcal{M}_H(G, C) \to \mathbf{B} = \oplus_{i=1}^{l} H^0(C, K_C^{\otimes d_i}) \tag{3.174}$$

$$(E, \varphi) \mapsto (P_1(\varphi), \cdots, P_l(\varphi)), \tag{3.175}$$

where $l = \mathrm{rank}\mathfrak{g}$ and each polynomial $P_i$ has degree $d_i$, which are the exponents of $\mathfrak{g}$ plus 1. For each point $p \in \mathbf{B}$, the **Hichin fiber $\mathbf{F}_p$** is defiend by $\mathbf{F}_p = \pi^{-1}(p)$, which is known to be compact. The degrees $d_i$ of those polynomials satisfy

$$\sum_i (2d_i - 1) = \dim_{\mathbb{R}} G. \tag{3.176}$$

Since $\dim_{\mathbb{C}}\mathcal{M}_H(G,C) = 2(g-1)\dim_{\mathbb{R}}G$ and $\dim_{\mathbb{C}}H^0(X, K_C^{\otimes d_i}) = (2d_i - 1)(g-1)$, where $g > 1$ is genus of $C$, it turn out that

$$\dim_{\mathbb{C}}\mathbf{F} = \dim_{\mathbb{C}}\mathbf{B} = \frac{1}{2}\dim_{\mathbb{C}}\mathcal{M}_H(G,C) = (g-1)\dim_{\mathbb{R}}G. \tag{3.177}$$

Therefore $\mathbf{B} = \oplus_{i=1}^{l} H^0(C, K_C^{\otimes d_i}) \cong \mathbb{C}^{(g-1)\dim_{\mathbb{R}}(G)}$, since initially $\mathbf{B}$ is a linear space over $\mathbb{C}$. One of the main properties of $\mathcal{M}_H(G,C)$ is a **complete integrable system** in the complex structure $I$ as shown in [20]. In fact, such Hamiltonians are defined by linear functions on $\mathbf{B}$. For $G = SU(2)$, the Hamiltonians are, as given in [19],

$$H_a = \int_C \alpha_a \wedge \mathrm{Tr}\varphi^2, \quad a = 1, \cdots, 3(g-1), \tag{3.178}$$

where $\alpha_a$ are (0,1)-forms valued in the holomorphic tangent bundle to $C$. For a generic $G$, we simply replace $\mathrm{Tr}\varphi^2$ to the invariant polynomials $P_i(\varphi)$ and $\alpha_a$ to $\alpha_{a,i} \in H^1(C, K^{\otimes(1-d_i)})$.

**Theorem 3.2.2.** (Liouville and Arnold)
Let $(M, \omega, H)$ be a complete integrable system. Then for a set of maps $f = (f_1, \cdots, f_n) : M \to \mathbb{R}^n$ and any $c \in \mathrm{Image}(f)$,

$$M_f = f^{-1}(c) \cong T^k \times \mathbb{R}^{n-k}. \tag{3.179}$$

Moreover if $M_f$ is compact and connected, then

$$M_f \cong T^n = \mathbb{R}^n/\mathbb{Z}^n. \tag{3.180}$$

As a matter of fact $\mathbf{F}_p$ is compact and connected. Hence the Hitchin fiber $\mathbf{F}_p$ is homeomorphic to a torus $T^{2(g-1)\dim_{\mathbb{R}}(G)}$.

For example, the invariant polynomials on the classical Lie algebra $\mathfrak{g}$ are given in the following table. Here $X$ is an element in $\mathfrak{g}$.

| type | $\mathfrak{g}$ | Invariant polynomial $P$ | $i$ |
|---|---|---|---|
| $A_l$ | $sl_{l+1}(\mathbb{C})$ or $su_{l+1}$ | $\mathrm{Tr}(X^i)$ | $2 \le i \le l+1$ |
| $B_l$ | $so_{2l+1}(\mathbb{C})$ or $so_{2l+1}$ | $\mathrm{Tr}(X^{2i})$ | $1 \le i \le l$ |
| $C_l$ | $sp_l(\mathbb{C})$ or $sp_l$ | $\mathrm{Tr}(X^{2i})$ | $1 \le i \le l$ |
| $D_l$ | $so_{2l}(\mathbb{C})$ or $so_{2l}$ | $\mathrm{Tr}(X^{2i})$, $(-1)^l\det(X)$ | $1 \le i \le l-1$ |

For instance, for $G = SU(N)$, the degrees of $\mathrm{Tr}(X^i)$ are $i$ $(2 \leq i \leq N)$, and then surely we can verify the above formula (3.176) $\sum_{i=2}^{N}(2i-1) = N^2 - 1 = \dim_{\mathbb{R}}(SU(N))$ is correct.

Even if we switch $G$ to its Langlands dual group ${}^LG$ and we write ${}^L\mathbf{B}$ for the dual of $\mathbf{B}$, still it is possible to identify ${}^L\mathbf{B}$ as $\mathbf{B}$, which is essentially based on Chevalley's theorem. We can verify the truth of this statement for the examples in the above table, since the dual of $B_l$ is $C_l$, and $A_l$ and $D_l$ are self-dual respectively.

Consequently we obtain the two Hitchin fibrations over the same base space $\mathbf{B}$:

$$\begin{tikzcd} \mathcal{M}_H(G,C) \arrow[dr] & & \mathcal{M}_H({}^LG,C) \arrow[dl] \\ & \mathbf{B} & \end{tikzcd} \tag{3.181}$$

Moreover for a generic $p \in \mathbf{B}$ we have two torus $\mathbf{F}_p$ and ${}^L\mathbf{F}_p$. As stated in [46], the fibers of the two mirror fibrations over the same point in the base space are $T$-dual to each other. In this sense, we say that the tori $\mathbf{F}_p$ and ${}^L\mathbf{F}_p$ are $T$-dual.

Surprisingly this $T$-duality originates from $S$-duality in four dimensions as discussed in [4, 17]. And this is actually in accord with our construction of sigma model, showing of which is our next purpose. That is a remarkable accomplishment, in the sense that SYZ approach to mirror symmetry [46], Hausel and Thaddeus showed in [18] that $\mathcal{M}_H(G,C)$ and $\mathcal{M}_H({}^LG,C)$ are indeed mirror partners if G is a type $A_l$ group, and they predicted this would be true for any reductive group. And Hitchin proved in [21] this expectation for the exceptional group $G_2$.

To summarize, in the context of geometric Langlands correspondence, we claim that the $S$-duality is equivalent to the mirror symmetry.

### 3.2.5 $A$-models and $B$-models

We want to construct a topological field theory (TFT) from our sigma model $\Phi : \Sigma \to \mathcal{M}_H(G,C)$. We start with revisiting the family of the complex structures $I_w$ of the Hitchin moduli space $\mathcal{M}_H(G,C)$. Recall $I_w$ is parametrized by $w \in \mathbb{CP}^1$ and defined as

$$I_w = \frac{1 - w\overline{w}}{1 + w\overline{w}} I + \frac{i(w - \overline{w})}{1 + w\overline{w}} J + \frac{w + \overline{w}}{1 + w\overline{w}} K. \tag{3.182}$$

The linear holomorphic functions in $I_w$ are $A_z + w^{-1}\phi_z$ and $A_{\overline{z}} - w\phi_{\overline{z}}$.

Since $\mathcal{M}_H$ is hyper Kählerian and it has (4,4) supersymmetry, all we have to do is to pick up a pair of complex structures $(J_+, J_-)$ form $\mathcal{M}_H$, which automatically is identified with a structure of (2,2) supersymmetry. If $J_+ = -J_- = \hat{J}$, we obtain $A$**-model** in complex structure $\hat{J}$. In contrast, if $J_+ = +J_- = \hat{J}$, we obtain $B$**-model** in complex structure $\hat{J}$.

To chose $(J_+, J_-)$, we assign a pair of parameters $(w_+, w_-) = (-t, t^{-1})$. An $A$-model is obtained if and only if $t$ is real. In contrast, we obtain a $B$-mode If and only if $w_+ = w_-$, i.e. $t = \pm i$.

For examples, the value $t = -1$ corresponds to the complex structures $(I_1, I_{-1}) = (K, -K)$, so it is $A$-model in complex structure $K$.

Since $SL_2(\mathbb{Z})$ acts on $t$ by

$$t \mapsto t\frac{c\tau + d}{|c\tau + d|} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}), \tag{3.183}$$

the acton of $SL_2(\mathbb{Z})$ on $(w_+, w_-)$ is

$$w_+ \mapsto w_+\frac{c\tau + d}{|c\tau + d|}, \tag{3.184}$$

$$w_- \mapsto w_-\frac{|c\tau + d|}{c\tau + d}. \tag{3.185}$$

Therefore the $A$-model in complex structure $K$ is mapped to $(w_+, w_-) = (-i, -i)$, which is the $B$-model in $J$. Thus $S$-duality shows the $A$-model for $\mathcal{M}_H(G, C)$ in complex structure $K$ is equivalent to the $B$-model for $\mathcal{M}_H({}^LG, C)$ in $J$. Furthermore, recall that the Higgs field $\varphi = \phi_z dz$ and the complex structure rotate in such a way that

$$\varphi \mapsto \lambda\varphi, \tag{3.186}$$

$$I_w \mapsto I_{\lambda^{-1}w}. \tag{3.187}$$

So setting $\lambda = i$ make the complex structure $K$ into $J$. Thus, we get a mirror symmetry that maps the $A$-model in $J$ to $B$-model in $J$, namely $\mathcal{M}_H(G, C)$ and $\mathcal{M}_H({}^LG, C)$ are a mirror pair in $J$.

### 3.2.6 Wilson and 't Hooft operators

The geometric Langlands duality can be seen physically via the duality of **Wilson** and **'t-Hooft operators** as pointed out by Kapustin in [24]. Since these are the most elementally operators in gauge theory, it is worthwhile to investigate them.

### (a) Wilson operators

Let $G$ be a gauge group and $A$ be a connection on $G$-bundle $\pi : E \to M$. A Wilson loop $W_0$ is the holonomy of $A$ in an irreducible representation $R$ around a loop $\gamma$ in $M$:

$$W_0(R,\gamma) = \mathrm{Tr}_R P \exp\left(-\oint_\gamma A\right). \tag{3.188}$$

When we return to the twisted $\mathcal{N} = 4$ super Yang-Mills theory, the connection $A$ obeys the transformation law (3.97)

$$\delta_T A_\mu = iu\psi_\mu + iv\tilde{\psi}_\mu, \tag{3.189}$$

under which the Wilson loop is not invariant unless $(u, v) = (0, 0)$. However we have another adjoint valued one-form $\phi$, which transforms as (3.98)

$$\delta_T \phi_\mu = iv\psi_\mu - iu\tilde{\psi}_\mu. \tag{3.190}$$

If $t = v/u$ is equal to $\pm i$, then a linear combination of $A$ and $\phi$ is invariant. More precisely, we obtain the invariant Wilson loop operators

$$W(R,\gamma) = \begin{cases} \mathrm{Tr}_R P \exp\left(-\oint_\gamma (A + i\phi)\right) & t = i \\ \mathrm{Tr}_R P \exp\left(-\oint_\gamma (A - i\phi)\right) & t = i \end{cases} \tag{3.191}$$

Rewriting these combinations as $\mathcal{A} = A + i\phi$ and $\bar{\mathcal{A}} = A - i\phi$, the equation (3.139) means they are flat connections $\mathcal{F} = \bar{\mathcal{F}} = 0$. The existence of a flat connection is important for the geometric Langlnads duality. Let $\mathcal{Y}(G, C)$ be the moduli space of flat $G$-bundle on $C$. Then a solution $(A, \phi)$ of the Hitchin equations determine a point in $\mathcal{Y}(G, C)$. In other words, the operator $\overline{\partial}$ defined by the (0,1) part of the derivative $D_A$ gives $E$ a holomorphic structure. Moreover, with respect to the (1,0) part $\varphi$ of $\phi$ and the canonical bundle $K$ of $C$, $\varphi$ become a holomorphic section of $K \otimes \mathrm{ad}(E)$. Therefore, the pair $(E, \varphi)$ defines a **Higgs bundle**. We will back to this topic later.

Though we have considered the Wilson loop operators around a closed curve $\gamma$, it is also possible to work on an open curve, in which the Wilson line operators is defined as the matrix of parallel transport of a section along the open curve.

**(b) 't Hooft operators**

Recalling the canonical parameter of the form (3.130)

$$\Psi = \mathrm{Re}(\tau) + i\mathrm{Im}(\tau)\frac{t^2-1}{t^2+1}, \tag{3.192}$$

the above case $t = \pm i$ correspond to $\Psi = \infty$. Since $\Psi = \infty$ goes to $\Psi = 0$ under the $S$-transformation, there must be the dual operators of the Wilson operators at $\Psi = 0$, which turn out to be the 't Hooft operators.

In order to explain the 't Hooft operator in a representation $R$ of $G$, we first restrict ourself to $G = U(1)$ gauge theory on $\mathbb{R}^4 = \mathbb{R} \times \mathbb{R}^3$, which is the most basic example. Let $(t, \overrightarrow{x})$ be local coordinates of $\mathbb{R} \times \mathbb{R}^3$ and $\gamma = \left\{(t, \overrightarrow{0}) : t \in \mathbb{R}\right\}$ be a line. The singular Dirac monopole is a solution of the Maxwell equations $dF = d \star F = 0$ on $\mathbb{R}^4 \setminus \gamma$ and has singularities at $\gamma$, where $F$ is the curvature of $U(1)$ connection $A$ and $\star$ is the Hodge star operator. Such a solution is given by the pull back of

$$F = \frac{i}{2} \star_3 d\left(\frac{1}{|\overrightarrow{x}|}\right) \tag{3.193}$$

to $\mathbb{R}^4 \setminus \gamma$, where $\star_3$ is the Hodge star operator in $\mathbb{R}^3$ and the form (3.193) is indeed a solution of the Maxwell equations in a constant time slice $\mathbb{R}^3 \setminus \{0\}$. $F$ is normalized so that $\frac{i}{2\pi}\int_{S^2} F = 1$, where $S^2$ is a two-sphere enclosing the point $\overrightarrow{x} = \overrightarrow{0}$.

We define the 't Hooft operator with charge $m$ in $U(1)$ gauge theory associated with a general curve $\gamma$ by requiring that the curvature near $\gamma$ pretends as

$$F \sim \frac{im}{2} \star_3 d\left(\frac{1}{r}\right), \tag{3.194}$$

where $r$ is the distance form $\gamma$ and $\star_3$ is an operation looks like the $\star$ operator on planes normal to $\gamma$.

Now we return to $\mathcal{N} = 4$ supersymmetric Yang-Mills theory with gauge group $U(1)$. We want to define the 't Hooft operator in a similar manner to the non supersymmetric case. However, we need to add a constraint in which the 't Hooft operator preserves the topological symmetry. Recall that we are motivated to introduce the 't Hooft operator by asking the dual of the Wilson operator at $\Psi = 0$, we find that the appropriate condition is to choose $t = 1$ and $\tau$ imaginary. At $t = 1$, as we have already seen in (3.133) and (3.134), a pair of fields $(A, \phi)$ should satisfy

$$(F + d\phi)^+ = (F - d\phi)^- = 0. \tag{3.195}$$

We take a curve $\gamma$ as a straight line for simplicity and then we have such a solution

$$\begin{aligned} F =& \frac{i}{2} \star_3 d\left(\frac{1}{|\overrightarrow{x}|}\right) \\ \phi =& \frac{i}{2}\frac{1}{|\overrightarrow{x}|}ds, \end{aligned} \tag{3.196}$$

where $s$ is the time coordinate along $\gamma$ and again $\star_3$ is the three-dimensional in the directions normal to $\gamma$. For a general curve $\gamma$ we define the 't Hooft operator as we did in the non supersymmetric case.

So far we have considered the $U(1)$ gauge theory, however we are more interested in a generic gauge group $G$. To construct an 't Hooft operator associated with $G$, we pick a homomorphism $\rho : U(1) \to G$ and define an 't Hooft operator by requiring that the fields should have a singularity along a curve $\gamma$ that looks like the image under $\rho$ of the singularity in (3.196). What is important for the Langlands correspondence is that the 't Hooft operators are classified by irreducible representation of ${}^LG$. Recall that if $G$ is a compact connected Lie group and $T$ is a maximal torus of $G$, then any element of $G$ is conjugate to an element of $T$, namely for any element $g \in G$, there is an element $x \in G$ such that $x^{-1}gx \in T$. Therefore when such a homomorphism $\rho : U(1) \to G$ is given, the conjugacy class of $\rho$ can be classified by a coweight of $G$, which is a weight of the dual group ${}^LG$. Hence **an 't Hooft operator of $G$ is dual to a Wilson operator of** ${}^LG$.

### (c) 't Hooft/Hecke correspondences

Now we explain relations between 't Hoof operators and Hecke transformations. We mainly focus on static 't Hooft operators and take our four-manifold as $M = I \times C \times \mathbb{R}$, where $I \subset \mathbb{R}$ is a closed interval and $C$ is a Riemann surface. We let $W$ denote the three-manifold of the form $W = I \times C$. The configurations of field associated with the supersymmetric 't Hooft operators that emerge at $\Psi = 0$ satisfy

$$\begin{aligned} F - \phi \wedge \phi + \star D\phi = 0 \\ D^*\phi = 0, \end{aligned} \tag{3.197}$$

which are obtained setting $t = 1$ in (3.139), (3.135) and (3.136).

We take the $G$-bundle $E$ and connection $A$ as pullbacks from $W$ and $\phi$ as $\phi = \phi_s ds$ (one can show that it is possible to take $A_s = \phi_y = 0$ [26]), where $\phi_s$ is an ad$(E)$-valued zero form on $W$ and $s$ is coordinate on $\mathbb{R}$. Then the equations become the Bogomoln'yi equation

$$F = - \star D\phi_s \tag{3.198}$$

as we have derived in (3.144). Let $(x_1, x_2, x_3)$ be local coordinates on $W$. Then the star operator $\star$ acts as $\star(dx_1) = dx_2 \wedge dx_3$. Now we parametrize $I$ by $y = x_1$ and choose $C = \mathbb{R}^2$, parametrized by $z = x_2 + ix_3$. Then we have

$$\begin{aligned} \star dy &= \frac{i}{2} dz \wedge d\bar{z} \\ \star dz &= -i dz \wedge dy \\ \star d\bar{z} &= i d\bar{z} \wedge dy. \end{aligned} \tag{3.199}$$

The left hand side of the Bogomoln'yi equation (3.198) is

$$F = F_{z\bar{z}} dz \wedge d\bar{z} + F_{yz} dy \wedge dz + F_{y\bar{z}} dy \wedge d\bar{z}. \tag{3.200}$$

And the right hand side is

$$\begin{aligned} - \star D\phi_s &= - \star (D_y \phi_s dy + D_z \phi_s dz + D_{\bar{z}} \phi_s d\bar{z}) \\ &= -\frac{i}{2} D_y \phi_s dz \wedge d\bar{z} + i D_z \phi_s dz \wedge dy - i D_{\bar{z}} \phi_s d\bar{z} \wedge dy. \end{aligned} \tag{3.201}$$

Compering (3.200) and (3.201), we obtain

$$\begin{aligned} F_{z\bar{z}} &= -\frac{i}{2} D_y \phi_s \\ F_{yz} &= -i D_z \phi_s \\ F_{y\bar{z}} &= i D_{\bar{z}} \phi_s \end{aligned} \tag{3.202}$$

For a generic Riemann surface $C$, we write a Kähler metric on $C$ as $h(z, \bar{z})|d^2 z|$ with a positive function $h$ on $C$. Then the first equation in (3.202) become

$$F_{z\bar{z}} = -\frac{ih}{2} D_y \phi_s, \tag{3.203}$$

and the others remain the same.

We work on $A_y = 0$ gauge, then the last equation of (3.202) results in

$$\frac{\partial A_{\bar{z}}}{\partial y} = iD_{\bar{z}}\phi_s, \tag{3.204}$$

which implies that the holomorphic structure of the bundle $E$ is independent of $y$, since the right hand side is the gauge transformation caused by $i\phi_s$.

Suppose there is an 't Hooft operator at $p_0 = (y_0, z_0)$. By definition, the 't Hooft operator creates a singularity there. We write the holomorphic bundle $E_-$ for $y < y_0$ and $E_+$ for $y > y_0$. When we across the singular point $y = y_0$, the bundle $E_-$ jumps to $E_+$, which is described via the Hecke modification.

$G = U(1)$ **case**

For $G = U(1)$, $E$ is simply a line bundle $\mathcal{L}$. We consider sufficiently small neighborhood $U$ of $p_0$, and we identify $\mathcal{L}_-$ with its the trivialization $\mathcal{O}$. $\mathcal{L}_+$ Since the holographic type of $\mathcal{L}$ does not depend on $y$ as we discussed before, the line bundles $\mathcal{L}_+$ and $\mathcal{L}_-$ are isomorphic on $C \setminus p_0$. We identify $\mathcal{L}_+$ with $\mathcal{O}(p_0)^q$, the line bundle whose local section near $p_0$ are holomorphic functions allowed to have a pole of order $q$.

An 't Hooft operator $T(m)$ for $G = U(1)$ is classified by the magnetic charge $m$, and the curvature $F$ behaves like (3.194) near $p = (p_0, y_0)$

$$F \sim \frac{im}{2} \star_3 d\left(\frac{1}{r}\right), \tag{3.205}$$

where $r$ is the distance from $p$ to a point $\overrightarrow{x} \in \mathbb{R} \times C$. Now $S$ be a small two-sphere which contains $p$. Then the first-Chern number is by definition

$$\int_S c_1(\mathcal{L}) = m. \tag{3.206}$$

Let $y_\pm$ be points in $\mathbb{R}$ such that $y_- < y_0 < y_+$, and $C_\pm$ be $y_\pm \times C$. Then when $\mathcal{L}$ is restricted on $C_\pm$, it gives $\mathcal{L}_\pm$. We write

$$\int_{C_\pm} c_1(\mathcal{L}) = q_\pm. \tag{3.207}$$

The boundary of $\mathcal{R} \times C \setminus p$ is the homology cycle $D = C_+ - C_- - S$ and then

$$0 = \int_D c_1(\mathcal{L}) = \int_{C_+} c_1(\mathcal{L}) - \int_{C_-} c_1(\mathcal{L}) - \int_S c_1(\mathcal{L}) = q_+ - q - m. \tag{3.208}$$

Therefore $q_+ = q_- + m$ and inserting the 't Hooft operator at $p$ cause the modification of the bundle in such a way that

$$\mathcal{L}_- = \mathcal{O}_- \to \mathcal{L}_+ = \mathcal{O}(p_0)^m, \tag{3.209}$$

which is nothing but the Hecke modification we described in section 2.2.

