Some (3+1) dimensional exact vortex solutions of the extended CP^N Skyrme-Faddeev model

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Motivation

- The low energy (non-perturbative) sector of pure Yang-Mills theory

- Effective classical models (description of some aspects of the low energy regime of the pure YM)
- Integrable sub-models (zero curvature formulation)
- Exact solutions

The Skyrme-Faddeev model

- Proposed by Faddeev [L. D. Faddeev, Princeton Report]

$$S = \int d^4x \left(M^2 (\partial_\mu \vec{n})^2 - \frac{1}{e^2} (\partial_\mu \vec{n} \times \partial_\nu \vec{n})^2 \right)$$

$$\vec{n} = n^a(x) T_a, \qquad \vec{n}^2 = 1, \qquad T_a = \frac{1}{2}\sigma^a$$

  - (3+1) dimensional field theory that can support finite-energy knotted solitons
  - local, Lorentz-invariant action quadratic in time derivatives
  - S^2 target space

- It has been conjectured [L. D. Faddeev, A. J. Niemi, PRL(1999)] that the SF model describes the low-energy pure sector of Yang-Mills theory.

The modified Skyrme-Faddeev model

- Recently became popular some modifications of the original Skyrme-Faddeev model
- The results obtained by H. Gies [Phys. Rev D (2001)] in calculation of the Wilsonian effective action of the $SU(2)$ Yang-Mills theory suggest unavoidable appearance of quartic term
- The extended Skyrme-Faddeev model [L. A. Ferreira, JHEP (2009)]

The Lagrangian

$$\mathcal{L} = M^2 \partial_\mu \vec{n}\, \partial^\mu \vec{n} - \frac{1}{e^2}(\partial_\mu \vec{n} \times \partial_\nu \vec{n})^2 + \frac{\beta}{2}(\partial_\mu \vec{n} \cdot \partial^\mu \vec{n})^2$$

The Ferreira's exact solution

Parametrization

- replace the real unit vector $\vec{n}(t,x)$ by complex scalar $u(t,x)$
- stereographic projection

$$\vec{n} = \frac{1}{1+|u|^2}\left[u+u^*, -i(u-u^*), |u|^2-1\right]$$

The complex scalar field model with CP^1 target space

The Lagrangian

$$\mathcal{L} = \underbrace{4M^2\frac{\partial_\mu u \partial^\mu u^*}{(1+|u^2|)^2}}_{CP^1 term} + \frac{8}{e^2}\left[\frac{(\partial_\mu u)^2(\partial^\mu u^*)^2}{(1+|u^2|)^4} + (\beta e^2 - 1)\frac{(\partial_\mu u \partial^\mu u^*)^2}{(1+|u^2|)^4}\right]$$

The Ferreira's exact solution

- Restrictions
  - integrability condition $\partial_\mu u\, \partial^\mu u = 0 \qquad \rightarrow \qquad F_{\mu\nu} = 0$
  - additional assumption $\beta e^2 = 1$
- The equation of motion $\partial_\mu \partial^\mu u = 0$
- The solution L. A. Ferreira, [JHEP 2009]
  - general solution

$$u = u(x^1 + i\varepsilon_1 x^2, x^0 + \varepsilon_2 x^3) \qquad \varepsilon_j = \pm 1$$

  - finite energy (vortex) solution

$$u(t, \vec{x}) = \underbrace{\left(\frac{\overbrace{x^1 + ix^2}^{z}}{a} \right)^n}_{CP^1 \text{ lump}} \underbrace{\exp\left(ik(\overbrace{x^3 + x^0}^{y_+})\right)}_{\text{wave}}$$

The solution:

- satisfies the equations

$$\partial_\mu \partial^\mu u = 0, \qquad \partial_\mu u \partial^\mu u = 0$$

- is simultaneously a solution of the modified Skyrme-Faddeev model and the CP^1 model (both in 3+1 Minkowski spacetime)
- can be interpreted as a set of vortices located top on each other with waves traveling along them

The energy per unit length

- The presence of waves generates additional contribution to the energy density
- Take the slightly modified solution

$$u(z, y_+) = (z - \delta)^n (z + \delta)^m e^{iky_+}$$

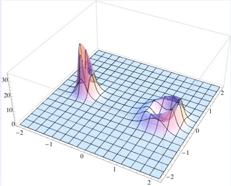
- The energy density per unit length

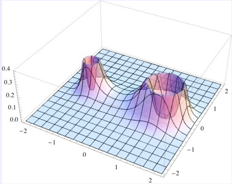
$$\mathcal{H}^{(1)} + \mathcal{H}^{(2)} = 8M^2 \left[\psi(x^1, x^2, \delta) + k^2\right] \underbrace{\frac{|u|^2}{(1 + |u|^2)^2}}_{\text{function of } \delta}$$

