# Complex Langevin simulation applied to a chiral model

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# Complexification approaches to the sign problem

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in collaboration with Kikukawa, Sano, ...

# Outline

- The sign problem
- Old & renewed approach:
  - Complex Langevin simulation
- New approach:
  - HMC simulation on Lefschetz thimble
  - No sign problem?
- Discussion and outlook

# QCD partition fn & sign problem

$$Z(T,\mu) = \mathrm{Tr}\; e^{-\beta(H-\mu N)} = \sum_N e^{-\beta E_N}\, 2\cosh(\beta\mu N)$$

- is real positive, and a function of $\mu^2$

$$Z(T,\mu) = \int dU\, d\psi\, d\bar{\psi}\, e^{-S} = \int dU\; e^{-S_B} \det D$$

- Importance sampling works if "$e^{-S}$ det D" >0
- At finite $\mu$, det $D(\mu)=\det D(-\mu)^*$ is complex: sign problem
- Attempts to overcome the problem
  - Taylor expansion at $\mu=0$, Imaginary $\mu$, Re-weighting,
  - Density of states, ...

# Old & new complexification approaches to the sign problem

- Complex Langevin simulation

- Simulation on the Lefschetz thimble

# Langevin dynamics

- Statistical sampling w/o explicit weight fn
  (the Fokker-Planck eqn is associated)
- Equilibrium state is thermal (Brown motion)

$$\frac{d}{dt}v_i(t) = -\gamma v_i(t) + \eta_i(t) \qquad \langle \eta_i(t)\eta_j(t')\rangle = 2kT\gamma\delta_{ij}\delta(t-t')$$

friction noise

$$\lim_{t\to\infty}\frac{1}{2}\langle v_i(t)v_j(t)\rangle = \frac{1}{2}\delta_{ij}kT$$

# Langevin dynamics

- Statistical sampling w/o explicit weight fn
  (the Fokker-Planck eqn is associated)
- Equilibrium state is the quantum vacuum

Parisi-Wu

$$\frac{\partial \phi(x,\theta)}{\partial \theta} = -\frac{\delta S[\phi]}{\delta \phi(x,\theta)} + \eta(x,\theta) \qquad \langle \eta(x,\theta)\eta(x',\theta')\rangle = 2\delta(x-x')\delta(\theta-\theta')$$

force          noise

# Complex Langevin dynamics

- Langevin algorithm is simple
- No obvious problem with complex action $S$

Parisi-Klauder

$$\frac{\partial\phi(x,\theta)}{\partial\theta} = -\frac{\delta S[\phi]}{\delta\phi(x,\theta)} + \eta(x,\theta) \qquad \langle\eta(x,\theta)\eta(x',\theta')\rangle = 2\delta(x-x')\delta(\theta-\theta')$$

force noise

- Price to pay:
  - Complex "force" $dS/d\phi$ makes $\phi$ also complex
  - True equilibribration is not formally guaranteed

# Results in $\phi$4 theory

Aarts et al.

- $S$ becomes complex at finite $\mu$

$$S = \int d^4x \Big[ \, |\partial_\nu \phi|^2 + (m^2 - \mu^2)|\phi|^2$$

$$+ \mu \left( \phi^* \partial_4 \phi - \partial_4 \phi^* \phi \right) + \lambda |\phi|^4 \Big]$$

# Results in $\phi$4 theory

Aarts et al.