$G = U(2)$ **case**

When $G = U(2)$, the Hecke modification of $G$-bundles is same as we considered in section 2.2, and it is parametrized by the Gassmannian manifold. Let us see this type of modification more precisely. Note ${}^L G = G = U(2)$.

We use the similar notations used in the previous example. The weight lattice of $U(2)$ is spanned by $(m_1, \cdots, m_N)$, where $m_1$ and $m_N$ take values in $\mathbb{Z}$. A dominant weight obeys $m_1 \geq m_2 \geq \cdots \geq m_N$. We choose a highest weight ${}^L w$ of ${}^L G = U(2)$ as ${}^L w = (1, 0)$. We take a special decomposition of $E_-$ as $\mathcal{O} \oplus \mathcal{O}$. Then a section of $E_+$ is a pair $(f, g)$ which is a section of $E_-$ away from $p_0$ and is allowed to have a simple pole at $p_0$. We choose a pair $(u, v) \neq (0, 0)$ of complex numbers and a pair $(f_0, g_0)$ of holomorphic functions on $C$. Let $z$ be a holomorphic function on $C$ with a simple zero at $p_0$. we require $(f, g)$ to satisfy the condition

$$(f, g) = (f_0, g_0) + \frac{\lambda}{z}(u, v), \tag{3.210}$$

where $\lambda$ is a nonzero complex number. Under this construction, the family of $E_+$ is parametrized by a copy of $\mathrm{Gr}(1, 2) = \mathbb{C}P^1$ since $E_+$ depends on the pair $(u, v)$ only up to overall scaling.

In general, an arbitrary highest weight of ${}^L G = U(2)$ is given by a pair of integers $(n, m)$ and a rank 2 bundle $E_- = \mathcal{O} \oplus \mathcal{O}$ undergoes the Hecke modification to $E_+ = \mathcal{O}(p_0)^n \oplus \mathcal{O}(p_0)^m$. So far we have considered the case $(n, m) = (1, 0)$ and the space of Hecke modifications is compact $\mathbb{C}P^1$. However, if $n - m \geq 2$, it is generically non-compact. The compactification of space of Hecke modifications for $n - m \geq 2$ accompanies with a singularity, which is physically interpreted as **monopole bubbling** [26, 30].

$G = SU(2)$ **and** ${}^LG = SO(3)$ **case**

Generally, the weight lattice of $SU(N)$ is spanned by $(m_1, \cdots, m_N)$, where $m_1$ and $m_N$ take values in $\mathbb{Z}/N$ such that $m_i - m_j \in \mathbb{Z}$ and $\sum_i m_i = 0$. A dominant weight obeys $m_1 \geq m_2 \geq \cdots \geq m_N$. A highest weight ${}^Lw$ of ${}^LG = SO(3)$ is a highest weight of $G = SU(2)$ which is divisible by 2. Hence it can be written with some integer $k$ as ${}^Lw = (k, -k)$. Therefore $T({}^Lw)$ acts on an $G = SU(2)$ bundle as

$$\mathcal{O} \oplus \mathcal{O} \to \mathcal{O}(p_0)^k \oplus \mathcal{O}(p_0)^{-k}. \tag{3.211}$$

$G = SO(3)$ **and** ${}^LG = SU(2)$ **case**

For our case $G = SO(3)$ and ${}^LG = SU(2)$, a dominant weight ${}^Lw$ is written as ${}^Lw = (n/2, -n/2)$ with integer $n$. A bundle $E_- = \mathcal{O} \oplus \mathcal{O}$ is modified by the 't Hooft/Hecke operator to $E_+ = \mathcal{O}(p_0)^{n/2} \oplus \mathcal{O}(p_0)^{-n/2}$. The nontrivial situation is when $n$ odd, since $G = SO(3)$ dose not have a two-dimensional representation. In this case we should consider a bundle in the adjoint representation of $SO(3)$. With respect to a rank 2 complex vector bundle $E$ and its dual $E^*$, we write $\mathrm{ad}(E)$ for $E \otimes E^*$ and $\mathrm{ad}_0(E)$ for the traceless part of $E \otimes E^*$. Then what we consider is the $SO(3)$ bundle $V = \mathrm{ad}_0(E)$ and the action of $T({}^Lw)$ on $V_- = \mathrm{ad}_0(E_-) = \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}$ results in $V_+ = \mathcal{O} \oplus \mathcal{O}(p_0)^n \oplus \mathcal{O}(p_0)^{-n}$.

### 3.2.7 Electric brane and Magnetic brane

**(a) Eigenbranes**

A brane $\mathcal{B}$ is understood as a boundary condition imposed on the boundary of $\Sigma$ when the original theory on $M = \Sigma \times C$ is compactified on $\Sigma$. A line operator $L$ approaching a boundary with boundary condition $\mathcal{B}$ creates a new boundary condition $L\mathcal{B}$. We say that the brane $\mathcal{B}$ is an **eigenbrane** of the line operator $L$ if

$$L\mathcal{B} = \mathcal{B} \otimes V \tag{3.212}$$

for a vector space $V$. If $V$ is of dimension $n$, $\mathcal{B} \otimes V$ is roughly the sum of $n$ copies of $\mathcal{B}$. A brane is constructed using a vector bundle, called the Chan-Paton bundle, on a manifold. Tensoring a brane $\mathcal{B}$ with a vector space $V$ means that we tensor the Chan-Paton bundle with $V$. For a given brane $\mathcal{B}$, there is a corresponding sheaf $\mathcal{F}$ over $\mathcal{M}_H$ and $\mathcal{B} \otimes V$ is the brane associated with the sheaf $\mathcal{F} \otimes V$ over $\mathcal{M}_H$.

Let $L$ and $L'$ be line operators supported at different points $p$ and $p'$ in $C$. Then they can commute each other in $\Sigma$ without picking up any singularity. Moreover a line operator is locally independent on $C$ in the context of four-dimensional topological field theory. Hence line operators in two-dimensions which originate from loop operators in four-dimensions commute in $\Sigma$, namely we have

$$LL'\mathcal{B} = L'L\mathcal{B}. \tag{3.213}$$

Since the line operators do commute, we may consider a simultaneous eigenbranes for all Wilson operators or all 't Hooft operators. Eigenbranes for Wilson operators are called **electric eigenbranes** and those for 't Hooft operators are called **magnetic eigenbranes**.

The geometric Langrands correspondence is all about the duality between the Wilson operators and the 't Hooft operators and it is natural to see that the associated eigenbranes are dual to each other. We make concrete surveys for those eigenbranes in the rest of this section.

**(b) Electric Eigenbranes**

We first observe the action of Wilson operators on branes. The **universal bundle** over $C$ is a bundle

$$\mathcal{E} \to M_H({}^LG, C) \times C \tag{3.214}$$

with a connection $\hat{A}$ that for all $m \in M_H({}^LG, C)$, the restriction of $\mathcal{E}$ to $m \times C$ is isomorphic to $E_m$, where $E_m$ is a corresponding ${}^LG$-bundle $E_m \to C$ for a given $m \in M_H({}^LG, C)$. Let $\mathcal{B}$ be a $B$-brane associated with a coherent sheaf

$$\mathcal{F} \to M_H({}^LG, C). \tag{3.215}$$

Let $W_p({}^LR)$ be a Wilson operator in representation ${}^LR$ supported at $p \in C$ and $\mathcal{E}({}^LR)|_p$ be the restriction of $\mathcal{E}({}^LR)$ to $M_H({}^LG, C) \times p$, where we write $\mathcal{E}({}^LR)$ for the associated bundle in representation ${}^LR$. Then $W_p({}^LR) \cdot \mathcal{B}$ is the $B$-brane associated with the sheaf $\mathcal{F} \otimes \mathcal{E}({}^LR)|_p$, namely the action of the Wilson operator $W_p({}^LR)$ on the coherent sheaf $\mathcal{F}$ is

$$\mathcal{F} \to \mathcal{F} \otimes \mathcal{E}({}^LR)|_p. \tag{3.216}$$

Comparing the definition of an eigenbrane (3.212), we see that $\mathcal{E}({}^LR)|_p$ must be a constant vector space, in other words the gauge fileds there must be trivial, when restricted to the support of $\mathcal{B}$.

This happens if and only if $\mathcal{B}$ is a zero brane supported at a point $m \in \mathcal{M}_H(^LG, C)$. Hence $\mathcal{F}$ is a skyscraper sheaf supported at $m$.

In order to obtain a corresponding magnetic eigenbrane, we simply operate $S$-duality to an electric eigenbrane. Then a magnetic eigenbrane turn out to be a rank 1 $A$-brane supported on a fiber of the Hitchin fibration as we will observe next.

**(c) Magnetic Eigenbranes**

We describe how 't Hoof operators act on $A$-branes. For more detail one may want to consult [55]. We consider a brane $\mathcal{B}$ supported on a fiber of the Hitchin fibration and endowed with a flat Chan-Paton line bundle. We call such a brane of type **F**. Recall that an electric-eigenbrane is a zero-brane supported on a fiber $^L\mathfrak{F}$ of the Hitchin fibration of $^LG$, and the $T$-duality transforms $^L\mathfrak{F}$ to the dual fiber $\mathfrak{F}$ of that of $G$, in accord with which the electric-eigenbrane is mapped to a rank 1 brane supported on $\mathfrak{F}$, which we call the **magnetic-eigenbrane** corresponding to the electric-eigenbrane. Namely, we regard magnetic-eigenbranes as branes of type **F**. As an electric-eigenbrane is an eigenbrane of a Wilson operator, we expect that a magnetic-eigenbarn should be a eigenbarane of an 't Hooft operator. So a magnetic-eigenbrane obeys

$$T(^Lw, p)\mathcal{B} = \mathcal{B} \otimes V, \tag{3.217}$$

where $V$ is the "eigenvalue". For $G = U(N)$ case the eigenvalues are obtained. For more detail, see section 7 of [55].

### 3.2.8 $A$-branes to $\mathcal{D}$-modules

Let $C$ be a algebraic curve over $\mathbb{C}$ and $G$ be a complex reductive Lie group. What the geometric Langlands correspondence claims is that with respect to each holomorphic $^LG$-bundle $E$ with a holomorphic connection on $C$, there exists a Hecke eigensheaf $\mathrm{Aut}_E$ (with the eigenvalue $E$) on the moduli space $\mathrm{Bun}_G(C)$ of holomorphic $G$-bundles on $C$. Here $\mathrm{Aut}_E$ is a $\mathcal{D}$-module and related with an $A$-brane in physical sense, which brings about the 't Hooft/Hecke correspondence. In terms of the Hitchin moduli space $\mathcal{M}_H(^LG, C)$, the geometric Langlands duality is interpreted that the derived category of coherent sheaves on $\mathcal{Y}({}^LG, C)$, the subset of $\mathcal{M}_H({}^LG, C)$ which consists of flat connections, is equivalent to the derived category of $\mathcal{D}$-module on $\mathrm{Bun}_G(C)$.

We begin with observation on coisotropic $A$-branes [23] on symplectic manifolds. Let $X$ be a symplectic manifold and $Y$ be a its submanifold. $Y$ is **coisotropic** if $I_Y = \{f \in C^\infty(X) : f|_X = 0\}$ is closed under the Poisson bracket. Generically $Y$ satisfies $\dim Y \geq \frac{1}{2}\dim X$. We consider an brane supported on a coisotropic submanifold, called coisotropic brane. Though precise conditions on the curvature of a vector bundle on a coisotropic brane is not known except for the rank one case, it is sufficient for us to get a rough idea for connecting an $A$-brane and a $\mathcal{D}$-module since we only work on the case that $Y = X$ and the $A$-brane has rank one. In this case the Chan-Paton bundle is $U(1)$ bundle whose curvature $F$ satisfies

$$(\omega^{-1}F)^2 = -1, \tag{3.218}$$

where $\omega$ is the symplectic form on $X$. Hence $\hat{N} = \omega^{-1}F$ is an almost complex structure. Moreover $\omega^{-1}F$ turns out to be integrable from the fact that $F$ and $\omega$ are closed 2-from.

Now we move to the case $Y = X = \mathcal{M}_H(G, C)$. We choose $F = (\mathrm{Im}\ \tau)\ \omega_J$ and $\omega = (\mathrm{Im}\ \tau)\ \omega_K$, then we have $\hat{N} = \omega_K^{-1}\omega_J = I$. Here $\omega_I$, $\omega_J$ and $\omega_K$ are those we defined by (3.164), (3.165) and (3.166). The $A$-brane on $\mathcal{M}_H(G, C)$ with this $F$ is called the **canonical coisotropic brane** and abbreviated as $\mathcal{B}_{c.c.}$. Notice that $(\omega_I^{-1}\omega_J)^2 = -1$, which implies $\mathcal{B}_{c.c.}$ is an $A$-brane in complex structure $I$.

Our strategy to relate $\mathcal{B}_{c.c.}$ and a $\mathcal{D}$-module is to see that supersymmetric strings attached to $\mathcal{B}_{c.c.}$ correspond to differential operators on $\mathcal{M}_H(G, C)$. Let us begin with our action

$$S = \int_\Sigma \Phi^*(\omega - iF) + (\text{BRST-exact terms}). \tag{3.219}$$

Since the closed two-forms $\omega_I$ and $\omega_K$ are exact, the action becomes

$$S = -i(\mathrm{Im}\ \tau)\int_{\partial\Sigma} \Phi^*(\varpi) + (\text{BRST-exact terms}), \tag{3.220}$$

where $\varpi$ is a canonical holomorphic one-form on $\mathcal{M}_H(G, C)$:

$$\varpi = \frac{1}{\pi}\int_C |d^2z| \mathrm{Tr}(\phi_z \delta A_{\bar{z}}). \tag{3.221}$$

Let $q^\alpha$ be local holomorphic coordinates on $\mathcal{M}_H(G,C)$ and $p_\alpha$ be linear functions on the fibers of $T^*\mathcal{M}_H(G,C)$ which are canonically conjugate to $\phi_\alpha$. Then it is possible to write $\varpi$ as

$$\varpi = \sum_\alpha p_\alpha dq^\alpha \tag{3.222}$$

and hence we obtain the action of the form

$$S = -i(\text{Im } \tau) \int p_\alpha dq^\alpha + (\text{BRST-exact terms}). \tag{3.223}$$

To see this action gives a module of differential operators, we simply put $\mathcal{L} = -i(\text{Im}\tau)p_\alpha\dot{q}^\alpha$ and action $S = -i(\text{Im}\tau)\int p_\alpha\dot{q}^\alpha dt$. Recall wave function is defined by the path integral

$$\psi(q_f) = \int_{q(t_f)=q_f} DqDp\ e^S, \tag{3.224}$$

where $q_f$ is a final position at time $t = t_f$. The conjugate momentum is by definition $p_\alpha = \frac{\partial\mathcal{L}}{\partial\dot{q}^\alpha}$. Operating the derivative with respect to $q^\alpha$ gives

$$\frac{\partial}{\partial q_f}\psi(q_f) = \int DqDp\ (-i\text{Im}\tau)p_f e^S, \tag{3.225}$$

which implies the correspondences between differential operators $\frac{\partial}{\partial q^\alpha}$ and functions $p_\alpha$:

$$\frac{\partial}{\partial q^\alpha} \leftrightarrow -i(\text{Im}\tau)p_\alpha. \tag{3.226}$$

Hence the operators $\hat{p}_\alpha$ defined by

$$\hat{p}_\alpha = \frac{i}{\text{Im}\tau}\frac{\partial}{\partial q^\alpha} \tag{3.227}$$

produce the non commutative algebra

$$[\hat{q}^\alpha, \hat{p}_\beta] = -\frac{i}{\text{Im}\tau}\delta^\alpha_\beta,\ [\hat{q}^\alpha, \hat{q}^\beta] = [\hat{p}_\alpha, \hat{p}_\beta] = 0, \tag{3.228}$$

where $\hat{q}^\alpha$ act on wave functions as usual. Essentially this is the story to obtain the $\mathcal{D}$-module associated with the canonical coisotropic $A$-brane $\mathcal{B}_{c.c.}$. However this is not a general method to attain $\mathcal{D}$-modules from given $A$-branes since generic $A$-branes are not holomorphic.

### 3.2.9 Categories of $A$-branes and $B$-branes

We partially follow arguments in [10]. From now we assume $\Sigma$ has a boundary $\partial\Sigma$. Let $M$ be a certain manifold with nice properties. Branes are topological spaces associated with boundary conditions for the maps $\Phi : \Sigma \to M$ on $\partial\Sigma$. Let $M' \subset M$ be a submanifold which include $\Phi(\partial\Sigma)$. In the previous subsection, we obtained TFT with $\mathcal{N} = (2, 2)$ supersymmetries:

$$\text{Charges:} \quad Q_+, \ Q_-, \ \overline{Q}_+, \ \overline{Q}_-. \tag{3.229}$$

If there a brane exists, a half of them are preserved and they result in (1,1) supersymmetries:

$$Q_A = Q_- + \overline{Q}_+, \quad \overline{Q}_A = Q_+ + \overline{Q}_-, \tag{3.230}$$

or

$$Q_B = \overline{Q}_+ + \overline{Q}_-, \quad \overline{Q}_B = Q_+ + Q_-. \tag{3.231}$$

In a classical picture, however this is enough for our model, $(Q_A, \overline{Q}_A)$ is preserved if and only if the target space $M$ is Kähler manifold and $M'$ is Lagrangian submanifold. We call this $A$**-model**. While $(Q_B, \overline{Q}_B)$ is preserved if and only if $M$ is a complex manifold. The supersymmetries are protected if $M'$ is a complex submanifold. We call this $B$**-model**.

In general, the category of branes in the $A$-model, called $A$-branes, would contain **Fukaya category**, which consists of pairs $(L, \nabla)$, where $L \subset M$ is a Lagrangian submanifold and $\nabla$ is a flat unitary vector bundle on $L$. On the other hand the category of $B$-branes is the **derived category** of coherent sheaves on $M$. The latter is better understood than the former one.

We write $\mathcal{Y}(^LG, C)$ for $\mathcal{M}_H(^LG, C)$ with respect to the complex structure $J$. What the **homological mirror symmetry** [28] expects is there should be equivalence between the derived categories of $B$-branes on $\mathcal{Y}(^LG, C)$ and that of $A$-branes on $\mathcal{M}_H(G, C)$.

The Langlands correspondence predicts that derived category of $\mathcal{O}$-modules on $\mathrm{Loc}_{^LG}$, which is the moduli stack of flat $^LG$-bundle on $C$, is equivalent to the derived category of $\mathcal{D}$-modules on $\mathrm{Bun}_G$, which is the moduli stack of holomorphic $G$-bundle with holomorphic connection on $C$. Those two (derived) categories are supposed to be mapped by non-abelian Fourier-Mukai transformation. For abelian case $G = GL_1$, this equivalence was shown by G. Laumon [33] and M. Rothstein [42]. Under the Fourier-Mukai transformation, the skyscraper sheaf $\mathcal{O}_{\mathcal{E}}$ supported at $\mathcal{E} \in \mathrm{Loc}_{^L G}$ is transformed into the Hecke eigensheaf $\mathcal{F}_{\mathcal{E}}$ with eigensheaf $\mathcal{E}$. In physical perspective, the skyscraper sheaf $\mathcal{O}_{\mathcal{E}}$ corresponds to the electric-eigenbrane, which is the eigenbrane of the Wilson operator (functor), and the Hecke eigensheaf corresponds to the magnetic-eigenbrane, which is the eigenbrane of the 't Hooft operator (functor). Hence the prediction made by Kapustin and Witten is expressed by the following picture:

$$\begin{tikzcd}
A\text{-branes on } \mathcal{M}_H(G,C) \arrow[rr, leftrightarrow] \arrow[dr] & & B\text{-branes on } \mathcal{Y}(^L G, C) \arrow[dl] \\
& \mathcal{D}\text{-modules on } \mathrm{Bun}_G(C) &
\end{tikzcd}$$

The upper equivalence is due to the homological mirror symmetry, the right lower arrow comes from the suggestion by Kapustin and Witten, and the right lower arrow is a consequence of the geometrical Langlands correspondence.

# Chapter 4

# Conclusion and future direction

Let us close this review note with a few words. The Langlands program encloses several mysterious aspects hidden in mathematics, which is why the author was fascinated by and jumped in this program. Looking back the history, we find the world the Langlands program embodies is so huge that even with a lot of examples or proofs, which are almost enough for the program to capture one's hart, yet one may still wonder why they should be so. With those things in my mind, it is surprising that only recently the physical applications has been considered. At this time, all of the well known examples in physics are written in terms of the supersymmetry, which have been a main stream in modern physics, however at the same time we know that such a feature does not explicitly appear at least the energy level of our physics in laboratory today. Hence the meaningful question we address is what is a non-supersymmetric example of the geometric Langlands correspondence, and how it is applicable for technology. When we talk about its application for technology, we soon come up with usage of the elliptic curve cryptography is established as fundamental cryptography systems in various fields today. Recalling that elliptic curves are at the one side of the Langlands program, it would be natural to try and find the corresponding application based on other side.

If I were allowed to make some comments on the technical detail of the geometric Langlands correspondence, I would feel some uncertainty in the way to construct the geometrical analogues of automorphic representations. As mentioned previously or in other standard reviews, we treated the Hecke eigensheaves as $\mathcal{D}$-modules on the associated moduli stack by identifying them with perverse sheaves as a consequence of the Riemann-Hilbert correspondence, which is established for complex manifolds and algebraic varieties. But generically such a moduli stack is neither a complex manifold nor an algebraic variety, and hence we are not confident enough to apply the Riemann-Hilbert correspondence for general cases. But, in different words, it can be said that the Langlands correspondence suggests the Riemann-Hilbert correspondence is true for a general algebraic stack.

