カシミア効果の繰り込み群アプローチと物質中の電磁場の幾何学的取り扱い[2]

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2012 年 8 月 23 日
京都大学基礎物理学研究所・研究会「熱場の量子論とその応用」
湯川記念館 Panasonic 国際交流ホール

[2] arXiv:1203.2708, Proc. of Int. Tribology Conf.2011Hiroshima; arXiv:1004.2573

1. Introduction(1)

Casimir force , Casimir-Polder force , Van der Waals force ,
Forces caused by fluctuation due to microdynamics

- (nearly) free theory is involved. Main part is independent of the coupling(s)
- depends on the boundary parameters and the topology, macro property
- quantum effect (zero-point oscillation), micro property
- highly-delicate regularization is required, IR and UV, Summation formula
- Ambiguity of energy origin.

1. Introduction (2)

Electro-Magnetic field in substance

$$\mathbf{D} = \varepsilon(\omega)\mathbf{E} \quad , \quad \mathbf{B} = \mu(\omega)\mathbf{H} \quad . \tag{1}$$

dielectric function (permittivity) ε, magnetic permeability μ: shows the substance property effectively. Can we approach it geometrically ?

2. Maxwell Equation in Substance(1)

Electric and Magnetic Field

$$\text{Upper-index Fields} \ : \ \hat{\mathbf{D}}(t,\mathbf{x}) = (\hat{D}^i(x)) \quad , \quad \hat{\mathbf{B}}(t,\mathbf{x}) = (\hat{B}^i(x)),$$
$$\text{Lower-index Fields} \ : \ \hat{\mathbf{E}}(t,\mathbf{x}) = (\hat{E}_i(x)) \quad , \quad \hat{\mathbf{H}}(t,\mathbf{x}) = (\hat{H}_i(x)),$$
$$i = 1,2,3 \quad , \quad \mathbf{x} = (x,y,z) \quad , \quad (x^\mu) = (t,\mathbf{x}) \quad , \quad \mu = 0,1,2,3 \qquad (2)$$

Dielectric function, Magnetic permeability (general form)

$$\hat{D}^i(x) = \hat{\varepsilon}^{ij}(x)\hat{E}_j(x) \quad , \quad \hat{B}^i(x) = \hat{\mu}^{ij}(x)\hat{H}_j(x) \quad . \qquad (3)$$

$$\text{div}\hat{\mathbf{D}} = \partial_i \hat{D}^i = 0 \quad \text{electric charge density} = 0$$
$$\text{div}\hat{\mathbf{B}} = \partial_i \hat{B}^i = 0 \quad \text{magnetic charge density} = 0 \quad . \qquad (4)$$

2. Maxwell Equation in Substance(2)

Ampére's Law

$$\partial_t \hat{D}^i - \epsilon^{ijk}\partial_j \hat{H}_k = 0 \quad \text{or} \quad \partial_t \hat{\mathbf{D}} - \nabla \times \hat{\mathbf{H}} = 0$$
$$(\text{electric current density} = 0) \qquad (5)$$

Faraday's Law

$$\partial_t \hat{B}^i + \epsilon^{ijk}\partial_j \hat{E}_k = 0 \quad \text{or} \quad \partial_t \hat{\mathbf{B}} + \nabla \times \hat{\mathbf{E}} = 0 \quad . \qquad (6)$$

Faraday's law is solved by the vector and scalar potentials.

$$\hat{\mathbf{E}} \ , \ \hat{\mathbf{B}} \ \rightarrow \ \hat{\mathbf{A}} \ , \ \hat{\phi}$$
$$\hat{E}_i(\mathbf{x},t) = -\partial_t \hat{A}_i(\mathbf{x},t) - \partial_i \hat{\phi}(\mathbf{x},t)$$
$$\hat{B}^i(\mathbf{x},t) = \epsilon^{ijk}\partial_j \hat{A}_k(\mathbf{x},t) \quad . \qquad (7)$$