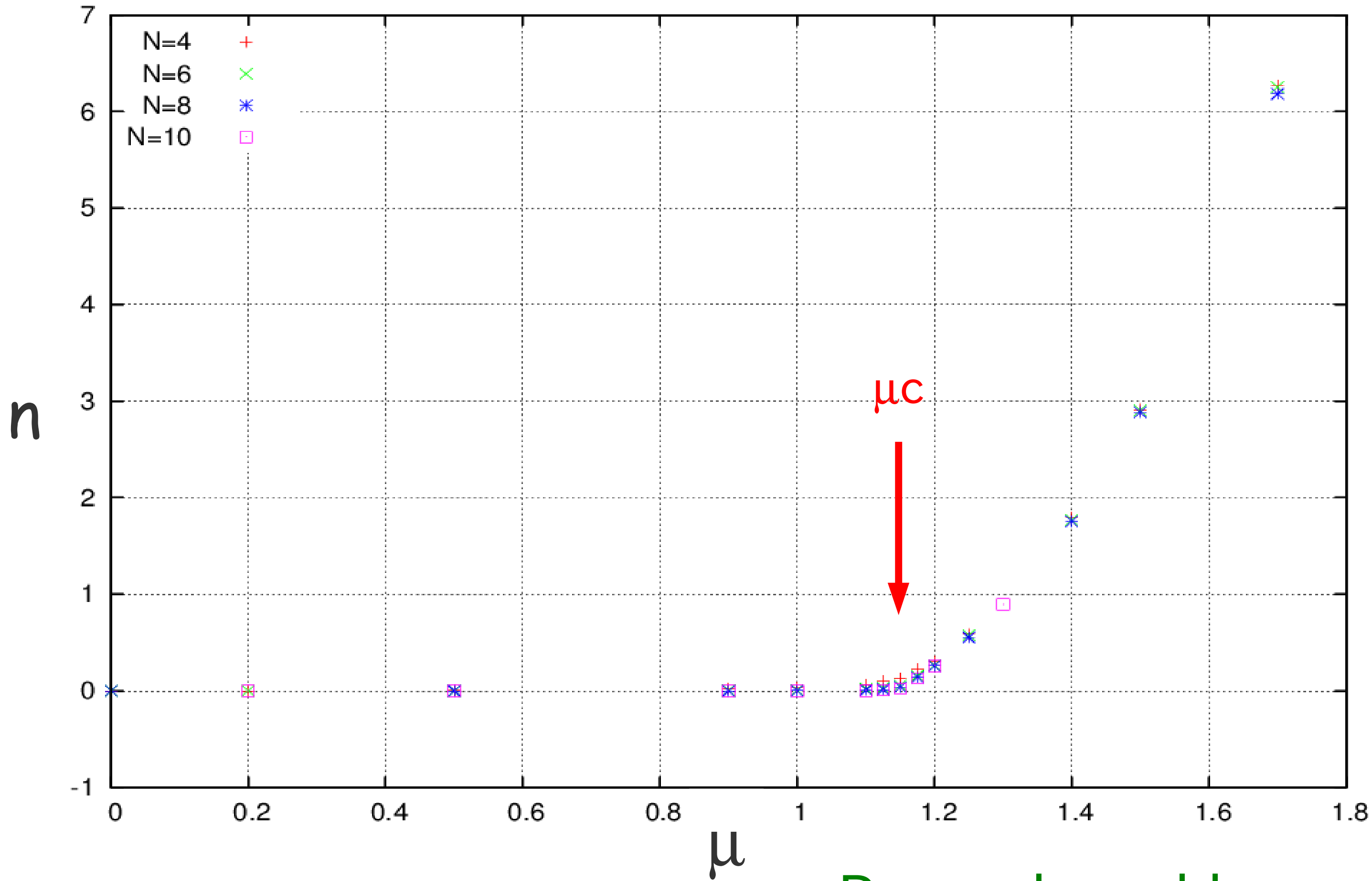
- $S$ becomes complex at finite $\mu$

$$
\begin{aligned}
S = \sum_{x \in \mathbb{L}^n} \Big[ & -\phi_1(x)\left[\phi_1(x+\hat{0})\cosh(\mu) - \phi_2(x+\hat{0})\sinh(\mu) I\right] \\
& -\phi_2(x)\left[\phi_2(x+\hat{0})\cosh(\mu) + \phi_1(x+\hat{0})\sinh(\mu) I\right] \\
& -\sum_{\hat{k}} \left(\phi_1(x)\phi_1(x+\hat{k}) + \phi_2(x)\phi_2(x+\hat{k})\right) \\
& +(D + \frac{1}{2}\kappa)\left(\phi_1(x)\phi_1(x) + \phi_2(x)\phi_2(x)\right) + \frac{1}{4}\lambda\left(\phi_1(x)\phi_1(x) + \phi_2(x)\phi_2(x)\right)^2 \Big]
\end{aligned}
$$

# Results in $\phi 4$ theory

Aarts et al.