For physical interpretation of the geometric Langlands, a lot of works will be needed to ensure our perspective is really correct. Hence it is a great chance for a young scientist or a beginner who wants to contribute from this approach. For example, in mathematical side, the Hecke eigensheaves are difficult to calculate, however in our view of $A$-branes, there would be relative advantage when one consider some concrete examples since there are a lot of surveys on barnes, though there still remains some ambiguity in deciding which is the corresponding $\mathcal{D}$-module. Of course needless to say, much more hard work will be required when one wants to say the mirror symmetry is certainly connected to the Langlands conjecture in more direct way.

With great expectation for further development of the Langlands program, I would like to say goodbye to readers.

# Appendix A

# Langlands dual groups

In section A.1 we will give a brief introduction to a Langlands dual group, and in section A.2 we will investigate the Langlands dual groups of linear Lie groups over $\mathbb{C}$ in quite some detail. The meanings of the terminologies mentioned in section A.1 will become clear in section A.2.

## A.1 Introduction to Langlands dual groups

Let us first consider a **connected reductive algebraic group** $G$ over the closure $\overline{k}$ of a field $k$. Let $T$ be a maximal torus of $G$ defined over $\overline{k}$. We associate to $T$ two lattices: the **weight lattice** $X^*(T) = \mathrm{Hom}(T, \overline{k}^\times)$ and the **coweight lattice** $X_*(T) = \mathrm{Hom}(\overline{k}^\times, T)$. They contain the set of **roots** $\Delta \subset X^*(T)$ and **coroots** $\Delta^\vee \subset X_*(T)$, respectively. The quadruple$(X^*(T), \Delta, X_*(T), \Delta^\vee)$ is called the **root data** for $G$ over $\overline{k}$. Thanks to the Chevally's theorem, the root data completely determines the connected reductive group $G$ up to an isomorphism over $\overline{k}$. Choose a Borel subgroup of G over $\overline{k}$ in such a way $T \subset B$. Then we can find a basis in $\Delta$, namely the set of simple roots $\Delta_s$, and the corresponding basis $\Delta_s^\vee$ in $\Delta^\vee$. We call the quadruple

$$R_\circ(G, B, T) = (X^*(T), \Delta_s, X_*(T), \Delta_s^\vee) \tag{A.1}$$

the **based root data**.

Given $\gamma \in \mathrm{Gal}(\overline{k}/k)$, there is $g \in G(\overline{k})$ such that $\gamma(B) = gBg^{-1}$. Moreover $\gamma(T) = gTg^{-1}$ since $\gamma(T)$ is a maximal torus contained in $\gamma(B)$. For $\xi \in X^*(T)$, let us define $\gamma\xi \in X^*(\gamma(T))$

as $(\gamma\xi)(\gamma(t)) = \gamma(\xi(t))$, where $t \in T$. Then we obtain $\xi_\gamma \in X^*(T)$ defined in such a way that $\xi_\gamma(t) = (\gamma\xi)(g^{-1}tg)$. The map $\xi \mapsto \xi_\gamma$ is an automorphism of $X^*(T)$ which maps a positive root to a positive root. Thus we obtain an action of $\mathrm{Gal}(\overline{k}/k)$ on the based root data $R_\circ(G, B, T)$.

Now let us exchange the lattices of weights and coweights and the set of simple roots and coroots. Then we obtain the based root data

$$(X_*(T), \Delta_s^\vee, X^*(T), \Delta_s) \tag{A.2}$$

of a connected reductive group over $\mathbb{C}$, which is denoted by ${}^LG^\circ$. Again by the Chevalley's theorem, ${}^LG^\circ$ is uniquely determined up to the isomorphism class of $G$ over $\overline{k}$. The action of $\mathrm{Gal}(\overline{k}/k)$ on the based root data $R_\circ(G, B, T)$ gives rise to its action on ${}^LG^\circ$. The **semi-direct product**

$${}^LG = {}^LG^\circ \rtimes \mathrm{Gal}(\overline{k}/k) \tag{A.3}$$

is called the **Langlands dual group** of $G$. Here semi-direct product ${}^LG^\circ \rtimes \mathrm{Gal}(\overline{k}/k)$ means that every $g \in {}^LG$ can be written uniquely as

$$g = nh, \quad n \in {}^L G^\circ, \ h \in \mathrm{Gal}(\overline{k}/k). \tag{A.4}$$

**Remark A.1.1.** If $k = \mathbb{C}, X^*(T) = \mathrm{Hom}(T, \mathbb{C}^\times)$ $=\mathrm{Hom}(T, U(1))$ since $U(1)$ is the only compact connected subgroup of $\mathbb{C}^\times$.

## A.2 Examples : Semisimple linear Lie groups over $\mathbb{C}$

### A.2.1 Semisimple groups

In the following part of this section we give some examples of the Langlands dual group of a **classical linear semisimple Lie groups**. Let $k = \mathbb{C}$ then ${}^LG = {}^LG^\circ$ since $\mathrm{Gal}(\bar{k}/k) = \mathrm{id}_\mathbb{C}$. The general linear group $GL_n(\mathbb{C})$ of invertible complex $n \times n$ matrices has a natural structure of an affine variety, namely

$$GL_n(\mathbb{C}) = \{X \in M_n(\mathbb{C}) : \mathrm{det}X \neq 0\}, \tag{A.5}$$

with coordinate ring

$$\mathcal{O}(GL_n) = \mathbb{C}[x_{ij}, \mathrm{det}^{-1}], \quad \mathrm{det} = \mathrm{det}(x_{ij}) \in \mathbb{C}[x_{ij}]. \tag{A.6}$$

A closed subgroup $G \subset GL_n(\mathbb{C})$ is called an **algebraic group** or **linear algebraic group**. A subgroup $G$ of $GL_n(\mathbb{C})$ is called a **reductive Lie group** if for all $g \in G$, ${}^t\overline{g}$ is contained in $G$ and the number of connected subgroups of $G$ is finite. For example, $GL_n(\mathbb{C})$ and $SL_n(\mathbb{C})$ are reductive. A subgroup G of $GL_n(\mathbb{C})$ is called **semisimple** if its radial is trivial. For example, $SL_n(\mathbb{C})$ and $SO_n(\mathbb{C})$ are semisimple, but $GL_n(\mathbb{C})$ is not so. A semisimple Lie group is reductive by definition. The semisimple Lie groups are classified in terms of root systems of the corresponding Lie algebras. The following table gives the some of the examples. We denote by $\mathfrak{g}$ the Lie algebra of $G$ and by $1_n$ the identity element of $M_n(\mathbb{C})$.

| $G$ | $\mathfrak{g}$ | type |
|---|---|---|
| $SL_{n+1}(\mathbb{C})$ | $sl_{n+1}(\mathbb{C}) = \{X \in M_n(\mathbb{C}) : \mathrm{tr}X = 0\}$ | $A_n$ |
| $SO_{2n+1}(\mathbb{C})$ | $so_{2n+1}(\mathbb{C}) = \{X \in M_n(\mathbb{C}) : {}^tX + X = 0, \mathrm{tr}M = 0\}$ | $B_n$ |
| $Sp_n(\mathbb{C})$ | $sp_n(\mathbb{C}) = \{X \in M_{2n}(\mathbb{C}) : {}^tXJ_{2n} + XJ_{2n} = 0\}$ | $C_n$ |
| $SO_{2n}(\mathbb{C})$ | $so_{2n}(\mathbb{C})$ | $D_n$ |

The matrix $J$ which appears in definition $sp_n(\mathbb{C})$ is defined by

$$J = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}. \tag{A.7}$$

### A.2.2 Root data

To explain the root data of semisimple Lie group, we first state the definition of semisimple Lie algebra. Once given a $n$-dimensional Lie algebra, we can choose its basis $\{v_1, \cdots, v_n\}$. Let $K$ be a complex bilinear form called the **Killing form**

$$K(v_i, v_j) = \mathrm{tr}(\mathrm{ad}v_i, \mathrm{ad}v_j), \tag{A.8}$$

where ad is **adjoint representation** $\mathrm{ad}(v_i)(v_j) = [v_i, v_j]$. The Lie algebra $\mathfrak{g}$ is called **semisimple** if the Killing form is nondegenerate, namely $K(X,Y) = 0$ for all $Y \in \mathfrak{g}$ if and only if $X = 0$. One can check that a Lie group is semisimple if the associated Lie algebra is semisimple and vice versa. A maximal abelian subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is called the **Cartan subalgebra** if for all $H \in \mathfrak{h}$ $\mathrm{ad}(H)$ is diagonalizable. A **root** $\alpha$ of a semisimple Lie algebra $\mathfrak{g}$ is a linear map $\alpha : \mathfrak{h} \to \mathbb{C}$ such that there is $0 \neq X \in \mathfrak{g}$ for all $H \in \mathfrak{h}$

$$\mathrm{ad}(H)(X) = \alpha(H)X. \tag{A.9}$$

We denote by $\Delta$ the set of nonzero roots and call it **root system**. Since $\mathrm{ad}(H)$ $(H \in \mathfrak{h})$ is simultaneous diagonalizable, for a given root $\alpha$ the associated one dimensional eigenspace

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} : \mathrm{ad}(H)(X) = \alpha(X)H, \ {}^{\forall}H \in \mathfrak{h}\} \tag{A.10}$$

$$= \{ke_\alpha : k \in \mathbb{C}\} \cong \mathbb{C}, \quad {}^{\exists!}e_\alpha \in \mathfrak{g} \tag{A.11}$$

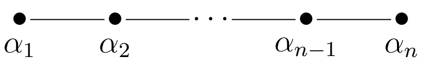
is determined. The generator $e_\alpha$ of $\mathfrak{g}_\alpha$ will play an fundamental role to construct a Langlands dual group. Then $\mathfrak{g}$ can be decomposed into the direct sum

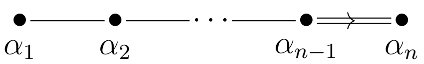
$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha\in\Delta} \mathfrak{g}_\alpha. \tag{A.12}$$

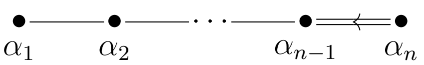
Now we define a **simple root** in the following way. Firstly note that every nonzero root has $\dim\mathfrak{g} - \dim\mathfrak{h}$ number of components as a vector in the $\dim\mathfrak{g}$ dimensional vector space. However since a nonzero root is an element of $\mathfrak{h}^*$ =Hom$(\mathfrak{h}, \mathbb{C}^\times)$, the number of linearly independent nonzero roots is equal to $\dim\mathfrak{h}^*(= \dim\mathfrak{h} = \mathrm{rank}\ G)$. So let $\Delta_s$ be the set of the linearly independent nonzero roots, which we call the **simple roots**. Equivalently this is the set of positive roots.

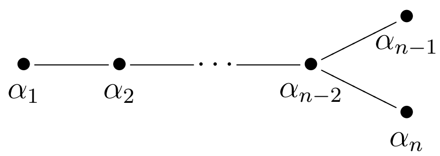
| type | $G$ (complex) | $G$ (compact) | Root | rank |
|---|---|---|---|---|
| $A_n$ | $SL_{n+1}(\mathbb{C})$ | $SU_{n+1}$ | $e_i - e_j \quad (1 \leq i \neq j \leq n+1)$ | $n$ |
| $B_n$ | $SO_{2n+1}(\mathbb{C})$ | $SO_{2n+1}$ | $\pm e_i \pm e_j, \ \pm e_i \quad (1 \leq i \neq j \leq n)$ | $n$ |
| $C_n$ | $Sp_n(\mathbb{C})$ | $Sp_n$ | $\pm e_i \pm e_j, \ \pm 2e_i \quad (1 \leq i \neq j \leq n)$ | $n$ |
| $D_n$ | $SO_{2n}(\mathbb{C})$ | $SO_{2n}$ | $\pm e_i \pm e_j \quad (1 \leq i \neq j \leq n)$ | $n$ |
| $E_6$ | $E_6^c$ | $E_6$ | $\pm e_i \pm e_j \quad (1 \leq i \neq j \leq 5)$, | 6 |
| | | | $\frac{1}{2}(\pm e_1 \pm \cdots \pm e_5 \pm \sqrt{3}e_6)$ even# of + | |
| $E_7$ | $E_7^c$ | $E_7$ | $\pm e_i \pm e_j, \ \pm \sqrt{2}e_7 \quad (1 \leq i \neq j \leq 6)$, | 7 |
| | | | $\frac{1}{2}(\pm e_1 \pm \cdots \pm e_6 \pm \sqrt{2}e_7)$ even# of + | |
| $E_8$ | $E_8^c$ | $E_8$ | $\pm e_i \pm e_j \quad (1 \leq i \neq j \leq 8)$, | 8 |
| | | | $\frac{1}{2}(\pm e_1 \pm \cdots \pm e_8)$ even# of + | |
| $F_4$ | $F_4^c$ | $F_4$ | $\pm e_i \pm e_j, \ \pm 2e_i \quad (1 \leq i \neq j \leq 4)$, | 4 |
| | | | $\pm e_1 \pm \cdots \pm e_4$ | |
| $G_2$ | $G_2^c$ | $G_2$ | $e_i - e_j, \ \pm(e_i + e_j - 2e_k) \quad (1 \leq i \neq j \neq k \leq 3)$ | 2 |

Thanks to the well known discussion of the seisimple linear Lie algebra, we know that the Dynkin diagrams, which can be obtained by calculating the Cartan matrices which we will explain later, associated with the corresponding Lie algebras, classify the set of simple roots $\Delta_s = \{\alpha_1, \cdots, \alpha_n\}$, $(n = \dim\mathfrak{h}^*)$ as follows.

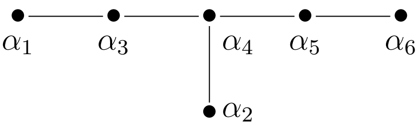
Type $A_n$ : 


Type $B_n$ : 

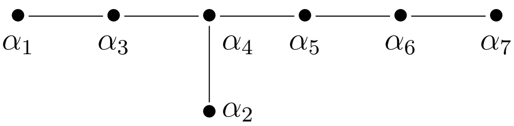

Type $C_n$ : 


Type $D_n$ : 

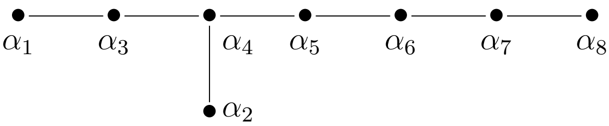
Type $E_6$ :



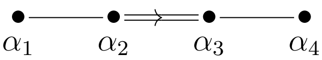

Type $E_7$ :




Type $E_8$ :




Type $F_4$ :




Type $G_2$ :


Here note that $A_1 = B_1 = C_1, C_2 = B_2, D_3 = A_3,$ and $D_2 = A_1 + A_1$. The Dynkin diagram has necessary and sufficient information to reconstruct the original semisimple Lie algebra $\mathfrak{g}$. That is to say, in order to to get comprehensive knowledge of a semisimple Lie algebra, one should consult with its Dynkin diagram.

Now let us define the coroot of a corresponding root in the following way. Recall that the Killing form $K$ of a semisimple Lie algebra $\mathfrak{g}$ is nondegenerate, then it allows us to choose a isomorphism $h : \mathfrak{h}^* \to \mathfrak{h}$ such that

$$K(h(\gamma), H) = \gamma(H), \quad \gamma \in \mathfrak{h}^*, H \in \mathfrak{h}. \tag{A.13}$$

The restriction of $t$ to the root system $\Delta$ defines the **coroot** $\alpha^\vee$ of a root $\alpha$ by

$$\alpha^\vee = \frac{2h(\alpha)}{(\alpha, \alpha)}, \quad \alpha \in \Delta, \tag{A.14}$$

where we defined a inner product $(\ ,\ ) : \mathfrak{h} \times \mathfrak{h} \to \mathbb{R}$ by

$$(\alpha, \beta) = K(h(\alpha), h(\beta)) = \alpha(t(\beta)). \tag{A.15}$$

Then it is possible to calculate a product $< \ , \ >: \mathfrak{h} \times \mathfrak{h}^* \to \mathbb{R}$ as

$$< \alpha, \beta^\vee > = \frac{2\alpha(t(\beta))}{(\alpha, \alpha)} = \frac{2(\alpha, \beta)}{(\beta, \beta)}. \tag{A.16}$$

We write this simply $\alpha(\beta^\vee) =< \alpha, \beta^\vee >$.

The set $\Delta^\vee$ of coroots is called the **dual root system**. It is known that the dual root system of a semisimple Lie algebra $\mathfrak{g}$ is also the root system of a certain semisimple Lie algebra $\mathfrak{g}'$.

**Lemma A.2.1.**

The map $\Delta \mapsto \Delta^\vee$ preserves the type $A_n, D_n$ and exchanges the type $B_n$ and $C_n$.

**Proof .** This reason is simple. First of all, the **Cartan matrix** $C = (c_{ij})$ is defined by

$$c_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)} \in \mathbb{Z}, \quad \alpha_i, \alpha_j \in \Delta_s. \tag{A.17}$$

Note that

$$\frac{2(\alpha_j^\vee, \alpha_i^\vee)}{(\alpha_i^\vee, \alpha_i^\vee)} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)}. \tag{A.18}$$

So, exchanging the root system and the dual root system means that $c_{ij}$ goes to $c_{ji}$. Recall that the Cartan matrices of $A_n$ and $D_n$ are both symmetric, and the transported Cartan matrix of $B_n$ is that of $C_n$. □

To construct the Langlands dual groups ${}^LG$ of $G$, we need one more step, because there is an ambiguity in choosing a group which has the same Lie algebra. In other words, for a given Lie group $G$ with Lie albegra $\mathfrak{g}$, quotients of a universal covering Lie group $\tilde{G}$ by a discrete normal subgroup $\Gamma$ have the same $\mathfrak{g}$; this is essentially Lie's second and third theorem. What we are missing here is information of weight.

To go further we review the idea of weight, which is generalization of the idea of root. Let $(\mathfrak{g}, \rho)$ be a representation $\rho : \mathfrak{g} \to gl_m(\mathbb{C})$. A **weight** $\lambda$ of $(\mathfrak{g}, \rho)$ is a linear map $\lambda : \mathfrak{h} \to \mathbb{C}$ such that

$$\rho(H)(X) = \lambda(H)X \quad 0 \neq^\exists X \in \mathfrak{g},^\forall H \in \mathfrak{h}. \tag{A.19}$$

We denote by $\Delta(\rho)$ the set of weights of representation $\rho$. Since $\rho(H)$ $(H \in \mathfrak{h})$ is simultaneous diagonalizable, for a given weight $\lambda$ the associated eigenspace $V_\lambda = \{v \in \mathbb{C}^m : \rho(H)(v) = \lambda(H)v, \ \forall H \in \mathfrak{h}\}$

is determined uniquely, and then we obtain the form of

$$\mathbb{C}^m = \sum_{\lambda \in \Delta(\rho)} V_\lambda. \tag{A.20}$$

One may have noticed that a root $\alpha$ is a weight of adjoint representation. In contrast to an eigenspace of a root, whose dimension is one, the dimension of an eigenspace of a weight is not necessarily one.

With respect to a set of simple roots $\Delta_s = \{\alpha_1, \cdots, \alpha_n\}$ and that of dual $\Delta_s^\vee = \{\alpha_1^\vee, \cdots, \alpha_n^\vee\}$, we consider the **root lattice** $L_0$, which is generated by the elements of $\Delta_s$, and the **root lattice** $L_0^\vee$, which is generated by the elements of $\Delta_s^\vee$. Then we define the **weight lattice** $L_1$ spanned by **fundamental weight vector**s $\omega_i$ and **coweight lattice** $L_1^\vee$ spanned by **fundamental coweight vector**s $\omega_i^\vee$ as follows:

$$L_1 = \{m_1\omega_1 + \cdots + m_n\omega_n : (\omega_i, \alpha_j) = \delta_{ij}, m_i \in \mathbb{Z}, 1 \leq^\forall i, j \leq n\}. \tag{A.21}$$

$$L_1^\vee = \{m_1\omega_1^\vee + \cdots + m_n\omega_n^\vee : (\omega_i^\vee, \alpha_j^\vee) = \delta_{ij}, m_i \in \mathbb{Z}, 1 \leq^\forall i, j \leq n\}. \tag{A.22}$$

By the definition of the Cartan matrix, we have

$$c_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)} =< \alpha_i, \alpha_j^\vee > . \tag{A.23}$$

Hence

$$\alpha_i = \sum_j^n c_{ij}\omega_j \tag{A.24}$$

$$\alpha_i^\vee = \sum_j^n c_{ij}\omega_j^\vee. \tag{A.25}$$

Thus, there are the following relations between those lattices

$$L_0 \subset L_1, \;\; L_0^\vee \subset L_1^\vee, \tag{A.26}$$

$$[L_1 : L_0] = [L_0^\vee : L_1^\vee] = \det C. \tag{A.27}$$

The meanings of the last relations become clear in the next subsection.

**Lemma A.2.2.**

The lattice generated by all weights of a faithful representation on $\mathbb{C}^m$ forms a lattice $L_V$ between $L_0$ and $L_1$.

**Proof .** For any nonzero root $\alpha \in \Delta$, there exists $0 \neq v \in V_\lambda$ for some weight $\lambda$ with $e_\alpha v \in V_{\lambda+\alpha}$. Therefore $\lambda + \alpha$ is a weight of $\mathbb{C}^m$, and then $\alpha = (\lambda+\alpha) - \lambda \in L_V$. This shows $L_0 \subset L_V$. $L_V \subset L_1$ is held since $L_1$ is the lattice generated by the all weights of all representations. $\square$

In particular, the lattice $L_0$ corresponds to the adjoint representation, and the lattice $L_1$ corresponds to the natural representation.

**Remark A.2.3.** $X^*(T) \simeq L_V$, and $X_*(T) \simeq L_V^\vee$. With respect to the set of weights $\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{Z}^n$, we identify $\lambda \in L_V$ with a homeomorphism $\chi_\lambda \in \mathrm{Hom}(T, U(1)) =: X^*(T)$ by

$$\chi_\lambda(t) = t_1^{\lambda_1} \cdots t_n^{\lambda_n}, \tag{A.28}$$

where we denoted coordinates on $T \simeq \mathbb{T}^{\dim \mathfrak{h}}$ by $t = (t_1, \cdots, t_n) = (e^{i\theta_1}, \cdots, e^{i\theta_n})$ ($\dim \mathfrak{h} = n,\ \theta_i \in \mathbb{R}$). Moreover, since $\lambda$ is written as

$$\lambda = \lambda_1 \mu^1 + \cdots + \lambda_n \mu^n, \tag{A.29}$$

where weight vectors $\mu^i$ is normalized as $\mu^i(H_j) = \delta^i_j$ with basis $H_i$ of $L_V$, then for all $H \in \mathfrak{h}$

$$\chi_\lambda(\exp H) = e^{\lambda(H)} \tag{A.30}$$

is to be satisfied.