2. Maxwell Equation in Substance(3)

time t *to* frequency ω (Fourier expansion)

$$\hat{\mathbf{D}}(t,\mathbf{x})=\int_{-\infty}^{\infty}\mathbf{D}(\omega,\mathbf{x})\mathrm{e}^{i\omega t}d\omega \quad , \quad \hat{\mathbf{E}}(t,\mathbf{x})=\int_{-\infty}^{\infty}\mathbf{E}(\omega,\mathbf{x})\mathrm{e}^{i\omega t}d\omega$$

$$\hat{\mathbf{B}}(t,\mathbf{x})=\int_{-\infty}^{\infty}\mathbf{B}(\omega,\mathbf{x})\mathrm{e}^{i\omega t}d\omega \quad , \quad \hat{\mathbf{H}}(t,\mathbf{x})=\int_{-\infty}^{\infty}\mathbf{H}(\omega,\mathbf{x})\mathrm{e}^{i\omega t}d\omega \quad . \quad (8)$$

$$D^i(\omega,\mathbf{x})=\varepsilon^{ij}(\omega)E_j(\omega,\mathbf{x}) \quad , \quad B^i(\omega,\mathbf{x})=\mu^{ij}(\omega)H_j(\omega,\mathbf{x}) \quad . \qquad (9)$$

$$\hat{\mathbf{A}}(t,\mathbf{x})=\int_{-\infty}^{\infty}\mathbf{A}(\omega,\mathbf{x})\mathrm{e}^{i\omega t}d\omega \ , \ \hat{\phi}(t,\mathbf{x})=\int_{-\infty}^{\infty}\phi(\omega,\mathbf{x})\mathrm{e}^{i\omega t}d\omega$$

$$E_i(\omega,\mathbf{x})=-i\omega A_i(\omega,\mathbf{x})-\partial_i\phi(\omega,\mathbf{x}) \text{ or } \mathbf{E}(\omega,\mathbf{x})=-i\omega\mathbf{A}(\omega,\mathbf{x})-\nabla\phi(\omega,\mathbf{x})$$

$$B^i(\omega,\mathbf{x})=\epsilon^{ijk}\partial_jA_k(\omega,\mathbf{x}) \quad \text{or} \quad \mathbf{B}(\omega,\mathbf{x})=\nabla\times\mathbf{A}(\omega,\mathbf{x})$$

2. Maxwell Equation in Substance(4)

$\mathbf{E}(\omega, \mathbf{x})$ and $\mathbf{B}(\omega, \mathbf{x})$ are unchanged under the gauge transformation.

$$A_i \quad \to \quad A_i + \partial_i \Lambda \quad , \quad \phi \quad \to \quad \phi - i\omega\Lambda \quad \text{where } \Lambda = \Lambda(\omega, \mathbf{x}) \quad . \quad (11)$$

For simplicity, we consider the diagonal case.

$$\varepsilon^{ij} = \varepsilon(\omega)\delta^{ij} \quad , \quad (\mu^{-1})_{ij} = \mu^{-1}(\omega)\delta_{ij} \quad . \qquad (12)$$

2. Maxwell Equation in Substance(5)

Gauge 1. $\partial_i\{i\omega\phi + (\varepsilon\mu)^{-1}\mathrm{div}\mathbf{A}\} = 0$
Ampére's law gives the field eq. of $\mathbf{A}$

$$(\Delta + \omega^2\varepsilon\mu)\mathbf{A}(\omega, \mathbf{x}) = 0 \quad , \quad \mathbf{E} = -i\omega\mathbf{A} - \frac{i}{\omega\varepsilon\mu}\nabla(\mathrm{div}\mathbf{A}) \quad . \qquad (13)$$

Note: When ε and μ are constants, $\hat{\mathbf{A}}(t, \mathbf{x})$ satisfies the free wave equation with velocity of light $v = \frac{1}{\sqrt{\varepsilon\mu}}$.
Gauge 2. $\partial_i\phi = 0$ (Landau-Lifshitz textbook)
The fields eq. of $\mathbf{A}$

$$\Delta\mathbf{A} - \nabla(\mathrm{div}\mathbf{A}) + \omega^2\varepsilon\mu\mathbf{A} = 0 \quad ,$$
$$\mathbf{E} = -i\omega\mathbf{A} \quad . \qquad (14)$$