- Density n stays zero up to $\mu = \mu c$ ; Silver Blaze




Reproduced by ourselves

# When Complex Langevin works?

- Longstanding problems
  - instability – only numerical? Aarts et al
  - wrong equilibrium – what's the key physics?
- TrLog(D) is unique to fermion theories, whose effects are to be studied
  - Chiral Random Matrix model (see Sano's talk)
  - Nambu-Jona-Lasinio model (in progress)
  - QCD, …, etc.

# Path-integral in complexified phase space

- Complex Langevin = sampling algorithm in complexified phase space

- In complex space, one can choose the integration contour on which ImS=const !
  - Method of steepest descent – deform the path around critical points, dS/dz=0

$$\int_{-\infty}^{\infty} dx\, e^{i\kappa x^2} = \int_C dz\, e^{i\kappa z^2} = e^{i\pi/4} \int_{-\infty}^{\infty} dt\, e^{-\kappa t^2}$$

# Path-integral in complexified phase space

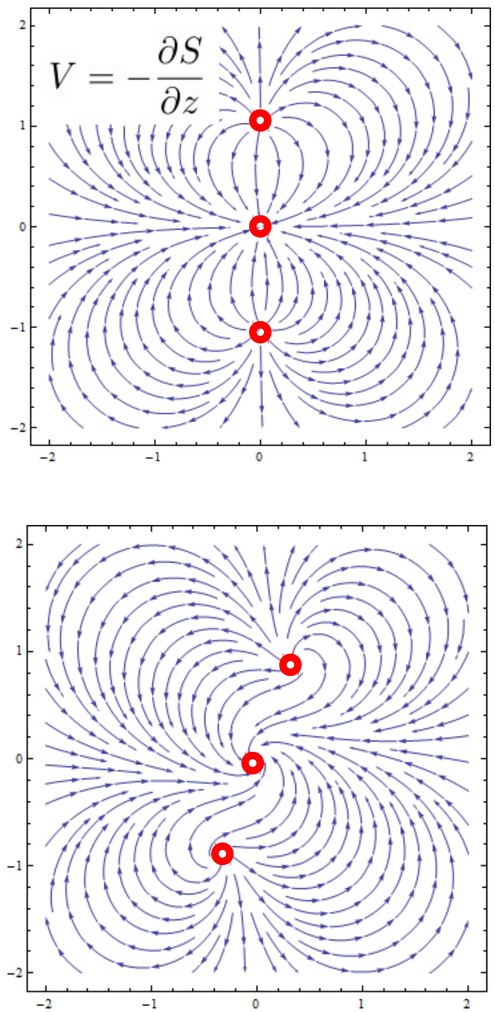
- Illustration $\quad S = \frac{\kappa}{2}x^2 + \frac{\lambda}{4}x^4 \to \frac{\kappa}{2}z^2 + \frac{\lambda}{4}z^4 \qquad \kappa \in \mathbb{C}$