### A.2.3 Chevalley groups

In the beginning of this section we mentioned the Cartan decomposition of a given Lie algebra $\mathfrak{g}$, and we saw that there is a generator $e_\alpha$ ($\alpha \in \Delta$) of an one dimensional eigenspace $\mathfrak{g}_\alpha$. Given a representation $(\mathfrak{g}, \rho)$, we define a exponential map of the form

$$x_\alpha(\tau) := \exp(\tau \rho(e_\alpha)), \quad \tau \in \mathbb{C}. \tag{A.31}$$

Our main object is the group $G$ generated by all of $\alpha$ and $x_\alpha(\tau)$, namely

$$G = \{x_\alpha(\tau) : \tau \in \mathbb{C}, \alpha \in \Delta\}. \tag{A.32}$$

We call it a **Chevalley group**.

**Example A.2.4.** Type $A_n$

Below we construct the type $A_n$ Chevalley groups on general field $k$. Let $I$ be the $n \times n$ unit matrix, and $e_{ij}$ be a $(n+1) \times (n+1)$ matrix whose $(i,j)$ component is 1 and the others are 0. Since the type $A_n$ Lie algebra is the set of $(n+1) \times (n+1)$ matrices whose diagonal sum is 0, its basis $B_{A_n}$ is

$$B_{A_n} = \{e_{ii} - e_{i+1,i+1},\ e_{ij}\ (i \neq j)\}. \tag{A.33}$$

We call the basis like $B_{A_n}$ a **Chevalley basis**. Then the Chevalley group by the fundamental representation is generated by

$$x_\alpha(\tau) = 1_{n+1} + te_{ij}\ \ (\alpha = e_i - e_j,\ \tau \in k). \tag{A.34}$$

**Theorem A.2.5.**

The type $A_n$ Chevalley group by the fundamental representation is isomorphic to $SL_{n+1}(k)$.

To show this theorem, we need some preparations. Let $V$ be a $n+1$ dimensional vector space and $H$ be a hyperplane, $n$ dimensional sub space of $V$.

**Definition A.2.6.** $1 \neq \tau \in GL(V)$ is **transvection** on $H$ if $\tau$ satisfies the following:

1. $^\forall v \in V,\ \tau(v) - v \in H$.

2. $^\forall u \in H,\ \tau(u) = u$.

Let $V^*$ be the dual space of $V$. Then there exists $\mu \in V^*$ such that $\ker\mu = H$. For a given $a \in H$, we define a map $\tau(\mu, a)$ as

$$\tau(\mu, a) : v \mapsto v + \mu(v)a. \tag{A.35}$$

If $a \neq 0$, $\tau(\mu, a)$ turns out to be a transvection on $H$.

**Proposition A.2.7.**

Any transvection on $H$ can be written in the form $\tau(\mu, a)$. Moreover, $\tau(\mu, a) \in SL(V)$

**Proof .** Let $\tau$ be a transvection on $H$. We choose $v \in V$ so as to $\mu(v) = 1$ and put $a = \tau(v) - v$, then $a \in H$ follows by the definition. Since $v \notin H$, any $x \in V$ can be decomposed into

$$x = \lambda v + u\ \ (\lambda \in k,\ u \in H). \tag{A.36}$$

Then

$$\mu(x) = \lambda\mu(v) + \mu(u) = \lambda\mu(v) = \lambda, \tag{A.37}$$

and

$$\tau(x) = \lambda\tau(v) + \tau(u) = \lambda(a+v) + u = \lambda a + x = x + \mu(x)a, \tag{A.38}$$

namely we obtained $\tau = \tau(\mu, a)$.

When we choose $\{a, v_3, \cdots, v_{n+1}\}$ as a basis of $H$, and $\{v, a, v_3, \cdots, v_{n+1}\}$ as a basis of $H$, then the matrix representation of $\tau(\mu, a)$ is

$$(v + a, a, v_3, \cdots, v_{n+1}) = (v, a, v_3, \cdots, v_{n+1})\tau(\mu, a) \tag{A.39}$$

$$\Longleftrightarrow (v, a, v_3, \cdots, v_{n+1}) \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 1 & 1 & 0 & \cdots \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \cdots & 1 \end{pmatrix} = (v, a, v_3, \cdots, v_{n+1})\tau(\mu, a), \tag{A.40}$$

which shows $\tau(\mu, a) \in SL(V)$. $\square$

**Proposition A.2.8.**

The subgroup $G$ of $GL(V)$ generated by the set $T$ of all transvections coincides with $SL(V)$.

**Proof .** Write $G = \langle T \rangle$ and assume $G \neq SL(V)$. We have seen $G \subset SL(V)$ above. So we choose $\sigma \in SL(V)$ such that $\sigma \notin G$ and $I(\sigma) = \{v \in V : \sigma(v) = v\}$ is maximal in $V$. We write a additive group generated by $I(\sigma)$ and $v \in V$ as $U(v) = \langle I(\sigma), v \rangle$

We show $\sigma(v) \in V$ for all $v \in V$. We assume that $\sigma(x) \notin U(x)$ for an element $x \in V$ and put $\sigma(x) - x = a$. If $x \in U(a)$, then $\sigma(x) = x + a \in U(a) = U(x)$, which contradicts our assumption. So $x \notin U(x)$. Therefore there exists $\mu \in V^*$ such that $\mu(x) = 1$ and $\mu(y) = 0$ ($^\forall y \in U(a)$). This $\mu$ is obtained as follows. Let $\{u_1, \cdots, u_m\}$ ($m < n$) be a basis of $I(\sigma)$ and $u_{m+1} = x$. Moreover we choose $u_{m+2}, \cdots, u_{n+1}$ so as to $\{u_1, \cdots, u_{n+1}\}$ become a basis of $V$. Let $\{f_1, \cdots, f_{n+1}\}$ be the standard dual basis of $\{u_1, \cdots, u_{n+1}\}$ and we define $\mu$ as $\mu = f_{m+1}$. Then this $\mu$ satisfies the properties

$$\mu(x) = f_{m+1}(x) = f_{m+1}(u_{m+1}) = 1. \tag{A.41}$$

$$\mu(y) = f_{m+1}(y) = 0 \ ^\forall y \in I(\sigma). \tag{A.42}$$

Now we use a transvection $\tau = \tau(\mu, a)$ (A.35), then

$$\tau(x) = x + \mu(x)a = x + a = \sigma(x), \tag{A.43}$$

$$\tau(y) = y + \mu(y)a = y = \sigma(y). \tag{A.44}$$

Therefore $I(\tau^{-1}\sigma) \supset U(x) \supsetneq I(\sigma)$. However this contradicts $I(\sigma)$ is maximal in $V$. Hence $\sigma(v) \in V$ for all $v \in V$, equivalently $\sigma(U(v)) = U(v)$.

Next, we show that $I(\sigma)$ is a hyperplane in $V$. Since $\sigma \neq 1$, $I(\sigma) \subsetneq V$. So we choose a hyperplane $H$ in $V$ which contains $I(\sigma)$. It is sufficient to show that any transvection $\tau$ on $H$ satisfies

$$\tau(x) - x \in I(\sigma) \quad ({}^{\forall}x \in V). \tag{A.45}$$

If $x \in H$, then $\tau(x) = x$, which satisfy the above relation. Assume $x \notin H$ and note that $U(x) = \tau\sigma(U(x)) = \tau(U(x))$ since $\tau\sigma \notin G$ and $\tau\sigma$ leaves each element of $I(\sigma)$ invariant. With respect to $\tau(x) \in U(x)$, we obtain the relation $\tau(x) - x \in U(x) \cap H = I(\sigma)$. By the proposition (A.2.7), The group of transvections on $H$ with identity map is isomorphic to an additive group $H$ generated by $\tau(x) - x$. Therefore $U = H$ follows, namely $I(\sigma)$ is a hyperplane in $V$.

Note that $\sigma$ induces a linear map $\bar{\sigma}$ on $V/I(\sigma)$. Recall we have shown that $\sigma$ is identity map on $I(\sigma)$, then $\det\bar{\sigma} = \det\sigma = 1$. Furthermore $\sigma(v) - v \in I(\sigma)$ for all $v \in V$ since $V/I(\sigma)$ is 1 dimensional space. This means $\sigma$ is a transvection on $I(\sigma)$, which contradicts our assumption $\sigma \notin G$. Thus $G = SL(V)$. $\square$

**Proof .** (Theorem A.4) Now we are ready to prove the theorem A.4. Let $G'$ be the group generated by $x_\alpha(\tau)$ (A.34). We claim $G' = SL_{n+1}(k)$. Since $SL_{n+1}(k)$ naturally acts on a $n+1$ dimensional vector space $V$ with a basis $\{u_1, \cdots, u_{n+1}\}$, to complete the proof, it is sufficient to show that $G'$ contains all of transvections of the elements in $V$. Let $H_0$ be the hyperplane spanned by $u_2, \cdots, u_{n+1}$. Since any transvection on $H_0$ is given by

$$I + \Sigma\alpha_i e_{i1} = \Pi(I + \alpha_i e_{i1}) \tag{A.46}$$

is contained in $G'$.

Next we show $G'$ freely acts on the set of all hyperplanes in $V$. To do so, we introduce the dual space $V^*$ of $V$. Choose $\{f_1, \cdots, f_{n+1}\}$ as a natural basis ($f_i(u_j) = \delta_{ij}$) of $V^*$, then action of

$M \in SL_{l+1}(k)$ on $V^*$ is given by the transpose matrix ${}^t\!M$ of $M$, which implies $G'$ acts on $V^* - 0$ freely. For any given hyperplane $H$ in $V$, there exists $0 \neq \mu \in V^*$ such that $H = \ker\, \mu$. Therefore if we choose $\sigma \in G'$ such that ${}^t\sigma(\mu) = f_1$, then $\sigma(H_0) = H$ follows. Thus $G'$ freely acts on the set of all hyperplanes in $V$.

Lastly, we show that any transvection $\tau$ on $H$ is an element of $G'$. With respect to the $\sigma$, $\sigma^{-1}\tau\sigma$ is a transvection on $H_0$. That is to say $\sigma^{-1}\tau\sigma \in G'$ then $\tau \in G'$, which completes our proof. □

**Theorem A.2.9.**

The type $A_n$ Chevalley group by the adjoint representation is isomorphic to $PSL_{n+1}(k)$.

**Proof .** If $\Sigma\alpha_{ij}e_{ij} \in SL_{n+1}(k)$ commute with each element of the type $A_n$ Lie algebra,

$$(\Sigma\alpha_{ij}e_{ij})e_{kl} = e_{kl}(\Sigma\alpha_{ij}e_{ij}) \tag{A.47}$$

should be satisfied for all $k, l$ $(k \neq l)$. Then $\alpha_{lj} = 0$ $(l \neq j)$ and $\alpha_{ll} = \alpha_{kk} = \lambda$ with $\lambda^{n+1} = 1$ follow. Thus the center $Z$ of $SL_{n+1}(k)$, which is the kernel of the adjoint representation, is

$$Z = \{\lambda 1_{n+1} : \lambda \in k,\ \lambda^{n+1} = 1\} \tag{A.48}$$

and $SL_{n+1}(k)/Z = PSL_{n+1}(k)$ follows by the homomorphism theorem. This proves our claim. □

The following theorem gives us a method to construct the Langlands dual group ${}^LG$ of $G$. We admit this theorem without proof. For its detail see [45].

**Theorem A.2.10.**

Let $G_V$ be a Chevalley group associated with a lattice $L_V$ and $G_{V'}$ be a different one with $L_{V'}$. Here we assume both of $G_V$ and $G_{V'}$ are constructed from the same Lie algebra $\mathfrak{g}$ and field $\mathbb{C}$. If $L_{V'} \subset L_V$, then there exists a homomorphism $\varphi : G_{V'} \to G_V$ such that for all $\alpha, t$, and $\ker\, \varphi \subset Z(G_{V'})$. If $L_{V'} = L_V$, then $\varphi$ is an isomorhism.

From this theorem, for a Chevalley group $G_V$, we have homomorphisms $\varphi, \psi$ such that $\varphi : G_1 \to G_V$, and $\psi : G_V \to G_0$ with $\ker\, \psi = Z(G_V) \cong L_V/L_0$. Therefore by the isomorphism theorem we obtain $G_V/Z(G_V) \cong G_0$. Furthermore, since there is a short exact sequence

$$1 \to \pi_1(G_V) \to G_1 \to G_V \to 1, \tag{A.49}$$

we obtain $G_1/\pi_1(G_V) \cong G_V$. This isomorphic allows us to identify $\ker\varphi = \pi_1(G_V)$. We call $G_0$ **adjoint group**, $G_1$ **universal covering group**.

Here are some examples of the center $Z(G) = \{g \in G : gh = hg, {}^\forall h \in G\}$:

$$Z(SL_n(\mathbb{C})) = \{a1_n : a \in \mathbb{C}, a^n = 1\} \simeq \mathbb{Z}_n, \tag{A.50}$$

$$Z(SO_n(\mathbb{C})) = \begin{cases} 1_n & (n\text{: odd}) \\ SO_2(\mathbb{C}) & (n = 2) \\ \pm 1_n & (n \geq 4 : \text{ even}) \end{cases}, \tag{A.51}$$

$$Z(Sp_n(\mathbb{C})) = \{\pm 1_n\} \simeq \mathbb{Z}_2. \tag{A.52}$$

The following table is a list of Chevalley groups for semisimple connected complex linear Lie groups and some data.

| Type | $L_1/L_0$ | $G_0$ | $G_V$ | $G_1$ |
|---|---|---|---|---|
| $A_n$ | $\mathbb{Z}_{n+1}$ | $PSL_{n+1}(\mathbb{C})$ | | $SL_{n+1}(\mathbb{C})$ |
| $B_n$ | $\mathbb{Z}_2$ | $PSO_{2n+1}(\mathbb{C}) = SO_{2n+1}(\mathbb{C})$ | | $Spin_{2n+1}(\mathbb{C})$ |
| $C_n$ | $\mathbb{Z}_2$ | $PSp_n(\mathbb{C})$ | | $Sp_n(\mathbb{C})$ |
| $D_{2n+1}$ | $\mathbb{Z}_4$ | $PSO_{4n+2}(\mathbb{C})$ | $SO_{4n+2}(\mathbb{C})$ | $Spin_{4n+2}(\mathbb{C})$ |
| $D_{2n}$ | $\mathbb{Z}_2 \times \mathbb{Z}_2$ | $PSO_{4n}(\mathbb{C})$ | $SO_{4n}(\mathbb{C})$ | $Spin_{4n}(\mathbb{C})$ |
| $E_6$ | $\mathbb{Z}_3$ | | | |
| $E_7$ | $\mathbb{Z}_2$ | | | |
| $E_8$ | $\mathbb{Z}_1$ | $G_0$ | $=$ | $G_1$ |
| $F_4$ | $\mathbb{Z}_1$ | $G_0$ | $=$ | $G_1$ |
| $G_2$ | $\mathbb{Z}_1$ | $G_0$ | $=$ | $G_1$ |

### A.2.4 Langlands dual groups

Now we are ready to construct the Langlands dual group ${}^L G_V$ for a given semisimple connected Lie group $G_V$. Let $\mathfrak{g}$ be the Lie algebra with a root data $(L_V, \Delta_s, L_V^\vee, \Delta_s^\vee)$, and ${}^L\mathfrak{g}$ be its Langlands dual Lie algebra with the root data $(L_V^\vee, \Delta_s^\vee, L_V, \Delta_s)$. There are two steps

1. Choose the natural representation for ${}^L\mathfrak{g}$, and obtain the universal group $({}^LG)_1$, whose center is $Z\left(({}^LG)_1\right) = L_1^\vee/L_0^\vee$.

2. Then ${}^LG_V \cong ({}^LG)_1/\pi_1({}^LG_V)$, where $\pi_1({}^LG_V) \cong Z(G_V)^* \cong L_V^\vee/L_0^\vee$.

**Proposition A.2.11.**

The Langlands dual of $G_1$ and $G_0$ are $({}^LG)_0$ and $({}^LG)_1$, respectively.

**Proof .**

1. Since $\pi_1({}^LG_1) \cong L_1^\vee/L_0^\vee = Z\left(({}^LG)_1\right)$, then ${}^LG_1 \cong ({}^LG)_1/Z\left(({}^LG)_1\right) = ({}^LG)_0$.

2. Since $\pi_1({}^LG_0) \cong L_0^\vee/L_0^\vee = 1$, then ${}^LG_0 \cong ({}^LG)_1$.

□

Therefore we obtain the following table.

| $G$ | ${}^LG$ |
|---|---|
| $SL_{n+1}(\mathbb{C})$ | $PSL_{n+1}(\mathbb{C})$ |
| $SO_{2n+1}(\mathbb{C})$ | $Sp_n(\mathbb{C})$ |
| $Sp_n(\mathbb{C})$ | $SO_{2n+1}(\mathbb{C})$ |
| $Spin_{2n}(\mathbb{C})$ | $PSO_{2n}(\mathbb{C})$ |
| $SO_{2n}(\mathbb{C})$ | $SO_{2n}(\mathbb{C})$ |

# Appendix B

# Mirror symmetry and Category

## B.1 Complex manifolds and Kähler manifolds

### B.1.1 Basic concepts on complex manifolds

Since we mainly work on complex manifolds, here we give some fundamental concepts about complex manifolds.

**Definition B.1.1.** A $m$-dimensional complex manifold $M$ is a Hausdorff space with open coverings $\{U_j\}$ and maps $\phi_j : U_j \to \mathbb{C}^n$ which satisfy the followings:

1. each $U_j$ is homeomorphic to $\phi_j(U_j)$;

2. if $U_\alpha \cap U_\beta \neq \emptyset$, then $\phi_\beta \circ \phi_\alpha^{-1}$ is holomorphic.

**Example B.1.2.** Complex torus $T = \mathbb{C}^m/\Gamma$, where $\Gamma \subset \mathbb{C}^m$ is a **lattice** generated by vectors $v_1, \cdots, v_{2m}$ which are linearly independent over $\mathbb{R}$. $T = \mathbb{C}^m/\Gamma$ is a compact complex manifold homeomorphic to $(S^1)^m$. A structure of $T$ as a complex manifold depends on $\Gamma$. The case $m = 1$, homomorphic map $z_j$ is $z_j = z_j(z_i, \bar{z}_i) = z_j(z_i)$. A real 2-dim closed manifold $\Sigma$ is a complex manifold if and only if $\Sigma$ is orientable, this $\Sigma$ is a Riemann surface.

**Theorem B.1.3.** Any orientable 2-dimensional Riemannian manifold $\Sigma$ is a complex manifold.

**Proof .** Fix a point $p \in \Sigma$ and choose a neighborhood $U_p \subset \Sigma$. In 2-dimensions, it is always possible to choose coordinates $(x, y)$ in $U_p$ in such a way that the metric tensor is of the form

$$ds^2 = \eta(x, y)^2(dx^2 + dy^2), \tag{B.1}$$

where $\eta$ is a real function. In a complex coordinate $z = x + iy$, this correspond to

$$ds^2 = \eta(z, \bar{z})^2 dz d\bar{z}. \tag{B.2}$$

Let $V_p \subset \Sigma$ be another neighborhood of $p$ with a complex coordinate $w = u + iv$ such that

$$ds^2 = \xi(w, \bar{w})^2 dw d\bar{w}. \tag{B.3}$$

where $\xi$ is a real function. On $U_p \cap V_p \neq \emptyset$, consider coordinate transformations

$$dz = \frac{\partial z}{\partial w} dw + \frac{\partial z}{\partial \bar{w}} d\bar{w}, \tag{B.4}$$

$$d\bar{z} = \frac{\partial \bar{z}}{\partial w} dw + \frac{\partial \bar{z}}{\partial \bar{w}} d\bar{w}. \tag{B.5}$$

Then

$$\frac{\partial z}{\partial w}\frac{\partial \bar{z}}{\partial w} = 0 \tag{B.6}$$

is required. That is to say, $z$ is holomorphic or anti-holomorphic. If $z$ is anti-holomorphic, $\frac{\partial z}{\partial w} = 0$, we obtain

$$\frac{\partial x}{\partial u} = -\frac{\partial y}{\partial v}, \ \frac{\partial x}{\partial v} = \frac{\partial y}{\partial u} \tag{B.7}$$

from the Cauchy-Riemann equations. As a consequence, the Jacobi matrix is

$$J = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial x}{\partial v} & -\frac{\partial x}{\partial u} \end{pmatrix}, \tag{B.8}$$

which gives $\det J < 0$, and so $\Sigma$ is not orientable. This contradicts our assumption. $\square$

However, in a higher dimension, there is an orientable manifold $M$ which is not complex manifold.

**Example B.1.4.** $S^4$ cannot be a complex manifold.

### B.1.2 Kähler manifolds

**Definition B.1.5.** A $2n$ real dimensional Riemannian manifold $(M, g)$ is an **almost complex manifold** if there exists an endmorphism $J_M : TM \to TM$ such that $J^2 = -1$. This $J_M$ is called an **almost complex structure** on $M$.

**Example B.1.6.** It would be surprised to know that $S^2$ and $S^6$ are the only spheres which admit almost complex structures.

The **torsion** of $J_M$ is the type $(1, 2)$-tensor $N$, called **Nijenhuis tensor**, defined by

$$N(u, v) = [Ju, Jv] - [u, v] - J[u, Jv] - J[Ju, v], \tag{B.9}$$

where $u$ and $v$ are vector fields on $M$.

**Definition B.1.7.** An almost complex structure $J_M : TM \to TM$ is **integrable** if $J_M$ is torsion free $N = 0$.

**Example B.1.8.** The possible existence of an integrable almost complex structure on $S^6$ had been an open question for over 60 years, and Atiyah has recently denied the existence in 2016 [1].