3. Geometry in $(K^\mu)=(\omega,K^i)$ Space(1)

The energy of the substance is given by

$$\int d^3x \int d\omega\ \mathcal{E} = \int d^3x \int d\omega \frac{1}{2}(\varepsilon^{ij}E_iE_j + \mu^{-1}{}_{ij}B^iB^j)$$
$$= \int d^3x \int d\omega \frac{1}{2}\mu^{-1}\mathbf{A}\cdot(\Delta + \omega^2\varepsilon\mu)\mathbf{A} \quad . \qquad (15)$$

In terms of $(\omega, \mathbf{k})$,

$$\int d^3x \int d\omega\ \mathcal{E} = \int d^3k \int d\omega\ \bar{\mathcal{E}} =$$
$$\int d^3k \int d\omega \frac{1}{2}(\varepsilon^{ij}(\omega,\mathbf{k})\bar{E}_i(\omega,\mathbf{k})\bar{E}_j(\omega,\mathbf{k}) + \mu^{-1}(\omega,\mathbf{k})_{ij}\bar{B}^i(\omega,\mathbf{k})\bar{B}^j(\omega,\mathbf{k})$$
$$= \int d^3k \int d\omega \frac{1}{2}\mu^{-1}(\omega,\mathbf{k})\bar{\mathbf{A}}(\omega,\mathbf{k})\cdot(-\mathbf{k}^2 + \omega^2\varepsilon(\omega,\mathbf{k})\mu(\omega,\mathbf{k}))\bar{\mathbf{A}}(\omega,\mathbf{k})$$

3. Geometry in (K^{μ})=(ω,K^{i}) Space(2)

We propose

$$\varepsilon^{ij}(\omega,\mathbf{k}) = e_1 g^{ij}(\omega,\mathbf{k}), \quad \mu(\omega,\mathbf{k})^{ij} = m_1 g^{ij}(\omega,\mathbf{k}), \tag{17}$$

where e_1 and m_1 are constants and $g_{ij}(\omega,\mathbf{k})$ is some metric introduced next.
Under the coordinate transformation:

$$k^i \quad \to \quad k^{i'} \qquad (k^i)^2 = (p(\omega))^2,\ (k^{i'})^2 = (p'(\omega))^2 \tag{18}$$

the metric transforms as

$$g_{ij}{}' = \frac{\partial k^l}{\partial k^{i'}}\frac{\partial k^m}{\partial k^{j'}} g_{lm} \quad , \quad g^{ij'} = \frac{\partial k^{i'}}{\partial k^l}\frac{\partial k^{j'}}{\partial k^m} g^{lm} \quad . \tag{19}$$

3. Geometry in (K^μ)=(ω,K^i) Space(3)

General invariance of the scalar quantity $\bar{\mathcal{E}}$, (16), requires the following transformation of **E** and **B**.

$$\bar{E}_i = \frac{\partial k^{j'}}{\partial k^i}\bar{E}'_j \quad , \quad \bar{B}_i = \frac{\partial k^{j'}}{\partial k^i}\bar{B}'_j \quad . \qquad (20)$$

It says the change of the hyper-surface produces the change of fields **E** and **B**.
We would like to define the energy as

$$E = \int d[\text{hyper-surface (23)}]\sqrt{\det g_{ij}(\omega,\mathbf{k})}\bar{\mathcal{E}}(\omega,\mathbf{k}) \quad . \qquad (21)$$

3. Geometry in (K^μ)=(ω,K^i) Space(4)

Let us consider three typical metrics in (ω, K^i) space.