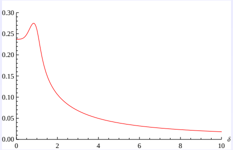
- The energy per unit length

$$\mathcal{E} = 8\pi M^2 \underbrace{K_{(n,m)}}_{\text{integer}} + 8\pi M^2 k^2 \underbrace{I_{(n,m)}}_{\text{function of } \delta}$$

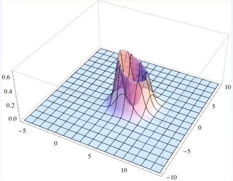
$nm > 0$, $\delta = 1.3$, $(n = 3,\ m = 2)$

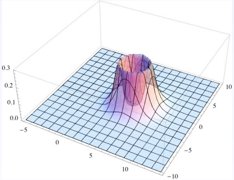


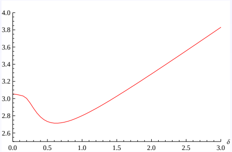




$nm < 0$, $\delta = 5.0$, $(n = 3,\ m = -1)$







$\mathcal{H}^{(1)}$ $\qquad$ $\mathcal{H}^{(2)}$ $\qquad$ $\int \mathcal{H}^{(2)} dx^1 dx^2$

Some comments

[L. A. Ferreira, P. Klimas, W. J. Zakrzewski, Phys Rev D83 (2011)]

- The energy per unit length varies with δ
- The total energy of the system is infinite
- As the result a dependence of the energy per unit length on δ does not lead to the force between vortices

Some more complex vortex solutions of the CP^N models: [L. A. Ferreira, P. Klimas, W. J. Zakrzewski, Phys Rev D84 (2011)]

Generalization to CP^N target space

- The mSF model on a CP^N target space $CP^N = \frac{SU(N+1)}{SU(N)\otimes U(1)}$

The Lagrangian

$$\mathcal{L} = \underbrace{-\frac{M^2}{2}\mathrm{Tr}(P_\mu^2)}_{CP^N\, term} + \frac{1}{e^2}\mathrm{Tr}([P_\mu, P_\nu])^2 + \frac{\beta}{2}\left[\mathrm{Tr}(P_\mu^2)\right]^2 + \underbrace{\gamma\left[\mathrm{Tr}(P_\mu P_\nu)\right]^2}_{\text{new term for } N=2,3,\ldots}$$

[L. A Ferreira, P. Klimas, JHEP 2010]

- Parametrization of P_μ in terms of complex scalar fields $u_i(x^0, x^1, x^2, x^3)$

$$P_\mu = \frac{2i}{1+u^\dagger\cdot u}\begin{pmatrix} 0_{N\times N} & \Delta\cdot\partial_\mu u \\ \partial_\mu u^\dagger\cdot\Delta & 0 \end{pmatrix} \qquad \text{where} \qquad u = \begin{pmatrix} u_1 \\ \vdots \\ u_N \end{pmatrix}$$

$$\Delta_{ij} = \sqrt{1+u^\dagger\cdot u}\,\delta_{ij} - \frac{u_i u_j^*}{1+\sqrt{1+u^\dagger\cdot u}}$$

- The most interesting case from the physical point of view is the case $N=2$

Equations of motion

Equation of motion

$$(1+u^\dagger\cdot u)\partial^\mu(C_{\mu\nu}\partial^\nu u_j) - C_{\mu\nu}\left[(u^\dagger\cdot\partial^\mu u)\partial^\nu u_j + (u^\dagger\cdot\partial^\nu u)\partial^\mu u_j\right] = 0$$

$$\begin{aligned}
C_{\mu\nu} &\equiv M^2\eta_{\mu\nu} - \frac{4}{e^2}\left[(\beta e^2-1)\tau^\rho_\rho\eta_{\mu\nu} + (\gamma e^2-1)\tau_{\mu\nu} + (\gamma e^2+2)\tau_{\nu\mu}\right] \\
\tau_{\mu\nu} &= -\frac{4}{(1+u^\dagger\cdot u)^2}\partial_\nu u^\dagger\cdot\Delta^2\cdot\partial_\mu u \\
\Delta^2_{ij} &= \Delta_{ik}\Delta_{kj} = (1+u^\dagger\cdot u)\delta_{ij} - u_i u_j^*
\end{aligned}$$