If a real $2n$-dimensional Riemannian manifold $M$ with local coordinates $(x^j, y^j)_{j=1,\cdots,n}$ can be identified with a complex manifold with local coordinates $z^j = x^j + iy^j$, then an almost complex structure $J_M$ defined by

$$J_M \left( \frac{\partial}{\partial x_j} \right) = \frac{\partial}{\partial y_j}, \quad J_M \left( \frac{\partial}{\partial y_j} \right) = -\frac{\partial}{\partial x_j} \tag{B.10}$$

is indeed integrable. So the following theorem guarantees equivalence that a Rienannian manifold $M$ becomes a complex manifold and that the $M$ has an integrable almost complex structure. In this case, we call $J_M$ a **complex structure** on $M$.

**Theorem B.1.9.** (Newlander-Nirenberg)
Any almost complex manifold $(M, J_M)$ with integrable $J_M$ has a holomorphic coordinate system $\{(\psi_i, U_i)\}$ whose associated almost complex structure corresponds to $J_M$.

**Definition B.1.10.** Let $g$ a Riemannian metric on an almost complex manifold $(M, J_M)$.

1. $g$ is an **Hermitian metric** if $g$ satisfies $g(J_M v, J_M w) = g(v, w)$ for all $v, w \in T_x M$ $(x \in M)$.

2. A **fundamental 2-form** $\omega$ is defined by $\omega(v, w) = g(J_M v, w)$ for all $v, w \in T_x M$ $(x \in M)$, where $g$ is an Hermitian metric.

The fundamental 2-form $\omega$ is 2-form since

$$\omega(v, w) = g(J_M v, w) = g(J_M J_M v, J_M w) = -g(v, J_M w) = -g(J_M w, v) = -\omega(w, v). \tag{B.11}$$

**Definition B.1.11.** An almost complex manifold $(M, J_M)$ with an Hermitian metric $g$ is a **Kähler manifold** if $d\omega = 0$. The $\omega$ is a **Kähler form** on $M$.

The first Chern class of $M$, $c_1(M) := c_1(T'M) = [c_1(R)]$, is the cohomology class of holomorphic tangent vector bundle $T'M$, which is a sub tangent bundle of $T^{\mathbb{C}}M = TM \otimes \mathbb{C}$ such that

$$T^{\mathbb{C}}M = T'M \oplus T''M, \tag{B.12}$$

where $T'M$ and $T''M$ are eigenspaces associated with eigenvalues $i$ and $-i$ respectively of the almost complex structure $J_M : T^{\mathbb{C}}M \to T^{\mathbb{C}}M, J^2 = -1$.

### B.1.3 Hodge numbers and Betti numbers

Let $M$ be a Kähler manifold of complex dimension $m = \dim_{\mathbb{C}} M$. We denote by $\Omega^{r,s}(M)$ the set of $(r, s)$-differential forms on M.

$$\partial : \Omega^{r,s}(M) \to \Omega^{r+1,s}(M) \tag{B.13}$$

$$\bar{\partial} : \Omega^{r,s}(M) \to \Omega^{r,s+1}(M) \tag{B.14}$$

We consider the following Dolbeault complex:

$$\Omega^{r,0}(M) \xrightarrow{\bar{\partial}} \Omega^{r,1}(M) \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \Omega^{r,m}(M). \tag{B.15}$$

Then $(r, s)$-degree $\bar{\partial}$-cohomology group $H^{r,s}_{\bar{\partial}}(M)$ is defined by

$$H^{r,s}_{\bar{\partial}}(M) = Z^{r,s}_{\bar{\partial}}(M)/B^{r,s}_{\bar{\partial}}(M), \tag{B.16}$$

where

$$Z_{\bar{\partial}}^{r,s}(M) = \{\omega \in \Omega^{r,s}(M) : \bar{\partial}\omega = 0\}, \tag{B.17}$$

$$B_{\bar{\partial}}^{r,s}(M) = \{\omega \in \Omega^{r,s}(M) :^{\exists} \eta \in \Omega^{r,s-1}(M) \ s.t.\ \bar{\partial}\eta = \omega\}. \tag{B.18}$$

The complex dimension

$$h^{r,s}(M) = \dim_{\mathbb{C}} H_{\bar{\partial}}^{r,s}(M) \tag{B.19}$$

is called the $(r,s)$-**Hodge number**

With respect to $d = \partial + \bar{\partial}$, the de Rham cohomology groups $H^k(X, \mathbb{C})$ $(k = 0, \cdots, 2m)$ are also defined in a similar way. We use the notation

$$b_p = \dim_{\mathbb{C}} H^p(M, \mathbb{C}), \tag{B.20}$$

the $p$-th **Betti number** of $M$.

**Theorem B.1.12.** The hodge numbers of a Kähler manifold $M$ with $\dim_{\mathbb{C}} X = m$ satisfy following properties:

1. $h^{r,s} = h^{s,r}$,

2. $h^{r,s} = h^{m-r,m-s}$.

**Theorem B.1.13.** On a Kähler manifold $M$ with $\dim_{\mathbb{C}} M = m$ and without boundary, there exist the following relations between the Hodge numbers and the Betti numbers:

1. $b_p = \sum_{s+r=p} h^{s,r}$ $(1 \leq p \leq 2m)$,

2. $b_{2p-1}$ are even $(1 \leq p \leq m)$,

3. $b_{2p} \geq 1$ $(1 \leq p \leq m)$.

**Example B.1.14.** Compact Riemann surface $M = \Sigma$

Any 2-form is closed since $\dim_{\mathbb{C}} M = 1$. Then $M$ is a Käheler manifold, and

$$H^1(M) = H^{1,0} + H^{0,1}, \ \overline{H}^{1,0}(M) = H^{0,1}(M). \tag{B.21}$$

Hence we obtain

$$b_0 = h^{0,0} = 1,\ b_1 = 2h^{1,0} = 2h^{0,1},\ b_2 = h^{1,1} = 1. \tag{B.22}$$

$h^{1,0}$ is called **genus** of $M$.

**Example B.1.15.** Complex projective space $M = \mathbb{C}P^m$

$$b_{2p} = 1,\ b_{2p+1} = 0,\ p = 0, 1, \cdots, m. \tag{B.23}$$

Since $h^{r,s} \geq 1$ on any Kähler manifold,

$$h^{r,s} = \delta_{rs} \tag{B.24}$$

is held.

**Example B.1.16.** Complex torus $M = \mathbb{C}^m/\Gamma$

$\mathbb{C}^m/\Gamma$ is identified as $(S^1)^m$. Then $b_p = \binom{2m}{p}$ follows. Let us introduce coordinates $z^1, \cdots, z^n$ of $\mathbb{C}^n$. A (r,s)-form $\alpha$ on $M$ is written uniquely as

$$\alpha = a_{IJ} dz^I \wedge d\bar{z}^J,\ \#I = s, \#J = s. \tag{B.25}$$

The dimension of vector space generated by $\alpha$ whose $a_{IJ}$ are constant is $\binom{m}{r} \times \binom{m}{s}$. Then $\binom{m}{p} \times \binom{m}{q} \geq h^{r,s}$, and

$$b_p = \sum_{r+s=p} h^{r,s} \leq \sum_{r+s=p} \binom{m}{r} \times \binom{m}{s}. \tag{B.26}$$

Now comparing coefficients of $x$ in $(1+x)^{2m} = (1+x)^m(1+x)^m$, we find

$$\binom{2m}{p} = \sum_{r+s=p} \binom{m}{r} \times \binom{m}{s}. \tag{B.27}$$

Since $b_p = \binom{2m}{p}$, we arrive at the formula

$$h^{r,s} = \binom{m}{r} \times \binom{m}{s}. \tag{B.28}$$

### B.1.4 Calabi-Yau manifold and Mirror symmetry

**Definition B.1.17.** A compact Kähler manifold with $c_1(M) = 0$ (or equivalently with a Ricci flat Kähler metric) is called an **Calabi-Yau manifold**

On a Kähler manifold, the (1,1)-form $c_1(R)$ is written by Ricci tensor $R_{k\bar{l}}$ as

$$c_1(R) = \frac{i}{2\pi}\mathrm{tr}(R) = \frac{i}{2\pi}R_{k\bar{l}}dz^k \wedge d\bar{z}^l. \tag{B.29}$$

By the famous theorem of Yau, if $c_1(M) = 0$, then there is a unique Kähler metric on $M$ such that the Ricci tensor $R_{k\bar{l}} = 0$ throughout $M$.

**Example B.1.18.** Torus $\mathbb{C}^m/\Gamma$ and K3 surface are Calabi-Yau manifolds.

**Definition B.1.19.** $m$-dimensional Calabi-Yau manifolds $M$ and $N$ are **mirror symmetric** if

$$h^{1,1}(M) = h^{m-1,1}(N),\ h^{m-1,1}(M) = h^{1,1}(N). \tag{B.30}$$

**Example B.1.20.** T-duality is mirror symmetry.

### B.1.5 Hyper Kähler manifolds

**Definition B.1.21.** A real $4n$-dimensional Riemannian manifold $(M, g)$ with three independent almost complex structures $I, J, K$ is a **Hyper Kähler manifold** if $(M, g)$ satisfies the following properties:

1. $g(Iv, Iw) = g(Jv, Jw) = g(Kv, Kw) = g(v, w),\quad {}^{\forall}x \in M,\ {}^{\forall}v, w \in T_xM$

2. $I^2 = J^2 = K^2 = IJK = -1$

3. $\nabla I = \nabla J = \nabla K = 0,\quad \nabla$ is the Levi-Civita connection of $g$, $\nabla g = 0$.

A hyper Kähler manifold $M$ is a complex manifold with respect to each of $I, J, K$, and $g$ becomes a Kähler metric. We denote Kähler forms associated with $I, J, K$ by $\omega_I, \omega_J, \omega_K$ respectively. Then $\omega_+ = \omega_I + i\omega_K$ is a holomorphic 2-form with respect to the complex structure $I$ and non-degenerate almost everywhere on $M$. We call $\omega_c$ like 2-form **holomorphic symplectic form** or **symplectic form** simply. $\omega_- = \omega_I - i\omega_K$ is an anti-holomorphic 2-form.

**Moment maps**

We introduce the Poisson structure $\{-,-\}_P : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ on $M$.

**Definition B.1.22.** Let $f, g, h$ be smooth functions on $M$.

1. $\{f, g\}_P = -\{g, f\}_P$
2. $\{f, gh\}_P = \{f, g\}_P h + g\{f, h\}_P$
3. $\{f, \{g, h\}_P\}_P + \{g, \{h, f\}_P\}_P + \{h, \{f, g\}_P\}_P = 0$

We call a manifold $M$ with a Poisson structure $(M, \{-,-\}_P)$ a Poisson manifold.

Let $M$ be a Poisson manifold and $H \in C^\infty(M)$ be a smooth function on $M$. A **Hamiltonian vector field** on $M$ generated by $H$ is defined by

$$X_H := \{H, -\}_P. \tag{B.31}$$

**Proposition B.1.23.**

$$[X_f, X_g] = X_{\{f,g\}_P} \quad {}^\forall f, g \in C^\infty(M) \tag{B.32}$$

**Proof .** The proof is straightforward since the following equations hold for any $h \in C^\infty(M)$.

$$[X_f, X_g]h = X_f X_g h - X_g X_f h- \tag{B.33}$$
$$= \{f, \{g, h\}_P\}_P - \{g, \{f, h\}_P\}_P \tag{B.34}$$
$$= \{f, \{g, h\}_P\}_P + \{g, \{h, f\}_P\}_P \tag{B.35}$$
$$= -\{h, \{f, g\}_P\}_P \tag{B.36}$$
$$= \{\{f, g\}_P, h\}_P \tag{B.37}$$
$$= X_{\{f,g\}_P} h \quad \square \tag{B.38}$$

**Theorem B.1.24.** Noether

Let $M$ be a symplectic manifold, $G$ be a Lie group acting on $M$ and $\mathfrak{g}$ the Lie algebra of $G$. Consider a $\mathfrak{g}^*$ valued $C^\infty$ function on $M$

$$\mu : M \to \mathfrak{g}^*. \tag{B.39}$$

We define

$$\mu_Y = \langle \mu, Y \rangle \quad Y \in \mathfrak{g}^*. \tag{B.40}$$

Then this function $\mu$ satisfies

$$\begin{aligned} X_{\{\mu_Y, \mu_Z\}_P} &= [X_{\mu_Y}, X_{\mu_Z}] && \text{(B.41)} \\ &= [Y, Z] && \text{(B.42)} \\ &= X_{\mu_{[Y,Z]}}. && \text{(B.43)} \end{aligned}$$

However $\{\mu_Y, \mu_Z\}_P = \mu_{[Y,Z]}$ is not always true.

**Definition B.1.25.** Let $(M, \omega)$ be a symplectic manifold and $G$ be a connected Lie group acting on $M$. Action of $G$ is **Hamiltonian action** if there exists $\mu : M \to \mathfrak{g}^*$ such that

1. $g^*\omega = \omega \ \ {}^\forall g \in G$
2. $\iota_Y = d\langle \mu, Y \rangle$
3. $\{\mu_Y, \mu_Z\}_P = \mu_{[Y,Z]}$

In particular we call this $\mu$ a **moment map**.

**Example B.1.26.** Let $A$ be a connection 1-form and $F_A$ be the curvature $F_A = dA + A \wedge A$. Let $\epsilon$ generate an infinitesimal gauge transformation and $V(\epsilon)$ be the corresponding vector field on $\mathcal{W}$, which acts by $\delta A = -d_A\epsilon$ and $\omega_A = \frac{1}{2}\int \delta A \wedge \delta A$.

We make a small transformation $A \to A + \delta A$, then

$$F_{A+\delta A} = F_A + F_{\delta A} + \delta A \wedge \delta A, \tag{B.44}$$

where $F_{\delta A} = d(\delta A) + A \wedge \delta A + \delta A \wedge A$. We define the moment map $\mu_A$ by

$$\mu_A = \int \mathrm{Tr}\epsilon F. \tag{B.45}$$

A small change in $\mu_A$ is

$$\delta\mu_A = \int \mathrm{Tr}\epsilon F_{\delta A} \tag{B.46}$$

$$= \int \mathrm{Tr}\epsilon(d(\delta A) + A \wedge \delta A + \delta A \wedge A) \tag{B.47}$$

$$= \int \mathrm{Tr}\epsilon d_A \delta A \tag{B.48}$$

$$= -\int \mathrm{Tr} d_A\epsilon \wedge \delta A. \tag{B.49}$$

Hence we obtain the formula

$$\iota_V \omega_A = \delta\mu_A. \tag{B.50}$$

**Hyper Käher Quotients**

**Definition B.1.27.** Action of $G$ on a hyper-Kähler manifold $(M, I, J, K)$ is **hyper Hamiltonian action** when $G$ is a hyper Hamiltonian action with respect to all of the three Käher forms $\omega_I, \omega_J, \omega_K$ respectively. In this case, we have three moment maps $\mu_I, \mu_J, \mu_K$. We write these as a single map

$$\mu : M \to \mathfrak{g}^* \otimes \mathbb{H}, \tag{B.51}$$

which we call a **hyper Käher moment map**.

**Theorem B.1.28.** (Hitchin et al.)
The quotient metric on $\mu^{-1}(0)/G$ is hyper Kählerian.

**Proof .** We first focus on a complex structure $I$ with Kähler form $\omega_I$. To complete the proof, repeat the following argument with respect to the remaining complex structures $J, K$.

Consider the complex function

$$\mu_+ := \mu_J + i\mu_K : M \to \mathfrak{g}^* \otimes \mathbb{C}. \tag{B.52}$$

Then we obtain

$$d\mu_{+Y}(X) = \omega_J(Y, X) + i\omega_K(Y, X) = g(JY, X) + ig(KY, X), \tag{B.53}$$

$$d\mu_{+Y}(IX) = g(JY, IX) + ig(KY, IX) = -g(KY, X) + ig(JY, X). \tag{B.54}$$

Thus

$$d\mu_{+Y}(X) = id\mu_{+Y}(IX) \tag{B.55}$$

for all vector field $Y$.

Let $\frac{\partial}{\partial \overline{z}^a}$ be a complex vector field arising from a local coordinate system $\{z^a\}$, holomorphic with respect to $I$ on $M$. Then

$$I\left(\frac{\partial}{\partial \overline{z}^a}\right) = -\frac{\partial}{\partial \overline{z}^a}, \tag{B.56}$$

therefore

$$i\frac{\partial \mu_{+Y}}{\partial \overline{z}^a} = d\mu_{+Y} I\left(\frac{\partial}{\partial \overline{z}^a}\right) = -i\frac{\partial \mu_{+Y}}{\partial \overline{z}^a}, \tag{B.57}$$

then

$$\frac{\partial \mu_{+Y}}{\partial \overline{z}^a} = 0, \tag{B.58}$$

which means $\mu_{+Y}$ is a holomorphic function. Thus $N = \mu_{+Y}^{-1}(0) = \mu_J^{-1}(0) \cap \mu_K^{-1}(0)$ is a complex sub manifold of $M$ with respect to the complex structure $I$, and so its induced metric Käherian. □

### B.1.6 Symplectic geometry

**Definition B.1.29.** A **symplectic form** $\omega$ on a $2n$ dimensional smooth manifold $M$ is a closed two-form ($d\omega = 0$) on $M$ and non degenerate ($\omega^n = \omega \wedge \cdots \wedge \omega \neq 0$). Such a pair $(M, \omega)$ is called a **symplectic manifold**. Assigning a symplectic form $\omega$ to a manifold $M$ is referred to as giving $M$ a **symplectic structure**.

**Definition B.1.30.** Let $(V, \omega)$ be a symplectic vector space, namely V is a vector space endowed with a non degenerate skew-symmetric bilinear form $\omega$, and $W \subset V$ be a linear subspace. We define the **symplectic perpendicular** by

$$W^{\perp} = \{v \in V |\ \omega(v, w) = 0\ {}^{\forall} w \in W\}. \tag{B.59}$$

$W$ is called

· **isotropic** if $W \subset W^{\perp}$

· **coisotropic** if $W^{\perp} \subset W$

· **Lagrangian** if $W = W^{\perp}$

· **symplectic** if $W \cap W^{\perp} = 0$.

Note that if $W$ is isotropic, then $\dim W \leq \frac{1}{2} \dim V$, on the other hand if it is coisotropic, then $\dim W \geq \frac{1}{2} \dim V$. Lagrangian subspaces are always middle-dimensional.

Let $(M, \omega)$ be a symplectic manifold and $N \subset M$ be a submanifold. $N$ is called **isotropic** (resp. **coisotropic**, **Lagrangian** and **symplectic**) if for every $p \in N$, $T_pN \subset T_pM$ is isotropic (resp. **coisotropic**, **Lagrangian** and **symplectic**).

# B.2 Categories

## B.2.1 Definition of a category and a functor

**Definition B.2.1.** A **categoly** $\mathcal{C}$ consists of a collection of **objects** $\mathrm{Ob}(\mathcal{C})$ and a collection of **morphisms** $\mathrm{Hom}_{\mathcal{C}}(A, B)$ for any two objects $A, B \in \mathrm{Ob}(\mathcal{C})$ satisfying the following conditions:

1. (composition law) For each $A, B, C \in \mathrm{Ob}_{\mathcal{C}}$, a composition

$$\circ : \mathrm{Hom}_{\mathcal{C}}(A, B) \times \mathrm{Hom}_{\mathcal{C}}(B, C) \to \mathrm{Hom}_{\mathcal{C}}(A, C) \tag{B.60}$$

$$(g, f) \mapsto f \circ g \tag{B.61}$$

satisfies

$$(f \circ g) \circ h = f \circ (g \circ h). \tag{B.62}$$

2. (identity morphism) For any $A \in \mathrm{Ob}(\mathcal{C})$ there exists an identity morphism $\mathrm{id}_A \in \mathrm{Hom}_{\mathcal{C}}(A, A)$ such that

$$\mathrm{id}_B \circ f = f \circ \mathrm{id}_A \tag{B.63}$$

for any $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$.

The idea of category is important and useful for modern science. For example, the category of sets is the category whose objects are sets and morphisms are maps between any two sets. Another important example is the category of commutative rings[1] whose objects are commutative rings and morphisms are homomorphisms between any two commutative rings. Moreover for a commutative ring $R$, the category of the $R$-module $\mathrm{mod}(R)$ is the category whose objects are $R$-modules and morphisms are $R$-module morphisms. If $R = \mathbb{Z}$, we denote by $Ab$ **$\mathbb{Z}$-mod** and call it **Abelian category**.

**Definition B.2.2.** For categories $\mathcal{C}_1$ and $\mathcal{C}_2$, $F : \mathcal{C}_1 \to \mathcal{C}_2$ is a **functor** (or **covariant functor**) if $F(E) \in \mathrm{Ob}(\mathcal{C}_2)$ for any $E \in \mathrm{Ob}(\mathcal{C}_1)$ and $F(f) \in \mathrm{Hom}_{\mathcal{C}_2}(F(A), F(B))$ for any $f \in \mathrm{Hom}_{\mathcal{C}_1}(A, B)$, and satisfies the following conditions:

1. $F(f \circ g) = F(f) \circ F(g)$.

2. $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$.

If $f \mapsto F(f)$ is a bijection, $F$ is called **fully faithful**.

For a category $\mathcal{C}$, the **opposite category** $\mathcal{C}^{\mathrm{op}}$ of $\mathcal{C}$ is a category such that $\mathrm{Ob}(\mathcal{C}^{\mathrm{op}}) = \mathrm{Ob}(\mathcal{C})$ and $\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(A, B) = \mathrm{Hom}_{\mathcal{C}}(A, B)$. A covariant functor form $\mathcal{C}_1^{\mathrm{op}}$ to $\mathcal{C}_2$ is called the **opposite functor** from $\mathcal{C}_1$ to $\mathcal{C}_2$.

## B.2.2 Definition of a sheaf

Sheaves are kind of generalization of functions on topological spaces. Before we state a sheaf, we define a **presheaf** as follow.

**Definition B.2.3.** Let $X$ be a topological space. A **presheaf** $\mathcal{F}$ on $X$ is defined in such a way that $\mathcal{F}$ associates an abelian group $\mathcal{F}(U)$ to every open set $U \subset X$ and there is a homomorphism $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$, called **restriction**, for each pair of open sets $V \subset U$ which satisfies

1. $\rho_{UU} = 1$

2. $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$ for a triad of open sets $W \subset V \subset U$

[1] We assume any ring in this article has an identity element.