1. Minkowski : $ds^2 = -d\omega^2 + \sum_{i=1}^{3} dK^{i2}$
2. dS_4 : $ds^2 = -d\omega^2 + \mathrm{e}^{2H\omega} \sum_{i=1}^{3} dK^{i2}$
3. AdS_4 : $ds^2 = (dK^3)^2 + \mathrm{e}^{-2H|K^3|}(-d\omega^2 + (dK^1)^2 + (dK^2)^2)$ (22)

To specify 3D metric $g_{ij}(\omega)$, introduce 3D *hypersurface*.

$$\text{Dispersion relation} \quad : \quad (k^i)^2 \;=\; p(\omega)^2 \quad , \tag{23}$$

The induced metric g_{ij} is given by

$$g_{ij}(\omega) = \delta_{ij} - \frac{k^i k^j}{(p\dot{p})^2} \quad . \tag{24}$$

3. Geometry in (K^μ)=(ω, K^i) Space(5)

some examples.

$$g_{ij}(\omega, \mathbf{k}) = \begin{cases} \delta_{ij} - \frac{c^4}{\omega^2} k^i_{(1)} k^j_{(1)}, & p_1(\omega) = \frac{\omega}{c},\ \dot{p}_1 = \frac{1}{c},\ (k^i_{(1)})^2 = \omega^2/c^2 \\ \delta_{ij} - \frac{c^4}{\omega^2} k^i_{(2)} k^j_{(2)}, & p_2(\omega) = \frac{\sqrt{\omega^2 - (mc^2)^2}}{c},\ \dot{p}_2 = \frac{\omega}{c^2} \frac{1}{p_2},\ (k^i_{(2)})^2 = (\omega^2 - (mc^2)^2)/c^2 \end{cases}$$

where c(light velocity) and m(mass) are some constants.

3. Geometry in (K^μ)=(ω,K^i) Space(6)

The proposal of the energy (21) is severely divergent (IR and UV). We regularize the divergence by introducing the damping factor from the geometric point of view. The area A of the hyper-surface (23) $\{p(\omega) : 0 \leq \omega \leq T\}$ is

$$A[p(\omega)] = \int \sqrt{\det g_{ij}} d^3k = \int_0^T \sqrt{\dot{p}^2+1}\ p^2 d\omega \quad . \tag{26}$$

The following damping (regularization) factor or the distribution $\Omega[p(\omega)]$ is regarded as the statistical ensemble due to the micro fluctuation.

$$\Omega[p(\omega)] = \frac{1}{N}\exp(-\frac{1}{2\alpha'}A[p(\omega)])$$
$$= \frac{1}{N}\exp\{-\frac{1}{2\alpha'}\int_0^T \sqrt{\dot{p}^2+1}\ p^2 d\omega\} \quad , \tag{27}$$

where a new model parameter α' is introduced.

3. Geometry in $(K^\mu)=(\omega,K^i)$ Space(7)

Hence the system energy is given by

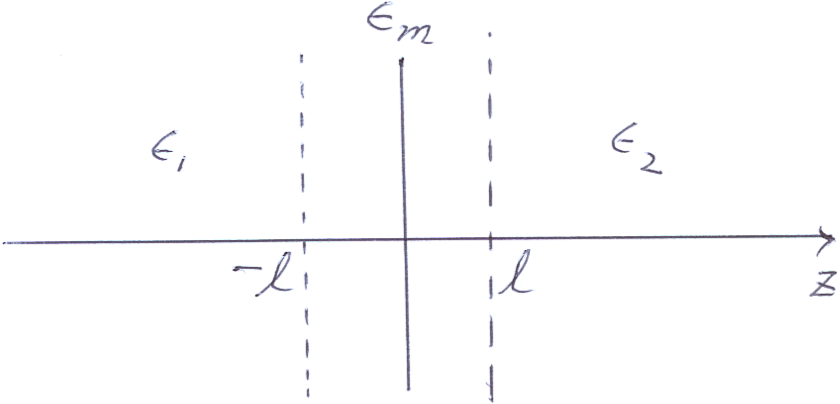
$$E(T) = \frac{1}{N}\int_0^\infty d\rho \int_{\substack{p(0)=\rho \\ p(T)=\rho}} \prod_{\omega,i} \mathcal{D}k^i(\omega) \times$$

$$\bar{\mathcal{E}}[\bar{\mathbf{A}}(\omega,\mathbf{k})] \exp\left[-\frac{1}{2\alpha'}\int_0^T \sqrt{\dot{p}^2+1}\, p^2 d\omega\right] ,$$