When $C_{\mu\nu} = M^2\eta_{\mu\nu}$: EOM $\rightarrow$ CP^N equation

Solutions

1. The integrable sector $\rightarrow$ $\partial_\mu u_i \partial^\mu u_j = 0, \qquad i,j = 1,\ldots,N$
   - infinitely many conserved currents
     $\partial^\mu J^G_\mu = 0$ where $G = G(u_i, u_j^*)$
2. Additional assumption: $\beta e^2 + \gamma e^2 = 2$ gives reduced EOM

$$\partial_\mu \partial^\mu u_i = 0$$

3. The exact solutions are configurations of the form

$$u_i = u_i(z, y_\pm), \qquad u_i^* = u_i^*(\bar{z}, y_\pm)$$

   where $z = x^1 + ix^2$ and $y_\pm = x^3 \pm x^0$
4. The class of solution is very large
   - we concentrate on a vortex solution (finite energy per unit length)

The reduced Hamiltonian

The Hamiltonian of the extended Skyrme-Faddeev model

$$\mathcal{H}_c = 8M^2(\underbrace{\mathcal{H}^{(1)}}_{CP^N\ term} + \mathcal{H}^{(2)}) + 64(\gamma - \beta)\mathcal{H}^{(3)}$$

$$\mathcal{H}^{(1)} \equiv \frac{\partial_{\bar{z}} u^\dagger \cdot \Delta^2 \cdot \partial_z u}{(1 + u^\dagger \cdot u)^2}, \qquad \mathcal{H}^{(2)} \equiv \frac{\partial_{y_+} u^\dagger \cdot \Delta^2 \cdot \partial_{y_+} u}{(1 + u^\dagger \cdot u)^2}$$

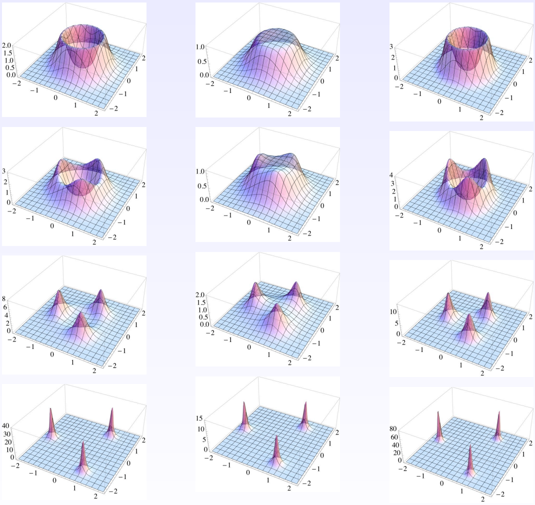
where $\Delta^2_{ij} = (1 + u^\dagger \cdot u)\delta_{ij} - u_i u_j^*$

$$\mathcal{H}^{(3)} \equiv \frac{1}{(1 + u^\dagger \cdot u)^4} \sum_{i,j,k,l=1}^{N} \Delta^2_{ij} \Delta^2_{kl} B^*_{ik} B_{jl},$$

$B_{jl} \equiv \underbrace{(\partial_z u_j \partial_{y_+} u_l - \partial_z u_l \partial_{y_+} u_j)}_{0\ for\ CP^1}$ and $B^*_{ik} \equiv (\partial_{\bar{z}} u_i^* \partial_{y_+} u_k^* - \partial_{\bar{z}} u_k^* \partial_{y_+} u_i^*)$

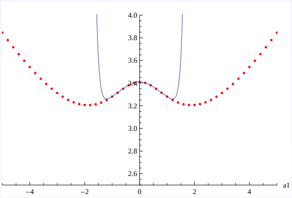
The energy density examples

$\mathcal{H}^{(1)}$ $\mathcal{H}^{(2)}$ $\mathcal{H}^{(3)}$



$$
\begin{aligned}
u_1 &= z^3 + a_1 e^{ik_1 y_+} \\
u_2 &= a_3 e^{ik_2 y_+}
\end{aligned}
$$

energy per unit length $\mathcal{E}(a_1)$



The special case

In the case $N = 2$ when

- $u_1(z, y_+)$
- $u_2(y_+) = a\, e^{ik_2 y_+}$

the Noether charge $\mathcal{Q}^{(2)}$ related with a transformation $u_k \to e^{i\alpha_k} u_k$ satisfies the following relation

$$\mathcal{Q}^{(2)} = \frac{k_2 a^2}{(1+a^2)^2} Q_{\text{Top}}$$

where

$$\frac{1}{\pi} \int dx^1 dx^2 \mathcal{H}^{(3)} = k_1 \underbrace{\mathcal{Q}^{(1)}}_{0} + k_2 \mathcal{Q}^{(2)}$$

The $\mathcal{H}_3$ term behaves exactly like a topological term !

Summary

- The extended Skyrme-Faddeev model has exact solutions
- Interpretation of the solutions: parallel vortices with waves traveling along them
- Nontrivial role of waves - additional contribution to the energy density
- The interesting dependence of energy per unit length on the mutual distance between vortices
- Holomorphic and anti-holomorphic solutions are simultaneously solutions of the CP^N and extended SF model

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