An trivial example of a presheaf $\mathcal{F}$ is that $\mathcal{F}$ associate $\mathcal{F}(U) = \mathbb{Z}$ for every open set $U \subset X$ and a homomorphism is defied $\rho_{UV} = 1$ everywhere. In this sense, an abelian group $\mathcal{F}(U) = \mathbb{Z}$ is identified with the set of constant functions from $U$ to $\mathbb{Z}$

Gluing presheaves in a natural way gives a **sheaf**.

**Definition B.2.4.** A presheaf $\mathcal{F}$ on $X$ is called a **sheaf** if $\mathcal{F}$ satisfies the following conditions for any open subset $U$ of $X$ and its open covering $\{U\}_{i\in I}$:

1. If $\rho_{U,U_i}(a) = 0$ for $a \in \mathcal{F}(U)$, then $a = 0$.

2. If $a_i \in \mathcal{F}(U_i)$, $i \in I$ satisfy

$$\rho_{U_i,U_i\cap U_j}(a_i) = \rho_{U_j,U_i\cap U_j}(a_j) \tag{B.64}$$

then there exists $a \in \mathcal{F}(U)$ such that

$$a_i = \rho_{U,U_i}(a). \tag{B.65}$$

We call elements of $\mathcal{F}(U)$ **sections** on $U$. Especially a section on $X$ is called a **global section** and denoted by $\Gamma(X, \mathcal{F})$. Moreover a **stalk** at each point $x \in X$ is defined by

$$\mathcal{F}_x = \varinjlim_{x\in U} \mathcal{F}(U), \tag{B.66}$$

where $U$ is all open subset of $X$ containing $x$.

A fundamental example of a sheaf is a vector bundle over a manifold.

**Example B.2.5.** Let $M$ be a smooth manifold and $\pi : E \to M$ be a vector bundle. For each open subset $U \subset X$, let $\mathcal{F}(U)$ be the set of smooth sections of the vector bundle over $U$. Then $\mathcal{F}$ becomes a sheaf on $M$. It is also possible to have a sheaf of $C^\infty$ functions. In any case, the restrictions are obvious.

Let $\mathcal{O}$ be a sheaf of commutative rings on a topological space $X$, namely $\mathcal{O}(U)$ are rings for each open subset $U \subset X$. The sheaf $U \mapsto \mathcal{F}(U)$ in Example B.2.5 is an standard example. The pair $(X, \mathcal{O}_X)$ is called a **ringed space**. If the stalks $\mathcal{O}_{X,x}$ at each point $x \in X$ are local rings, then $(X, \mathcal{O}_X)$ is called a **local ringed space**.

### B.2.3 Definition of a scheme

Let Spec$R$ be the set of all prime ideals of a commutative ring $R$. Then Spec$R$ has the **Zariski topology**, where open sets are defined by

$$U_f = \{\mathfrak{p} \in \mathrm{Spec}R : f \notin \mathfrak{p}\}, \quad f \in R. \tag{B.67}$$

From now we regard $X$ =Spec$R$ as a topological space with the Zariski topology. With respect to sheaf of the ring $\mathcal{O}_X$, the pair $(X, \mathcal{O}_X)$ is called an **affine scheme**, and $\mathcal{O}_X$ is called a **structure sheaf**. Moreover we call a point $\mathfrak{p} \in X$ a point of the affine scheme $(X, \mathcal{O}_X)$, and $X$ **base space**. For a point $\mathfrak{p}$ of an affine scheme $(X, \mathcal{O}_X)$, the **stalk** at $\mathfrak{p}$, written as $\mathcal{O}_{X,\mathfrak{p}}$, is defined by the left hand side of the equation

$$\varinjlim_{\mathfrak{p}\in U} \Gamma(U, \mathcal{O}_X) = R_{\mathfrak{p}}. \tag{B.68}$$

Assume a ring homomorphism $\phi : S \to R$ is given, then we obtain a continuous map

$$\begin{aligned} f : \mathrm{Spec}R &\to \mathrm{Spec}S \\ \mathfrak{p} &\mapsto \phi^{-1}(\mathfrak{p}). \end{aligned} \tag{B.69}$$

Moreover the ring homomorphism $\phi$ induces sheaf homomorphism $\mathcal{O}_{\mathrm{Spec}S} \to f_*\mathcal{O}_{\mathrm{Spec}R}$. And the pair $(f, \phi)$ is a morphism between the affine schemes. Therefore we obtain the opposite functor

$$\begin{aligned} \text{"Category of commutative rings"} &\to \text{"Category of affine schemes"} \\ R &\mapsto (\mathrm{Spec}R, \mathcal{O}_{\mathrm{Spec}R}) \end{aligned} \tag{B.70}$$

There is an inverse functor $(X, \mathcal{O}_X) \to \Gamma(X, \mathcal{O}_X)$, and then the above two categories are equivalent.

Likewise a algebraic variety, glueing affine schemes forms a scheme.

**Definition B.2.6.** A local ringed space $(X, \mathcal{O}_X)$ is called a **scheme**, if there exists an open covering $\{U_i\}_{i\in I}$ of $X$ such that $(U_i, \mathcal{O}_{U_i})$ are isomorphic as a local ringed space to an affine scheme. In this case, we call the topological space $X$ **base space** of the scheme and $\mathcal{O}_X$ its **structure sheaf**.

By definition, a scheme has affine schemes for its covering. We often consider an **algebraic scheme of finite type over** $\mathbb{C}$ (**algebraic scheme** in short), which looks like

$$X = \cup_{i=1}^{N} U_i, \quad U_i \cong \mathrm{Spec}R_i, \tag{B.71}$$

where $R_i$ are finitely generated $\mathbb{C}$-algebras. The definition of an algebraic scheme is close to that of an algebraic variety. For a commutative $R$, we define the **nilradical** of $R$ by

$$\sqrt{0} = \{f \in R : f^k = 0, \ ^{\exists}0 \leq k \in \mathbb{Z}\}. \tag{B.72}$$

Then a ring $R' = R/\sqrt{0}$ has the same maximal ideals, Spm$R'$ =Spm$R$. Let $X$ be an algebraic variety and $\{U_i\}_{i\in I}$ be its finite covering,

$$X = \cup_{i\in I}^N U_i, \quad U_i \cong \mathrm{Spm} R_i. \tag{B.73}$$

For each $R_i/\sqrt{0}$, let $U_i' \cong \mathrm{Spec} R_i'/\sqrt{0}$ be affine schemes. Then gluing $U_i$ naturally induces gluing $U_i'$ and it gives the scheme defined by

$$X' = \cup_{i\in I}^N U_i', \quad U_i' \cong \mathrm{Spec} R_i/\sqrt{0}. \tag{B.74}$$

This correspondence $X \to X'$ is indeed an fully faithful functor from the category of algebraic varieties to the category of algebraic schemes. In this sense we regard algebraic varieties as algebraic schemes.

### B.2.4 Coherent sheaves

#### Definition and examples

Let $\mathcal{F}$ be a sheaf on a scheme $X$. $\mathcal{F}$ is called a sheaf of $\mathcal{O}_X$-module if $\mathcal{F}(U)$ are $\mathcal{O}_X(U)$-modules for any open subset $U \subset X$ and they satisfy the natural consistency. A **coherent sheaf** $\mathcal{F}$ on a scheme $X$ is defined for the case $\mathcal{F}$ is a sheaf of $\mathcal{O}_X$-module. Rather than giving the original definition of a coherent sheaf, we introduce coherent sheaves in a different, but equivalent and more useful way.

Let $X$ be an algebraic scheme with coordinates (B.71). We assume $U_{i_1\cdots i_k} = U_{i_1} \cap \cdots \cap U_{i_k}$ to be affine schemes for all $i_1, \cdots, i_k$, and we write $R_{i_1\cdots i_k}$ for the associated commutative rings, namely $U_{i_1\cdots i_k} = \mathrm{Spec} R_{i_1\cdots i_k}$.

The following theorem is crucial for us.

**Theorem B.2.7.** The category Coh$(X)$ of coherent sheaves on algebraic scheme $X$ is equivalent to the category which consists of the following data: let $\{R_i\}_{i\in I}$ be a set of commutative rings. For

$M_i \in \mathrm{mod}(R_i)$, we define $M_i|_{U_{ij}} := M \otimes_{R_i} R_{ij}$.

$$M = (M_i, \phi_{ij}),\ M_i \in \mathrm{mod}(R_i),\ \phi_{ij} : M_j|_{U_{ij}} \cong M_i|_{U_{ij}}, \tag{B.75}$$

where $\phi_{ij}$ are isomorphisms of $R_{ij}$-modules which satisfy $\phi_{ii} = 1$ and the cocycle conditions

$$\phi_{ij} \circ \phi_{jk} \circ \phi_{ki} = 1. \tag{B.76}$$

Given another data $N = (N_i, \varphi_i)$, we define a morphism $\mathrm{Hom}(M, N)$ by

$$\mathrm{Hom}(M, N) = \left\{ f = (f_1, \cdots, f_N) \in \prod_{i=1}^{N} \mathrm{Hom}_{R_i}(M_i, N_i) : \varphi_{ij} \circ f_j = f_i \circ \phi_{ij} \right\}. \tag{B.77}$$

Moreover with respect to $f = (f_i)_{i=1}^{N}$, we define $\mathrm{Ker}(f)$ and $\mathrm{Coker}(f)$ by

$$\mathrm{Ker}(f) = (\mathrm{Ker}(f_i), \phi_{ij}), \tag{B.78}$$

$$\mathrm{Coker}(f) = (\mathrm{Ker}(f_i), \varphi_{ij}). \tag{B.79}$$

To uncover geometrical structure of an algebraic scheme $X$, making a survey on global sections of coherent sheaves on $X$ is effective. We identify $M \in \mathrm{Coh}(X)$ with $M$ of (B.75) by theorem B.2.7. Then there is a isomorphism

$$\Gamma(X, M) \cong \left\{ (s_1, \cdots, s_N) \in \prod_{i=1}^{N} M_i : s_i = \phi_{ij}(s_j) \right\}. \tag{B.80}$$

**Example B.2.8.** One of important examples of coherent sheaves is **algebraic vector bundles**. Let $X$ be an algebraic scheme. $M = (M_i, \phi_{ij}) \in \mathrm{Coh}(X)$ is called a **locally free sheaf** of rank $r$ if $M_i$ decomposes into $M_i = R_i^{\otimes r}$ by choosing the affine covering sufficiently small if necessary. In this case, $\phi_{ij} \in \mathrm{GL}_r(R_{ij})$ follows. We consider an vector bundle whose fiber is $\mathbb{C}^r$. To obtain an appropriate gluing rule for fibers, we define an equivalence relation between $(x_i, v_i) \in U_i \times \mathbb{C}^r$ and $(x_j, v_j) \in U_j \times \mathbb{C}^r$ by

$$(x_i, v_i) \sim (x_j, v_j) \Longleftrightarrow x_i = x_j \in U_i \cap U_j \text{ and } v_i = \phi_{ij}(v_j), \tag{B.81}$$

and then an algebraic scheme is defined by

$$\mathcal{M} = \left( \coprod_{i=1}^{N} (U_i \times \mathbb{C}^r) \right) / \sim . \tag{B.82}$$

The natural projections $U_i \times \mathbb{C}^r \to U_i$ induce a morphism of the affine scheme $\pi : \mathcal{M} \to X$, which is analogous to a vector bundle over a manifold and called an **algebraic vector bundle** on $X$. The algebraic vector bundles and the locally free sheaves can be identified since once given an algebraic vector bundle on $X$ defined by (B.82), one can construct a locally free sheaf $(R_i^{\otimes r}, \phi_{ij})$ on $X$. If $r = 1$, we call them **line bundle** and write $\mathrm{Pic}(X)$ for the set of all equivalent classes of line bundles on $X$.

**Example B.2.9.** Another important example of coherent sheaves is a sheaf of **the module of differentials**. Let $X$ be an algebraic scheme (B.71). We define modules of differentials $\Omega_{R_i/\mathbb{C}}$ for each $i$ and $\phi_{ij}$ to be natural isomorphisms. Then we obtain the sheaf of the module of differentials $\Omega_X \in \mathrm{Coh}(X)$. If $X$ is smooth, $\Omega_X$ is called a **cotangent bundle**, which is a locally free sheaf whose rank is $\mathrm{dim}X$, and its dual $\Omega_X^\vee$ is called a **tangent bundle**. The exterior product of cotangent bundles $\Omega_X^p = \wedge^p \Omega_X$ becomes a line bundle if $p = \mathrm{dim}X$, which is called a **canonical line bundle**.

The following statement distinguishes the categories of coherent sheaves as abelian categories.

**Lemma B.2.10.** Let $X$ and $Y$ be smooth projective algebraic varieties. Suppose thier categories $\mathrm{Coh}(X)$ and $\mathrm{Coh}(Y)$ of the coherent sheaves are equivalent as abelian categories. Then $X$ and $Y$ are isomorphic as algebraic varieties.

### Sheaf cohomology

As mentioned, global sections of a line bundle $L$ have much information to investigate the base algebraic variety $X$. One of strong instruments to know the existence of global sections is the sheaf cohomology.

Let $X$ be an algebraic variety and $L = (\mathcal{O}_{U_i}, \phi_{ij})$ be a line bundle on $X$. We restrict $L$ to a subvariety $Y \subset X$ and write it as $L|_Y$. Our strategy is that we consider whether a global section $s \in \Gamma(Y, L|_Y)$ of $L|_Y$ become a global section of $L$, that is to say we consider whether there exist an obstruction. If the sheaf cohomology is nontrivial, then it means that the global section of $L|_Y$ cannot become a global section of $L$. The cohomologies of an algebraic scheme $X$ is defined via the functors in the following theorem.

**Theorem B.2.11.** Let $X$ be a algebraic scheme. There exist functors

$$H^i(X,-) : \mathrm{Coh}(X) \to \text{"Category of complex vector space"} \tag{B.83}$$

for each $0 \leq i \in \mathbb{Z}$ such that

1. $H^0(X,-) = \Gamma(X,-)$

2. with respect to a exact sequence of coherent sheaves $0 \to M_1 \to M_2 \to M_3 \to 0$, there is a homomorphisms $\delta^i : H^i(X, M_3) \to H^{i+1}(X, M_1)$, which give the following long exact sequence

$$\begin{aligned} 0 \to H^0(X, M_1) \to H^0(X, M_2) \to H^0(X, M_3) \xrightarrow{\delta^0} H^1(X, M_1) \to \cdots \\ \cdots \to H^i(X, M_1) \to H^i(X, M_2) \to H^i(X, M_3) \xrightarrow{\delta^i} H^{i+1}(X, M_1) \to \cdots \end{aligned} \tag{B.84}$$

**Example B.2.12.** The sheaf cohomologies of the line bundle $\mathcal{O}_{\mathbb{P}^n}(k)$ on $\mathbb{P}^n$ are

$$H^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(k)) = \begin{cases} V_k & i = 0 \\ 0 & i \neq 0, k \\ V_{-k-n-1}^{\vee} & i = n \end{cases}, \tag{B.85}$$

where $V_k \subset \mathbb{C}[y_0, \cdots, y_n]$ is the vector space consisting of all homogeneous polynomials of degree $k$. We put $V_k = 0$ for $k < 0$.

Since the above method to consider a line bundle on algebraic schemes is almost parallel with the discussions about a line bundle on complex manifold, it would be natural to ask their connections. To relate the sheaf cohomology of an algebraic scheme $X$ with the Check cohomology, we write $X^h$ for a algebraic variety $X$ with standard Euclidean topology. Note that if $X$ is a smooth algebraic variety over $\mathbb{C}$, then $X^h$ is a complex manifold. The following theorem is paramount of importance to connect the Check cohomologies $H^n(X^h, \mathbb{C})$ of $X^h$ and the sheaf cohomologies.

**Theorem B.2.13.** Let $X$ be a smooth projective algebraic variety, and put $H^{p,q} = H^q(X, \Omega_X^p)$. Then there are isomorphisms

$$H^n(X^h, \mathbb{C}) \cong \oplus_{p+q=n} H^{p,q}(X). \tag{B.86}$$

The dimensions of $H^{p,q}(X)$ are the **Hodge numbers** $h^{p,q}(X)$ as defined in (B.19).

### B.2.5 Derived categories of abelian sheaves

We first give the generic definition of the derived category $D(\mathcal{A})$ of an abelian category $\mathcal{A}$. Let $C(\mathcal{A})$ be the category whose objects are complexes of objects of $\mathcal{A}$, namely objects are of the form

$$A = \cdots \xrightarrow{d^{i-2}} A^{i-1} \xrightarrow{d^{i-1}} A^i \xrightarrow{d^i} A^{i+1} \xrightarrow{d^{i+1}} \cdots , \tag{B.87}$$

where $A^i \in \mathrm{Ob}(\mathcal{A})$ and $d^{i-1} \circ d^i = 0$ for all $i$, and a morphism $f \in \mathrm{Hom}_{C(\mathcal{A})}(A, B)$ is a collection of maps $f^i : A^i \to B^i$ which satisfy the following commutative diagram

$$\begin{tikzcd} \cdots \arrow[r, "d^{i-2}"] & A^{i-1} \arrow[r, "d^{i-1}"] \arrow[d, "f^{i-1}"] & A^i \arrow[r, "d^i"] \arrow[d, "f^i"] & A^{i+1} \arrow[r, "d^{i+1}"] \arrow[d, "f^{i+1}"] & \cdots \\ \cdots \arrow[r, "d'^{i-2}"] & B^{i-1} \arrow[r, "d'^{i-1}"] & B^i \arrow[r, "d'^i"] & B^{i+1} \arrow[r, "d'^{i+1}"] & \cdots \end{tikzcd} \tag{B.88}$$

We say two morphisms $f, g \in \mathrm{Hom}_{C(\mathcal{A})}(A, B)$ are homotopy equivalent ($f \sim g$) if there exist $h^i \in \mathrm{Hom}_{\mathcal{A}}(A^i, B^{i-1})$ for every $i$ such that

$$d'^{i-1} \circ h^i - h^{i+1} \circ d^i = f^i - g^i. \tag{B.89}$$

The category of homotopy $K(\mathcal{A})$ of $C(\mathcal{A})$ is the category which has the same object with $C(\mathcal{A})$ and whose morphisms are the morphisms of $C(\mathcal{A})$ modulo the homotopy equivalence relation, namely $\mathrm{Hom}_{K(\mathcal{A})}(A, B) = \mathrm{Hom}_{C(\mathcal{A})}(A, B)/\sim$.

The cohomologies

$$H^i(A) = \mathrm{Ker} d^i / \mathrm{Im} d^{i-1} \tag{B.90}$$

of the complexes $\{A^i\}_{i \in I}$ do not depend on homotopy classes. A morphism $f \in \mathrm{Hom}_{K(\mathcal{A})}(A, B)$ is called a **quasi isomorphism** when induced maps

$$H^i(f) = H^i(A) \to H^i(B) \tag{B.91}$$

are isomorphisms for all $i$.

**Definition B.2.14.** The derived category $D(\mathcal{A})$ of the abelian category $\mathcal{A}$ is the category whose objects are complexes (B.87) of objects of $\mathcal{A}$ and morphisms $\mathrm{Hom}_{D(\mathcal{A})}(A, B)$ are given by the equivalence class of the following diagram

$$\begin{tikzcd} & A' \arrow[dl, "g"'] \arrow[dr, "h"] & \\ A & & B, \end{tikzcd} \tag{B.92}$$

where $g, h$ are morphisms in $K(\mathcal{A})$, and $g$ is a quasi isomorphism. The equivalence relation is defined by identifying the above diagram (B.93) with the following diagram for all quasi isomorphisms $g' : A'' \to A'$

$$\begin{tikzcd} & A'' \arrow[dl, "g \circ g'"'] \arrow[dr, "h \circ g'"] & \\ A & & B. \end{tikzcd} \tag{B.93}$$

Giving some constraints on the degrees of the cohomologies, we obtain several fully faithful subcategories $D^*(\mathcal{A}) \subset D(\mathcal{A})$ $(* = +, -, b)$, which are defined as follows: $D^+(\mathcal{A})$ consists of complexes (B.87) whose cohomologies $H^i(A)$ vanish for all sufficiently small $i \ll 0$. Similarly $D^-(\mathcal{A})$ consists of complexes (B.87) whose cohomologies $H^i(A)$ vanish for all sufficiently large $i \gg 0$. We put $D^b(\mathcal{A}) = D^+(\mathcal{A}) \cap D^-(\mathcal{A})$.

**Remark B.2.15.** Generally a derived category of an abelian category is not abelian category, which is distinguished as a **triangulated category**.

### B.2.6 Derived categories of coherent sheaves

Let $X$ be an algebraic variety, then the category $\mathrm{Coh}(X)$ of the coherent sheaf on $X$ is an abelian category, which allows us to construct the derived categories $D^*\mathrm{Coh}(X)$ of the coherent sheaf. The most important case for us is $D^b\mathrm{Coh}(X)$.

### B.2.7 Fourier-Mukai transformations

A notable feature of a derived category of a coherent sheaf is that, since $D^b\mathrm{Coh}(X)$ is not an abelian category, there is the possibility that there exists an algebraic variety $Y$ such that $D^b\mathrm{Coh}(X) \cong D^b\mathrm{Coh}(Y)$. This is in marked contrast to the case of $\mathrm{Coh}(X)$ as given in the lemma B.2.10. S. Mukai firstly considered those dualities in [38] for abelian varieties and there are indeed such dualities. Generally if $D^b\mathrm{Coh}(X) \cong D^b\mathrm{Coh}(Y)$ satisfied for two smooth projective algebraic varieties $X$ and $Y$, then $X$ and $Y$ are called **Fourier-Mukai partners** and the transformation to obtain $Y$ form $X$ is called the **Fourier-Mukai transformation**. This section is devoted to briefly recall the transformation [38].