$$\text{where} \quad \bar{\mathcal{E}}[\bar{\mathbf{A}}(\omega,\mathbf{k})] = \frac{1}{2}\left\{\varepsilon^{ij}(\omega,\mathbf{k})\bar{E}_i(\omega,\mathbf{k})\bar{E}_j(\omega,\mathbf{k}) + \mu^{-1}(\omega,\mathbf{k})_{ij}\bar{B}^i(\omega,\mathbf{k})\bar{B}^j(\omega,\mathbf{k})\right\}. \qquad (28)$$

4. Lifshitz Theory 1954(0)

Figure: Configuration of Substance $\varepsilon_1, \varepsilon_m, \varepsilon_2$.



4. Lifshitz Theory 1954(1)

Free energy formula for the $\varepsilon_1, \varepsilon_m, \varepsilon_2$ $(\mu = 1)$ substance of the structure: Fig.1 .
Simplified model of the previous Maxwell theory Kenneth and Klich '06

$$S = \frac{1}{2}\int d^3x \int \frac{d\omega}{2\pi}\phi^*_\omega(\Delta + \omega^2\varepsilon(\omega))\phi_\omega \ , \ \phi^*_\omega = \phi_{-\omega} \ , \ \varepsilon(\omega) = 1 + \chi(\omega).($$

When $\varepsilon(\omega) = \varepsilon_1$(const.), above is 3+1 massless complex free scalar.

$$S = \frac{1}{2}\int d^3x \int \frac{dt}{2\pi}\hat{\phi}^*(\mathbf{x}, t)(\Delta - \varepsilon_1\frac{\partial^2}{\partial t^2})\hat{\phi}(\mathbf{x}, t), \tag{30}$$

$$\hat{\phi}(\mathbf{x}, t) = \int_{-\infty}^{\infty}\phi_\omega(\mathbf{x})\mathrm{e}^{i\omega t}d\omega. \tag{31}$$

4. Lifshitz Theory 1954(2)

The field equation

$$(\Delta + \omega^2 \varepsilon(\omega))\phi_\omega = 0 \quad , \quad \phi_\omega(\mathbf{x}_\perp, z) = \tilde{\phi}_\omega(z) \mathrm{e}^{i\mathbf{q}\cdot\mathbf{x}_\perp} \quad . \tag{32}$$

This system is in the thermal equilibrium at temperature T.

$$\text{Periodicity} \quad : \quad t \;\rightarrow\; t + \frac{1}{T} \quad ,$$
$$\omega_n = \frac{2\pi T}{\hbar} n \quad . \tag{33}$$

In the plane perpendicular to z-axis,

$$\text{Periodicity} \quad : \quad \mathbf{x}_\perp = (x, y) \;\rightarrow\; (x + L, y + L) \quad ,$$
$$\mathbf{q}_{(n_x, n_y)} = (\frac{2\pi}{L} n_x, \frac{2\pi}{L} n_y) \quad . \tag{34}$$

$$(-\mathbf{q}^2 + \partial_z{}^2 + \omega^2 \varepsilon(\omega))\tilde{\phi}_\omega(z) = 0 \quad . \tag{35}$$

4. Lifshitz Theory 1954(3)

For z-dependence, take

$$\tilde{\phi}_\omega(z) = A(\omega)\mathrm{e}^{\rho z} + B(\omega)\mathrm{e}^{-\rho z} \quad , \quad -\mathbf{q}^2 + {\rho_\alpha}^2 + \omega^2\varepsilon_\alpha(\omega) = 0 \quad (\alpha = 1$$