To explain the reason for the "Fourier" appearing in the name of transformation, we recall the usual Fourier transformation. A typical example is $X = \mathbb{R}/\mathbb{Z} \cong S^1$ and $Y = \mathbb{Z}$, which we have already discussed in (2.72). A function $f \in L^2(\mathbb{R}/\mathbb{Z})$ corresponds to its Fourier transformation $\hat{f}$ by

$$\begin{aligned} L^2(\mathbb{R}/\mathbb{Z}) &\to L^2(\mathbb{Z}) \\ f(t) = \sum_{n\in\mathbb{Z}} \hat{f}(n)e^{2\pi int} &\mapsto \hat{f}(n) = \int_0^1 f(t)e^{-2\pi int}, \end{aligned} \tag{B.94}$$

which gives the isometry of the Hilbert spaces of square-integrable function on $\mathbb{R}/\mathbb{Z}$ and $\mathbb{Z}$, namely

$$\begin{aligned} L^2(\mathbb{R}/\mathbb{Z}) &\overset{\text{isometry}}{\cong} L^2(\mathbb{Z}) \\ f &\longleftrightarrow \hat{f}. \end{aligned} \tag{B.95}$$

Note that $\mathrm{Hom}(\mathbb{R}/\mathbb{Z}, \mathbb{C}^\times) \cong \mathbb{Z}$ by the map $t \to e^{2\pi int}$ for a given $n \in \mathbb{Z}$. That is to say, $Y = \mathbb{Z}$ is the moduli space of homomorphisms from $X = \mathbb{R}/\mathbb{Z}$ to $\mathbb{C}^\times$. In this sense $\hat{f}$ is a function on the moduli space.

Now we explain the Fourier-Mukai transformations. The following theorem is based on [40].

**Theorem B.2.16.** Let $X$ and $Y$ be smooth projective algebraic varieties, and

$$\Phi : D^b\mathrm{Coh}(X) \to D^b\mathrm{Coh}(Y) \tag{B.96}$$

be a fully faithful functor. Then there exists an object $\mathcal{E} \in \mathrm{Ob}(D^b\mathrm{Coh}(X \times Y))$, unique up to isomorphism, such that $\Phi$ is isomorphic to the functor

$$\Phi^{\mathcal{E}}_{X\to Y} : D^b\mathrm{Coh}(X) \to D^b\mathrm{Coh}(Y) \tag{B.97}$$

$$x \mapsto \mathbf{R}\mathrm{p}_{Y_*}(\mathcal{E} \overset{L}{\otimes} \mathrm{p}_X^*(x)), \tag{B.98}$$

where $\mathrm{p}_X$ and $\mathrm{p}_Y$ are projections from $X \times Y$ to $X$ and $Y$ respectively.

$\Phi^{\mathcal{E}}_{X\to Y}$ is called the **Fourier-Mukai transformation** if $\Phi^{\mathcal{E}}_{X\to Y}$ gives equivalences of the derived categories. The object $\mathcal{E}$ corresponds to the kernel function $e^{2\pi int}$ in the above example. The procedure of the Fourier transformation of $f$ to $\hat{f}$ is extended to the Fourier-Mukai transformation in such a way that $\mathrm{p}_X^*$ pull backs the complexes of $D^b\mathrm{Coh}(X)$ to those of $D^b\mathrm{Coh}(X \times Y)$ and $\mathbf{R}\mathrm{p}_{Y_*}$ push forwards the complexes of $D^b\mathrm{Coh}(X \times Y)$ to those of $D^b\mathrm{Coh}(Y)$.

**Example B.2.17.** Let $A$ be an **abelian variety**, which is a projective algebraic variety and can be written as

$$A = V/\Gamma, \tag{B.99}$$

where $V$ is a $n$-dimensional vector space and $\Gamma \subset \mathbb{C}^n$ is a free abelian group of rank $2n$. If $\dim A = 1$, then $A$ is an elliptic curve. We define the dual abelian variety $\hat{A}$ of $A$ by

$$\hat{A} = \overline{V}^{\vee}/\overline{\Gamma}^{\vee}, \tag{B.100}$$

where $\overline{V}^{\vee}$ is the dual of the complex conjugate of $V$ and $\overline{\Gamma}^{\vee}$ is given by

$$\overline{\Gamma}^{\vee} = \left\{ f \in \overline{V}^{\vee} : 2\mathrm{Re} f(\Gamma) \subset \mathbb{Z} \right\} \tag{B.101}$$

More precisely, when we denote by $\mathrm{Pic}(A)$ the moduli space of line bundles on $A$, then $\hat{A}$ is defined by the connected component of the identity $\mathrm{Pic}^0(A)$. If $\dim A = 1$, then $A \cong \hat{A}$, however, in higher dimensions $A$ and $\hat{A}$ are not isomorphic as algebraic varieties. Nevertheless, the derived categories of coherent sheaves are equivalent:

$$\mathbf{R}\hat{S} = \Phi^{\mathcal{P}}_{A \to \hat{A}} : D^b\mathrm{Coh}(A) \overset{\cong}{\to} D^b\mathrm{Coh}(\hat{A}), \tag{B.102}$$

where $\mathbf{R}\hat{S}$ is the Fourier-Mukai functor defined by (B.97). In our case the object $\mathcal{E}$ is the **Pincaré line bundle** $\mathcal{P} \in \mathrm{Pic}^0(A \times \hat{A})$, whose restriction $\mathcal{P}|_{A\times\hat{x}}$ on each point $\hat{x} \in \hat{A}$ is a line bundle on $A$. We normalize $\mathcal{P}$ so that both $\mathcal{P}|_{A\times\hat{0}}$ and $\mathcal{P}|_{0\times\hat{A}}$ are trivial. For $\hat{x} \in \hat{A}$, $P_{\hat{x}}$ denotes $\mathcal{P}|_{A\times\hat{x}}$. Note that to any coherent sheaf $F$ on $A$, we can associate the sheaf $\mathrm{p}_{\hat{A}_*}(\mathcal{P} \otimes \mathrm{p}_A^*(F))$ on $\hat{A}$, and so this correspondence gives the functor

$$\hat{S} : \mathrm{Coh}(A) \to \mathrm{Coh}(\hat{A}). \tag{B.103}$$

**Definition B.2.18.** Let $F$ be a coherent sheaf and $R^i\hat{S}(F)$ be the cohomologies of the derived complex $\mathbf{R}\hat{S}(F)$. We say that $F$ satisfies **W.I.T.** (weak index theorem) with index $i$ if

$$R^j\hat{S}(F) = 0 \tag{B.104}$$

for all $j \neq i$. We denote by $i(F)$ such one $i$. We write $\hat{F}$ for the sheaf $R^{i(F)}\hat{S}(F)$ and call it the Fourier-Mukai transform of $F$.

And we say that $F$ satisfies **I.T.** (index theorem) with index $i$ if $H^i(X, F \otimes P) = 0$ for all $P \in \mathrm{Pic}^0(X)$ and all but one $i$.

**Proposition B.2.19.** If $F$ obeys W.I.T., then $\hat{F}$ also does, and

$$i(\hat{F}) = \dim A - i(F). \tag{B.105}$$

Moreover $\hat{\hat{F}}$ is isomorphic to $(-1_A)^*F$, where $(-1_A)$ is the isomorphism $x \mapsto -x$ of $A$.

**Example B.2.20.** Let $k$ be an algebraically closed field and $k(\hat{x})$ be the one dimensional **skyscraper sheaf** supported at $\hat{x} \in \hat{A}$. Then one has $H^i(A, k(\hat{x}) \otimes P) = 0$ for each $i > 0$ and $P \in \mathrm{Pic}^0(\hat{X})$. Hence $k(x)$ obeys I.T. condition $i(k(x)) = 0$ and the Fourier-Mukai transformation of $k(\hat{x})$ is $\widehat{k(\hat{x})} \cong \mathcal{P}_{\hat{x}}$. On the other hand, $\widehat{P_{\hat{x}}} \cong k(-x)$ follows by the proposition B.2.19. Note that $P_{\hat{x}}$ obeys W.I.T. but does not satisfies the I.T. condition.

In summary, the following is held:

$$\text{Skyscraper sheaf} \overset{\text{FMT}}{\longleftrightarrow} \text{Line bundle}. \tag{B.106}$$

Recall the delta function $\delta(x)$ defined over $\mathbb{R}$ and 1 are transformed into each other under the Fourier transformation, the above correspondence is a natural extension.

# B.3 Mirror symmetry

## B.3.1 A-brane and B-brane

Let $X$ be a three-dimensional Calabi-Yau manifold, $B \in H^2(X, \mathbb{R}/\mathbb{Z})$ be a $B$-field and $\omega$ be a Kähler form on $X$.

The category of $B$-branes is defined by the derived category of the coherent sheaves $D^b\mathrm{Coh}(X)$. On the other hand the category of $A$-branes is known as the Fukaya category $\mathrm{Fuk}(X, \omega)$ (see [14] for its definition). Objects of $\mathrm{Fuk}(X, \omega)$ are a collection of Lagrange submanifolds of $X$ and morphisms between two Lagrange submaifolds $L_0$ and $L_1$ are given by the **Floer homology** $HF(L_0, L_1)$. We denote by $D^b\mathrm{Fuk}(X, \omega)$ the derived category of the Fukaya category.

The categorical mirror symmetry conjecture is proposed by Kontsevich[28].

**Conjecture B.3.1.** When $(X_1, B_1 + \sqrt{-1}\omega_1)$ and $(X_2, B_2 + \sqrt{-1}\omega_2)$ are mirror symmetry, then the two categories are equivalent:

$$D^b\mathrm{Fuk}(X_1, B_1 + \sqrt{-1}\omega_1) \cong D^b\mathrm{Coh}(X_2). \tag{B.107}$$

# Appendix C

# Spin representations for $SO(D)$ and $SO(D-t,t)$

## C.1 Spin representations for $SO(D)$

### C.1.1 Representations of gamma matrices in $D$ dimensions

We consider a set of gamma matrices $\{\Gamma^\mu\}_{\mu=0,\cdots,D-1}$, which satisfy the Clifford algebra $CL$ with

$$\{\Gamma^\mu, \Gamma^\nu\} = 2\delta^{\mu\nu}, \tag{C.1}$$

where $\delta^{\mu\nu} = \mathrm{diag}\{+1, +1, \cdots, +1\}$. We first work with the Euclidian signature, and we move on to more general Lorentzian signature later.

■ **Even dimension** $D = 2k+2$

We prefer to use convenient choices of combinations

$$\Gamma^{\mu\pm} = \frac{1}{2}(\Gamma^{2\mu} \pm i\Gamma^{2\mu+1}), \tag{C.2}$$

namely they are $k+1$ sets of raising and lowering operators which satisfy

$$\{\Gamma^{\mu+}, \Gamma^{\nu-}\} = \delta^{\mu\nu}, \tag{C.3}$$

$$\{\Gamma^{\mu+}, \Gamma^{\nu+}\} = \{\Gamma^{\mu-}, \Gamma^{\nu-}\} = 0. \tag{C.4}$$

In particular, $(\Gamma^{\mu+})^2 = (\Gamma^{\mu-})^2 = 0$ which implies there is a spinor $\zeta$ annihilated by all the $\Gamma^{\mu-}$,

$$\Gamma^{\mu-}\zeta = 0. \tag{C.5}$$

A general state $\zeta^{(s)}$ is created by operating $\Gamma^{\mu+}$s to $\zeta$

$$\zeta^{(s)} = (\Gamma^{k-})^{s_k+1/2} \cdots (\Gamma^{0-})^{s_0+1/2}\zeta, \tag{C.6}$$

where $s_\mu = \pm 1/2$. Then there are $2^{k+1}$ states which form the **Dirac representation**, **spin representation**, of the gamma matrices. What we want to find next is matrix representations of the gamma matrices.

<u>D=2</u>

For the case $D=2$, we have two states $\zeta$ and $\Gamma^{0+}\zeta$. By using the form $\Gamma^0 = \Gamma^{0+} + \Gamma^{0-}$, a matrix representation of $\Gamma^0$ is obtained as

$$\Gamma^0 \begin{pmatrix} \zeta \\ \Gamma^{0+}\zeta \end{pmatrix} = \begin{pmatrix} \Gamma^{0+}\zeta \\ \Gamma^{0-}\Gamma^{0+}\zeta \end{pmatrix} = \begin{pmatrix} \Gamma^{0+}\zeta \\ \zeta \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \zeta \\ \Gamma^{0+}\zeta \end{pmatrix}. \tag{C.7}$$

A matrix representation of $\Gamma^1$ is calculable by the same way. In summary

$$\Gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \Gamma^1 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{C.8}$$

<u>D=2k+2</u>

We first consider $\mu = 2k, 2k+1$ which are almost same as the $D=2$ case. We assume a spinor $\zeta_{k-1}$ and gamma matrices $\gamma_\mu$ $(\mu = 0, \cdots, 2k-1)$ is given in $D = 2k+1$ dimension. Then the $D = 2k+2$ dimensional spinor consists of $\zeta_{k-1}$ and $\Gamma^{k+}\zeta_{k-1}$. By using the form $\Gamma^{2k} = \Gamma^{k+} + \Gamma^{k-}$,

$$\Gamma^{2k} \begin{pmatrix} \zeta_{k-1} \\ \Gamma^{k+}\zeta_{k-1} \end{pmatrix} = \begin{pmatrix} \Gamma^{k+}\zeta_{k-1} \\ \zeta_{k-1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \zeta_{k-1} \\ \Gamma^{k+}\zeta_{k-1} \end{pmatrix}, \tag{C.9}$$

and since $\Gamma^{2k+1} = -i(\Gamma^{k+} - \Gamma^{k-})$,

$$\Gamma^{2k+1}\begin{pmatrix} \zeta_{k-1} \\ \Gamma^{k+}\zeta_{k-1} \end{pmatrix} = \begin{pmatrix} -i\Gamma^{k+}\zeta_{k-1} \\ i\zeta_{k-1} \end{pmatrix} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} \zeta_{k-1} \\ \Gamma^{k+}\zeta_{k-1} \end{pmatrix}. \tag{C.10}$$

Thus

$$\Gamma^{2k} = I_{2^k} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1_{2^k} \\ 1_{2^k} & 0 \end{pmatrix}, \tag{C.11}$$

$$\Gamma^{2k+1} = I_{2^k} \otimes \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & -i1_{2^k} \\ i1_{2^k} & 0 \end{pmatrix}, \tag{C.12}$$

where $1_{2^k}$ is the $2^k \times 2^k$ unit matrix. For $\mu = 0, \cdots, 2k-1$, we simply chose

$$\Gamma^{\mu} = \gamma^{\mu} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \gamma^{\mu} & 0 \\ 0 & -\gamma^{\mu} \end{pmatrix}. \tag{C.13}$$

Then the spinors with $\mu = 0, \ldots, 2k-1$ transform likewise in the $D = 2k$ dimensional case. Equivalently our gamma matrices are written in the form

$$\Gamma^0 = \sigma_1 \otimes \sigma_3 \otimes \sigma_3 \otimes \cdots \otimes \sigma_3 \otimes \sigma_3 \tag{C.14}$$

$$\Gamma^1 = \sigma_2 \otimes \sigma_3 \otimes \sigma_3 \otimes \cdots \otimes \sigma_3 \otimes \sigma_3 \tag{C.15}$$

$$\Gamma^2 = 1_2 \otimes \sigma_1 \otimes \sigma_3 \otimes \cdots \otimes \sigma_3 \otimes \sigma_3 \tag{C.16}$$

$$\Gamma^3 = 1_2 \otimes \sigma_2 \otimes \sigma_3 \otimes \cdots \otimes \sigma_3 \otimes \sigma_3 \tag{C.17}$$

$$\vdots \qquad\qquad \vdots$$

$$\Gamma^{2k-2} = 1_2 \otimes 1_2 \otimes \cdots \otimes 1_2 \otimes 1_2 \otimes \sigma_1 \tag{C.18}$$

$$\Gamma^{2k-1} = 1_2 \otimes 1_2 \otimes \cdots \otimes 1_2 \otimes 1_2 \otimes \sigma_2 \tag{C.19}$$

Here we used

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{C.20}$$

Generators of $SO(D)$ are given by

$$\Sigma^{\mu\nu} = -\frac{i}{4}[\Gamma^{\mu},\ \Gamma^{\nu}], \tag{C.21}$$

which satisfies

$$[\Sigma^{\mu\nu},\ \Sigma^{\rho\sigma}] = \delta^{\mu\rho}\Sigma^{\nu\sigma} + \delta^{\nu\sigma}\Sigma^{\mu\rho} - \delta^{\mu\sigma}\Sigma^{\nu\rho} - \delta^{\nu\sigma}\Sigma^{\mu\rho}. \tag{C.22}$$

Then

$$[\Sigma^{2\mu,2\mu+1},\ \Sigma^{2\nu,2\nu+1}] = 0, \tag{C.23}$$

which imply $\Sigma^{2\mu,2\mu+1}$ are simultaneous diagonalizable and they are part of Cartan sub algebra. So we introduce

$$S_\mu := \Sigma^{2\mu,2\mu+1} = -\frac{i}{2}\Gamma^{2\mu}\Gamma^{2\mu+1} = \Gamma^{\mu+}\Gamma^{\mu-} - \frac{1}{2}. \tag{C.24}$$

And each of $S_\mu$ acts on $\zeta^{(s)}$ as

$$S_\mu\zeta^{(s)} = s_\mu\zeta^{(s)} \quad s_\mu = \pm\frac{1}{2}. \tag{C.25}$$

Now we define $\Gamma := (-i)^{k+1}\Gamma^0\Gamma^1\cdots\Gamma^{2k+1}$, which satisfies the following properties:

1. $\Gamma^2 = 1$.

2. $\{\Gamma,\ \Gamma^\mu\} = 0$.

3. $[\Gamma,\ [\Gamma^\mu,\ \Gamma^\nu]] = 0$.

Using the equation (C.24), we rewrite $\Gamma$ as

$$\Gamma = 2^{k+1}S_0S_1\cdots S_k, \tag{C.26}$$

so $\Gamma$ takes eigenvalues $\pm 1$ and

$$\Gamma\zeta^{(s)} = +\zeta^{(s)} \Longleftrightarrow \text{even \# of } s_\mu = -\frac{1}{2}, \tag{C.27}$$

$$\Gamma\zeta^{(s)} = -\zeta^{(s)} \Longleftrightarrow \text{odd \# of } s_\mu = -\frac{1}{2}. \tag{C.28}$$

Therefore we are allowed to classify the set $\Delta$ of states $\zeta^{(s)}$ in such a way that

$$\Delta = \Delta^+ \oplus \Delta^-, \tag{C.29}$$

$$\Delta^+ = \{\zeta^{(s)} : \Gamma\zeta^{(s)} = +\zeta^{(s)}\}, \tag{C.30}$$

$$\Delta^- = \{\zeta^{(s)} : \Gamma\zeta^{(s)} = -\zeta^{(s)}\}. \tag{C.31}$$

Each of $\Delta^{\pm}$ is $2^k$ dimensional irreducible representation of $SO(D)$, called a **Weyl representation**. Irreducibility follows from

$$\begin{aligned}\Gamma\Sigma^{\mu\nu}\zeta^{(s)} &= \Sigma^{\mu\nu}\Gamma\zeta^{(s)} \\ &= \Sigma^{\mu\nu}\zeta^{(s)}\end{aligned} \tag{C.32}$$

for $\zeta^{(s)} \in \Delta^+$.

We call a spinor in $\Delta^+$ **Weyl spinor**. The Weyl spinors exist only in even dimensions by definition.

■ **Odd dimension** $D = 2k + 3$

We add a matrix $\Gamma = (-i)^{k+1}\Gamma^0\Gamma^1 \cdots \Gamma^{2k+1}$ to the gamma matrices $\{\Gamma^\mu\}_{\mu=0,\cdots,2k+1}$ in $2k+2$ dimension. The properties 1. and 2. of $\Gamma$ show that $\Gamma$ is compatible with $\Gamma^\mu$, so the $2^{k+1}$ dimensional Dirac representation of the gamma matrices in $D = 2k + 3$ dimension is formed by $\Gamma^0, \cdots, \Gamma^{D-1}, \Gamma$. This Dirac representation is now irreducible because of the property 3..

### C.1.2 Majorana spinor

■ **Even dimension** $D = 2k + 2$

In order to consider Majorana spinors, we introduce so called $B$-matrices $B_i$ $(i = 1,\ 2)$

$$B_1 = \Gamma^1\Gamma^3 \cdots \Gamma^{2k+1}, \tag{C.33}$$

$$B_2 = \Gamma B_1. \tag{C.34}$$

Those $2^{k+1} \times 2^{k+1}$ matrices satisfy

$$B_i^\dagger B_i = 1_{2^{(k+1)}}, \tag{C.35}$$

$$B_1\Gamma^\mu B_1^\dagger = (-1)^{k+1}\Gamma^{\mu*}, \tag{C.36}$$

$$B_2\Gamma^\mu B_2^\dagger = (-1)^{k+2}\Gamma^{\mu*}, \tag{C.37}$$

$$B_i\Sigma^{\mu\nu}B_i^\dagger = -\frac{i}{4}[\Gamma^{\mu*},\ \Gamma^{\nu*}] = -\Sigma^{\mu\nu*}. \tag{C.38}$$

where $\Gamma^{\mu *}$ are complex conjugate of $\Gamma^{\mu}$. To prove the above relations one should notice that non zero matrix elements of each of $\Gamma^3, \Gamma^5, \cdots, \Gamma^{2k+1}$ are pure complex.

We define the **charge conjugation** $\zeta_i^c$ of $\zeta$ by

$$\zeta_i^c = B_i^{\dagger}\zeta^* \tag{C.39}$$

Then $\zeta_i^c$ is also a Dirac spinor since $\zeta_i^c$ transformed in the same way of $\zeta$, namely

$$\delta\zeta = i\omega_{\mu\nu}\Sigma^{\mu\nu}\zeta \tag{C.40}$$

$$\Longrightarrow \delta\zeta^* = -i\omega_{\mu\nu}\Sigma^{\mu\nu *}\zeta^* = i\omega_{\mu\nu}B_i\Sigma^{\mu\nu}B_i^{\dagger}\zeta^* \tag{C.41}$$

$$\Longrightarrow \delta\zeta_i^c = i\omega_{\mu\nu}\Sigma^{\mu\nu}\zeta_i^c. \tag{C.42}$$

We call $\zeta$ a **Majorana spinor** if $\zeta$ satisfies

$$\zeta_i^c = \zeta \quad (\Longleftrightarrow \ \zeta^* = B_i\zeta). \tag{C.43}$$

This condition gives a constraints on $B_i$ as $B_i^* B_i = 1$ since $\zeta = B_i^*\zeta^* = B_i^* B_i\zeta$. On one hand

$$B_1^* B_1 = (-1)^{k+1}\Gamma^1\Gamma^3\cdots\Gamma^{2k+1}\Gamma^1\Gamma^3\cdots\Gamma^{2k+1} \tag{C.44}$$

$$= (-1)^{(k+1)(k+2)/2}, \tag{C.45}$$

here $(-1)^k$ comes out because $\Gamma^1,\ \Gamma^3, \cdots, \Gamma^{2k+1}$ are pure complex, and on the other hand

$$B_2^* B_2 = \Gamma^* B_1^*\Gamma B_1 \tag{C.46}$$

$$= (-1)^{(k+1)}\Gamma^*\Gamma B_1^* B_1 \tag{C.47}$$

$$= (-1)^{(k+1)} B_1^* B_1 \tag{C.48}$$

$$= (-1)^{(k+1)(k+4)/2}. \tag{C.49}$$

Hence the Majorana spinor conditions result in

$$\zeta_1^c = \zeta \Longleftrightarrow k = 2,\ 3 \bmod 4 \quad (D = 0,\ 6 \bmod 8), \tag{C.50}$$

$$\zeta_2^c = \zeta \Longleftrightarrow k = 0,\ 3 \bmod 4 \quad (D = 0,\ 2 \bmod 8). \tag{C.51}$$

In short a Majorana spinor exists in $D = 2,\ 6,\ 8,\ 10$ as for even dimensions. Next we force the Weyl condition $\Gamma\zeta = \zeta$ on a Majorana spinor.

$$B_1 \Gamma B_1^\dagger = B_2 \Gamma B_2^\dagger \tag{C.52}$$

$$= (-1)^{k+1}\Gamma. \tag{C.53}$$

The first equation follows by the definition of $B_2$ and the fast one follows since $\Gamma$ is diagonal and real matrix. Operating $B_i \Gamma B_i^\dagger$ to $\zeta^*$ by use of the formula

$$\zeta^* = (\Gamma\zeta)^* = \Gamma^* \zeta^* = \Gamma \zeta^*, \tag{C.54}$$

we observe the Majorana-Weyl conditions lead to

$$B_i \Gamma B_i^\dagger \zeta^* = (-1)^{k+1} \Gamma \zeta^* \tag{C.55}$$

$$\overset{\text{Weyl}}{\Longleftrightarrow} B_i \Gamma \zeta = (-1)^{k+1} \zeta^* \tag{C.56}$$

$$\overset{\text{Majorana}}{\Longleftrightarrow} B_i \zeta = (-1)^{k+1} \zeta^* \tag{C.57}$$

$$\overset{\text{Majorana}}{\Longleftrightarrow} \zeta^* = (-1)^{k+1} \zeta^*. \tag{C.58}$$

Hence a Majorana-Wely spinor exists only for $k = 3 \bmod 4$ ($D = 0 \bmod 8$).

■ **Odd dimension** $D = 2k + 3$

Recall that gamma matrices in odd dimension consist of $\Gamma^0, \cdots, \Gamma^{2k-1}, \Gamma$ and they satisfy the relations

$$B_1 \Gamma^\mu B_1^\dagger = (-1)^{k+1} \Gamma^{\mu *}, \tag{C.59}$$

$$B_2 \Gamma^\mu B_2^\dagger = (-1)^{k+2} \Gamma^{\mu *}, \tag{C.60}$$

$$B_1 \Gamma B_1^\dagger = B_2 \Gamma B_2^\dagger = (-1)^{k+1} \Gamma^*. \tag{C.61}$$

In this case $\Gamma^\mu$ and $\Gamma$ should transform in the same way under a Lorentz transformation, the consistent Majorana condition is given by $B_1$. Therefore we can find a Majorana spinor only in $k = 2,\ 3 \bmod 4$ ($D = 7,\ 9 \bmod 8$). Note that there are no Majorana-Weyl spinors since Weyl spinors live in even dimensions.

Here is a table for $SO(D)$ spinor. Each number in the box shows the degrees of freedom of spinors (we count 1 for a Majorana spinor in D=2) and empty boxes show there are no spinors.

| $D$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|
| Dirac | 2 | 2 | 4 | 4 | 8 | 8 | 16 | 16 | 32 |
| Majorana | 1 | | | | 4 | 4 | 8 | 8 | 16 |
| Weyl | 1 | | 2 | | 4 | | 8 | | 16 |
| Majorana-Weyl | | | | | | | 4 | | |

## C.2 Spin representations for $SO(D-t,t)$

### C.2.1 Majorana spinors in general

In this subsection we consider spinor representations for a more general Lorentz group $SO(D-t,t)$, where we denote space and time dimensions by $D-t$ and $t$ respectively. We extend our Clifford algebra to

$$\{\Gamma^M, \Gamma^N\} = 2\eta^{MN}, \tag{C.62}$$

where

$$\eta^{MN} = \mathrm{diag}\{\overbrace{-1,\cdots,-1}^{t},\overbrace{1,\cdots,1}^{D-t}\}. \tag{C.63}$$

We first consider even space-time dimension $D = 2k+2$. We work with convenient choices of combinations as we did in (C.2) with replacing $\Gamma^\mu$ to $i\Gamma^\mu$for the time directions $\mu$:

(1) $t = 2n$ (even)

$$\Gamma^{\mu\pm} = \frac{1}{2}(i\Gamma^{2\mu} \mp \Gamma^{2\mu+1}) \quad (\mu = 0,\cdots,n-1), \tag{C.64}$$

$$\Gamma^{a\pm} = \frac{1}{2}(\Gamma^{2a} \pm i\Gamma^{2a+1}) \quad (a = 2n,\cdots,k), \tag{C.65}$$

(2) $t = 2n+1$ (odd)

$$\Gamma^{\mu\pm} = \frac{1}{2}(i\Gamma^{2\mu} \mp \Gamma^{2\mu+1}) \quad (\mu = 0, \cdots, n-1), \tag{C.66}$$

$$\Gamma^{\mu\pm} = \frac{1}{2}(i\Gamma^{2n} \pm i\Gamma^{2n+1}), \tag{C.67}$$

$$\Gamma^{a\pm} = \frac{1}{2}(\Gamma^{2a} \pm i\Gamma^{2a+1}) \quad (a = 2n, \cdots, k), \tag{C.68}$$

where $\mu$ indicates the time directions and $a$ the space directions. Then they satisfy

$$\{\Gamma^{M+}, \Gamma^{N-}\} = \delta^{MN}, \tag{C.69}$$

$$\text{the others} = 0. \tag{C.70}$$

Let $\Gamma^M_E$ be representations of Euclidian Clifford algebra, namely

$$\{\Gamma^M_E, \Gamma^N_E\} = \delta^{MN}. \tag{C.71}$$

Then our gamma matrices $\Gamma^M$ are in relations with $\Gamma^M_E$ in such a way that

$$\Gamma^\mu = i\Gamma^\mu_E, \tag{C.72}$$

$$\Gamma^a = \Gamma^a_E. \tag{C.73}$$

Indeed, for $t = 2$ we introduce the spinor $\zeta$ which is to be annihilated by any $\Gamma^{M-}$ and use the formula $\Gamma^0 = -i(\Gamma^{0+} - \Gamma^{0-})$ and $\Gamma^1 = \Gamma^{0-} - \Gamma^{0+}$ then

$$\Gamma^0 \begin{pmatrix} \zeta \\ \Gamma^{0+}\zeta \end{pmatrix} = \begin{pmatrix} -i\Gamma^{0+}\zeta \\ -i\Gamma^{0-}\Gamma^{0+}\zeta \end{pmatrix} = \begin{pmatrix} -i\Gamma^{0+}\zeta \\ -i\zeta \end{pmatrix} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \begin{pmatrix} \zeta \\ \Gamma^{0+}\zeta \end{pmatrix}, \tag{C.74}$$

and

$$\Gamma^1 \begin{pmatrix} \zeta \\ \Gamma^{0+}\zeta \end{pmatrix} = \begin{pmatrix} -\Gamma^{0+}\zeta \\ \Gamma^{0-}\Gamma^{0+}\zeta \end{pmatrix} = \begin{pmatrix} -\Gamma^{0+}\zeta \\ \zeta \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \zeta \\ \Gamma^{0+}\zeta \end{pmatrix}. \tag{C.75}$$

So we obtain the matrix representations

$$\Gamma^0 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = -i\Gamma^0_E, \quad \Gamma^1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -i\Gamma^1_E. \tag{C.76}$$

To find $B_i$ ($i=1,2$), we review that $\Gamma_E^{2M}$ consist of real components and $\Gamma_E^{2M+1}$ consist of pure complex components. Therefore in our case, $\Gamma^{2\mu}$ are complex and $\Gamma^{2\mu+1}$ are real, and $\Gamma^{2a}$ are real and $\Gamma^{2a+1}$ are complex. So we should deal with two cases (1) $t$ is even and (2) $t$ is odd.

(1) $t = 2n$

We define $B_1$ and $B_2$ by

$$B_1 := \overbrace{\Gamma^0\Gamma^2\cdots\Gamma^{2n-2}}^{n}\overbrace{\Gamma^{2n+1}\Gamma^{2n+3}\cdots\Gamma^{2k+1}}^{k-n+1}, \tag{C.77}$$

$$B_2 := \Gamma B_1, \tag{C.78}$$

where $\Gamma = (-i)^{k+1}\Gamma^0\Gamma^1\cdots\Gamma^{2k+1}$. Then $\Gamma^* = \Gamma$ and for $i=1,2$

$$B_i^* = (-1)^{k+1}B_i, \tag{C.79}$$

$$^t\!B_i = (-1)^{-n+(k+1)(k+2)/2}B_i, \tag{C.80}$$

$$B_i^2 = (-1)^{n+k(k+1)/2}, \tag{C.81}$$

$$B_i^\dagger B_i = (-1)^{(k+1)(k+2)} = 1, \tag{C.82}$$

$$B_1^* B_1 = (-1)^{\frac{(k+1)(k+2)}{2}+n}, \tag{C.83}$$

$$B_2^* B_2 = (-1)^{k+1}B_1^*B_1 = (-1)^{\frac{k(k+1)}{2}+n}. \tag{C.84}$$

(2) $t = 2n+1$

We define $B_1$ and $B_2$ by

$$B_1 := \overbrace{\Gamma^0\Gamma^2\cdots\Gamma^{2n}}^{n+1}\overbrace{\Gamma^{2n+1}\Gamma^{2n+3}\cdots\Gamma^{2k+1}}^{k-n+1}, \tag{C.85}$$

$$B_2 := \Gamma B_1. \tag{C.86}$$

where $\Gamma = (-i)^{k+2}\Gamma^0\Gamma^1\cdots\Gamma^{2k+1}$. Then $\Gamma^* = \Gamma$ and

$$B_i^* = (-1)^{k+2}B_i, \tag{C.87}$$

$$^tB_i = (-1)^{-n+k(k+1)/2}B_i, \tag{C.88}$$

$$B_i^2 = (-1)^{n+1+(k+1)(k+2)/2}, \tag{C.89}$$

$$B_i^\dagger B_i = (-1)^{(k+1)(k+2)} = 1, \tag{C.90}$$

$$B_1^* B_1 = (-1)^{n+1+\frac{(k+2)(k+3)}{2}}, \tag{C.91}$$

$$B_2^* B_2 = (-1)^{k+1}B_1^* B_1 = (-1)^{n+1+\frac{k(k+3)}{2}}. \tag{C.92}$$

## ■ Odd dimension $D = 2k+3$

Next we consider odd space-time dimension $D = 2k+3$. The $B$-matrices for this case are the same we used in $D = 2k+2$.

<u>(3) $t = 2n$</u>

We define $B_1$ and $B_2$ by

$$B_1 := \overbrace{\Gamma^0\Gamma^2\cdots\Gamma^{2n-2}}^{n}\overbrace{\Gamma^{2n+1}\Gamma^{2n+3}\cdots\Gamma^{2k+1}}^{k-n+1}, \tag{C.93}$$

$$B_2 := \Gamma B_1, \tag{C.94}$$

where $\Gamma = (-i)^{k+1}\Gamma^0\Gamma^1\cdots\Gamma^{2k+2}$.

<u>(4) $t = 2n+1$</u>

We define $B_1$ and $B_2$ by

$$B_1 := \overbrace{\Gamma^0\Gamma^2\cdots\Gamma^{2n}}^{n+1}\overbrace{\Gamma^{2n+1}\Gamma^{2n+3}\cdots\Gamma^{2k+1}}^{k-n+1}, \tag{C.95}$$

$$B_2 := \Gamma B_1. \tag{C.96}$$

where $\Gamma = (-i)^{k+2}\Gamma^0\Gamma^2\cdots\Gamma^{2k+1}$.

Here is the complete table for pairs $(D,t)$ compatible with the Majorana spinor conditions $B_i^* B_1 = 1$. For instance $\times\circ$ in a box means that there is at least one Majorana spinor defined by $B_1$, but there are no Majorana spinors defined by $B_2$. A Majorana-Weyl spinor exists if and only if $\circ\circ$ is indicated in a box. Note that especially in odd $D$ dimensions the Majorana spinors are given by only $B_1$ likewise $SO(D)$.

| $t \diagdown D$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | $\circ\times$ | $\times\times$ | $\times\times$ | $\times\times$ | $\times\circ$ | $\times\circ$ | $\circ\circ$ | $\times\circ$ | $\circ\times$ | $\times\times$ | $\times\times$ |
| 1 | $\circ\circ$ | $\times\circ$ | $\circ\times$ | $\times\times$ | $\times\times$ | $\times\times$ | $\times\circ$ | $\times\circ$ | $\circ\circ$ | $\times\circ$ | $\circ\times$ |
| 2 | $\times\circ$ | $\times\circ$ | $\circ\circ$ | $\times\circ$ | $\circ\times$ | $\times\times$ | $\times\times$ | $\times\times$ | $\times\circ$ | $\times\circ$ | $\circ\circ$ |
| 3 | | $\times\times$ | $\times\circ$ | $\times\circ$ | $\circ\circ$ | $\times\circ$ | $\circ\times$ | $\times\times$ | $\times\times$ | $\times\times$ | $\times\circ$ |
| 4 | | | $\times\times$ | $\times\times$ | $\times\circ$ | $\times\circ$ | $\circ\circ$ | $\times\circ$ | $\circ\times$ | $\times\times$ | $\times\times$ |
| 5 | | | | $\times\times$ | $\times\times$ | $\times\times$ | $\times\circ$ | $\times\circ$ | $\circ\circ$ | $\times\circ$ | $\circ\times$ |
| 6 | | | | | $\circ\times$ | $\times\times$ | $\times\times$ | $\times\times$ | $\times\circ$ | $\times\circ$ | $\circ\circ$ |
| 7 | | | | | | $\times\circ$ | $\circ\times$ | $\times\times$ | $\times\times$ | $\times\times$ | $\times\circ$ |
| 8 | | | | | | | $\circ\circ$ | $\times\circ$ | $\circ\times$ | $\times\times$ | $\times\times$ |
| 9 | | | | | | | | $\times\circ$ | $\circ\circ$ | $\times\circ$ | $\circ\times$ |
| 10 | | | | | | | | | $\times\circ$ | $\times\circ$ | $\circ\circ$ |
| 11 | | | | | | | | | | $\times\times$ | $\times\circ$ |
| 12 | | | | | | | | | | | $\times\times$ |

### C.2.2 Charge conjugation matrix

In this subsection we refer to the charge conjugation matrix $C$. Let $\psi$ be a Majorana spinor of $SO(D-t,t)$, where $D$ is $2k+2$ or $2k+3$. We define $\overline{\psi}$ by

$$\overline{\psi} = \psi^\dagger \Gamma^0 \cdots \Gamma^{t-1}. \tag{C.97}$$

The charge conjugation $\zeta^c$ of $\zeta$ is given by $\zeta^c = B^\dagger \zeta$ (C.39) in $SO(D)$ case, and here we also define it in the same way, namely $\psi^* = B\psi^c = B\psi$ and $\psi^\dagger = {}^t(B\psi^c) = {}^t\psi\, {}^tB = {}^t\psi \epsilon B$. $\epsilon$ in the last equality comes from the facts we derived before

$${}^tB_i = \epsilon B_i = \begin{cases} (-1)^{-n+(k+1)(k+2)/2} B_i & (t = 2n) \\ (-1)^{-n+k(k+1)/2} B_i & (t = 2n+1). \end{cases} \tag{C.98}$$

The charge conjugation matrix $C$ is defined as

$$C = \epsilon B_1 \Gamma^0 \cdots \Gamma^{t-1} \tag{C.99}$$

$$= \eta \Gamma^1 \Gamma^3 \cdots \Gamma^{2k+1}, \tag{C.100}$$

$$\eta = \begin{cases} (-1)^{(k+1)(k+2)/2} & (t = 2n) \\ (-1)^{1+k(k+1)/2} & (t = 2n+1) \end{cases} \tag{C.101}$$

Therefore $\overline{\psi}$ is written in the form

$$\overline{\psi} =^t \psi C. \tag{C.102}$$

The gamma matrices are transformed under $C$ as

$$C\Gamma^{2m} = (-1)^{k+1}\Gamma^{2m}C = (-1)^{k+1}\ {}^t\Gamma^{2m}C, \tag{C.103}$$

$$C\Gamma^{2m+1} = (-1)^{k}\Gamma^{2m+1}C = (-1)^{k+1}\ {}^t\Gamma^{2m+1}C. \tag{C.104}$$

Hence $C\Gamma^M C^{-1} = (-1)^{k+1}\,{}^t\Gamma^M$. Moreover the transpose of $C$ is written in terms of $C$ as follows:

$${}^tC = \eta {}^t\Gamma^1 \cdots {}^t\Gamma^{2k+1} \tag{C.105}$$

$$= \eta(-1)^{(k+1)}\Gamma^1 \cdots \Gamma^{2k+1} \tag{C.106}$$

$$= \eta(-1)^{(k+1)+k(k+1)/2}\Gamma^{2k+1} \cdots \Gamma^1. \tag{C.107}$$

$$= (-1)^{(k+1)(k+2)/2}C. \tag{C.108}$$

Thus we obtain the following transformation rule of $\Gamma^M$ under $C$

$$C\Gamma^M = (-1)^{k+1}\ {}^t\Gamma^M C \tag{C.109}$$

$$= (-1)^{k+1}(-1)^{(k+1)(k+2)/2}\ {}^t\Gamma^M\ {}^tC \tag{C.110}$$

$$= (-1)^{k(k+1)/2}\ {}^t(C\Gamma^M), \tag{C.111}$$

which shows

$$C\Gamma^M : \begin{cases} \text{symmetric} & k = 0, 3 \bmod 4 \\ \text{anti-symmetric} & k = 1, 2 \bmod 4. \end{cases} \tag{C.112}$$

We introduce $\Gamma^{\sigma(0)\sigma(1)\cdots\sigma(m)}$ which are antisymmetric products of any combination of $\Gamma^{\sigma(0)}, \Gamma^{\sigma(1)}, \cdots, \Gamma^{\sigma(m)}$. For instance

$$\Gamma^{12} = \frac{1}{2!}(\Gamma^1\Gamma^2 - \Gamma^2\Gamma^1) = \Gamma^1\Gamma^2, \tag{C.113}$$

$$\Gamma^{123} = \frac{1}{3!}(\Gamma^1\Gamma^2\Gamma^3 + \Gamma^3\Gamma^1\Gamma^2 + \Gamma^2\Gamma^3\Gamma^1 - \Gamma^2\Gamma^1\Gamma^3 - \Gamma^3\Gamma^2\Gamma^1 - \Gamma^1\Gamma^3\Gamma^2) \tag{C.114}$$

$$= \Gamma^1\Gamma^2\Gamma^3. \tag{C.115}$$

In general, $\Gamma^{\sigma(0)\sigma(1)\cdots\sigma(m)}$ satisfies the following properties:

$$\Gamma^{\sigma(0)\sigma(1)\cdots\sigma(m)} = 0 \qquad \text{if } {}^{\exists}i, j\ \sigma(i) = \sigma(j), \tag{C.116}$$

$$\Gamma^{\sigma(0)\sigma(1)\cdots\sigma(m)} = \Gamma^{\sigma(0)} \cdots \Gamma^{\sigma(m)} \quad \text{if } {}^{\forall}i, j\ \mu_i \neq \mu_j. \tag{C.117}$$

Therefore $\Gamma^{\sigma(0)\sigma(1)\cdots\sigma(m)}$ undergoes the transformation under charge conjugation $C$ as follows:

$$C\Gamma^{\sigma(0)\sigma(1)\cdots\sigma(m)} = C\Gamma^{\sigma(0)} \cdots \Gamma^{\sigma(m)} \tag{C.118}$$

$$= C\Gamma^{\sigma(0)}C^{-1}C\Gamma^{\sigma(1)}C^{-1} \cdots C\Gamma^{\sigma(m)}C^{-1}C \tag{C.119}$$

$$= (-1)^{(m+1)(k+1)}\ {}^t\Gamma^{\sigma(0)} \cdots\ {}^t\Gamma^{\sigma(m)}C \tag{C.120}$$

$$= (-1)^{(m+1)(k+1)+m(m+1)/2+(k+1)(k+2)/2}\ {}^t\Gamma^{\sigma(m)} \cdots\ {}^t\Gamma^{\sigma(0)}\ {}^tC. \tag{C.121}$$

Hence for the Majorana spinors $\psi, \varphi$, we obtain the transfromation relation

$$\overline{\psi}\Gamma^{\sigma(0)\sigma(1)\cdots\sigma(m)}\varphi = {}^t\psi C\Gamma^{\sigma(0)\sigma(1)\cdots\sigma(m)}\varphi \tag{C.122}$$

$$= {}^t({}^t\psi C\Gamma^{\sigma(0)\sigma(1)\cdots\sigma(m)}\varphi) \tag{C.123}$$

$$= (-1)^{(m+1)(k+1)+m(m+1)/2+(k+1)(k+2)/2+1}\overline{\varphi}\Gamma^{\sigma(0)\sigma(1)\cdots\sigma(m)}\psi, \tag{C.124}$$

where the minus signs in the last second equalities come from exchanging two spinors $\psi, \varphi$.

For examples, in $D = 10$ $(k = 4)$ we have the following formulas:

$$\overline{\psi}\Gamma^{\sigma(0)\sigma(1)\cdots\sigma(m)}\varphi = (-1)^{(m+1)(m+2)/2}\overline{\varphi}\Gamma^{\sigma(0)\sigma(1)\cdots\sigma(m)}\psi, \tag{C.125}$$

$$\overline{\psi}\Gamma^M\psi = 0, \tag{C.126}$$

$$\overline{\psi}\Gamma^{MN}\psi = 0. \tag{C.127}$$

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