Wave function for each region

$$\begin{aligned} z < -l \qquad & \tilde{\phi}_\omega(z) = A(\omega)\mathrm{e}^{\rho_1 z} \quad , \quad \text{region } 1 \\ -l < z < l \qquad & \tilde{\phi}_\omega(z) = C_1(\omega)\mathrm{e}^{\rho_m z} + C_2(\omega)\mathrm{e}^{-\rho_m z} \quad , \quad \text{region m} \\ z > l \qquad & \tilde{\phi}_\omega(z) = B(\omega)\mathrm{e}^{-\rho_2 z} \quad \text{region } 2 \end{aligned} \tag{37}$$

$$\Delta = 1 - \frac{(\rho_1 - \rho_m)(\rho_2 - \rho_m)}{(\rho_1 + \rho_m)(\rho_2 + \rho_m)}\mathrm{e}^{-4\rho_m l} = 0 \quad . \tag{38}$$

4. Lifshitz Theory 1954(4)

Add periodicity to z-direction.

$$\text{Periodicity} \quad : \quad z \quad \to \quad z+L \quad ,$$
$$\mathbf{q}_{(n_x,n_y,n_z)} = (\frac{2\pi}{L}n_x, \frac{2\pi}{L}n_y, \frac{2\pi}{L}n_z) \quad . \qquad (39)$$

$$\mathrm{e}^{-F_\chi} = \int \mathcal{D}\phi_\omega \mathcal{D}\phi^*_\omega \mathrm{e}^{iS[\phi^*,\phi;\chi_1,\chi_m]}$$
$$= \det(\Delta + \omega^2\epsilon_\alpha(\omega)) = \exp \mathrm{Tr} \ln \ (\Delta + \omega^2(1+\chi_\alpha(\omega)) \ ) \quad , \qquad (40)$$

$\epsilon_\alpha = 1 + \chi_\alpha$ is given by

$$\text{In R}_1 \quad 1+\chi_1(\omega) = \frac{1}{\omega^2}(\frac{2\pi}{L})^2(n_x^2+n_y^2+n_z^2) \quad ,$$
$$\text{In R}_m \quad 1+\chi_m(\omega) = \frac{1}{\omega^2}(\frac{2\pi}{L})^2(n_x^2+n_y^2+n_z^2) \quad , \qquad (41)$$

4. Lifshitz Theory 1954(5)

REGULARIZATION 1

$$F_C \equiv F_\chi - F_{\chi=0} = -\mathrm{Tr}\ \ln(1 + \frac{\omega^2 \chi_\alpha(\omega)}{\Delta + \omega^2}) \quad . \tag{42}$$

REGULARIZATION 2 (Entanglement)

$$F \equiv F_C(R_1 \cup R_m) - F_C(R_1) - F_C(R_m) \quad . \tag{43}$$

Finally, well-defined (finite) quantity   Kenneth and Klich 2008

$$F = -\mathrm{Tr}\ \ln(1 - T_1 G_{1m} T_m G_{m1}) \quad ,$$

$$T_1 = \frac{\omega^2}{1 + \omega^2 \chi_1 G_{11}} \quad , \quad T_m = \frac{\omega^2}{1 + \omega^2 \chi_m G_{mm}} \quad . \tag{44}$$

4. Lifshitz Theory 1954(6)

G and $G_{\alpha\beta}$ is free propagators:

$$G = \begin{pmatrix} G_{11} & G_{1m} \\ G_{m1} & G_{mm} \end{pmatrix} \quad , \quad G(\mathbf{x}, \mathbf{x}') = < \mathbf{x} | \frac{1}{\Delta + \omega^2} | \mathbf{x}' > \quad ,$$

$$(\Delta + \omega^2) G(\mathbf{x}, \mathbf{x}') = < \mathbf{x} | \mathbf{x}' > = \delta(\mathbf{x} - \mathbf{x}') \quad , \quad \mathbf{x}, \mathbf{x}' \in R_1 \cup R_m \quad . \tag{45}$$

5. Final Comment(1)

幾何学（一般相対性理論）が物質科学、特に熱流体、粘弾性流体などの非平衡統計系の解析に非常に大切であることが最近わかってきている。

I. Bredberg, C. Keeler, V. Lysov and A. Strominger, arXiv:1101.2451
"From Navier-Stokes to Einstein"

V. Lysov and A. Strominger, arXiv:1104.5502
"From Petrov-Einstein to Navier-Stokes"

5. Final Comment(2)

Figure: Andromeda