Complex Langevin

Im

κ=1

κ=e^{-iπ/4}



# Path-integral in complexified phase space

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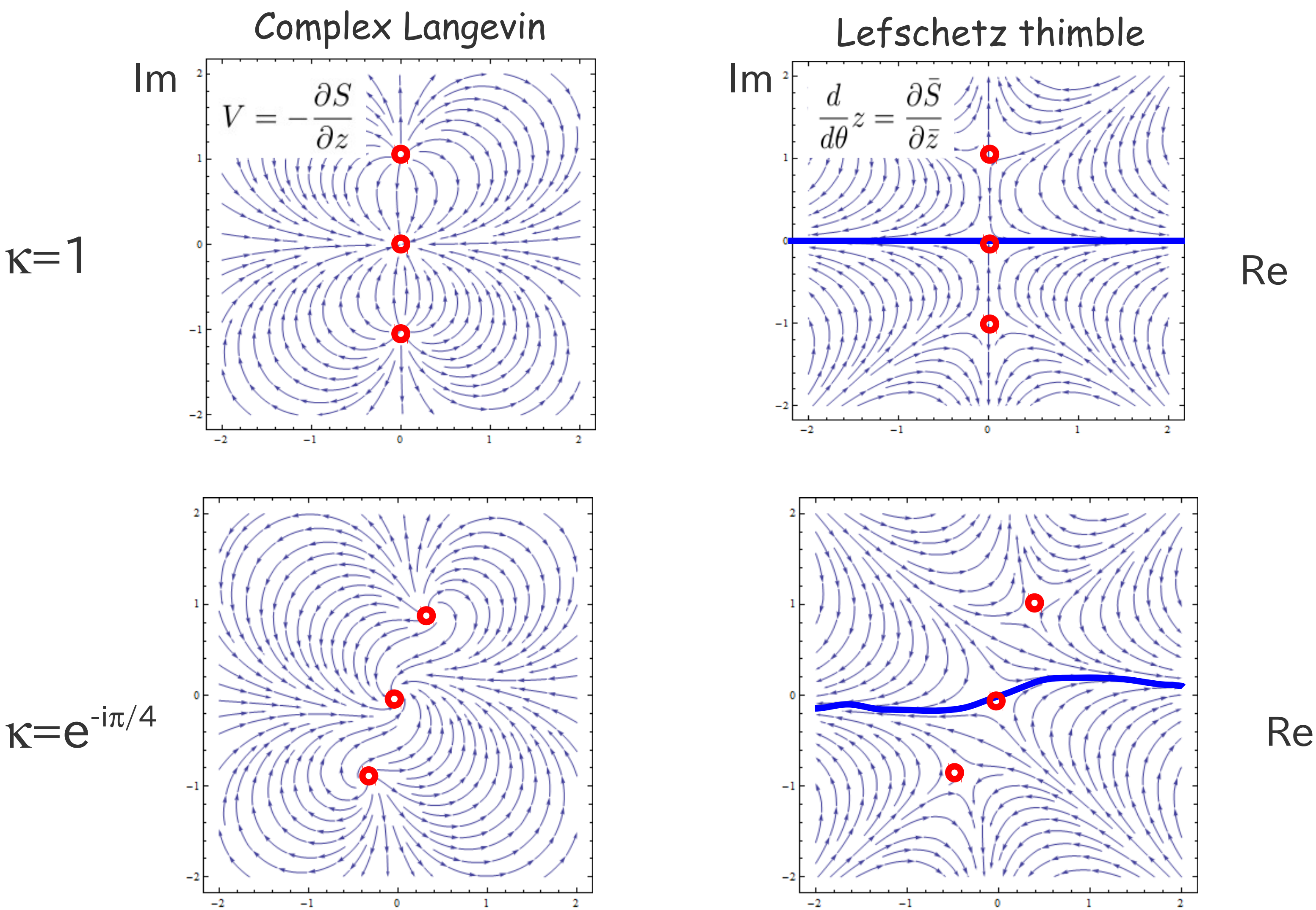
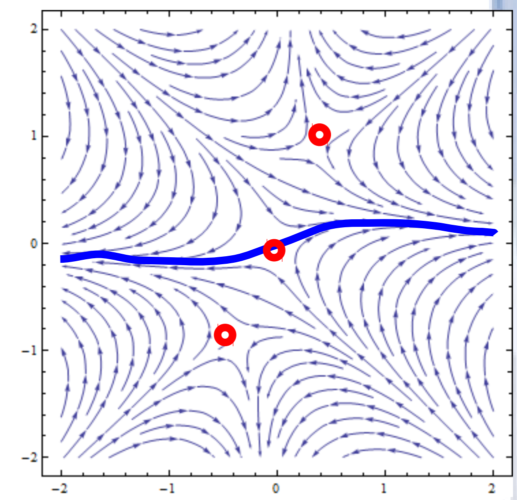


Witten, Honda, ...
Cristforetti et al (Aurora Coll.)

# Path integral on Lefschetz thimble

- Flow eqn: $\frac{d}{dt}z(t) = \frac{\partial \bar{S}[\bar{z}]}{\partial \bar{z}}, \quad \frac{d}{dt}\bar{z}(t) = \frac{\partial S[z]}{\partial z},$
- Along flow: $\mathrm{d(ReS)/dt} > 0 \quad \mathrm{d(ImS)/dt}=0$
- Critical pt.: $\left.\frac{\partial S[z]}{\partial z}\right|_{z=z_\sigma} = 0.$



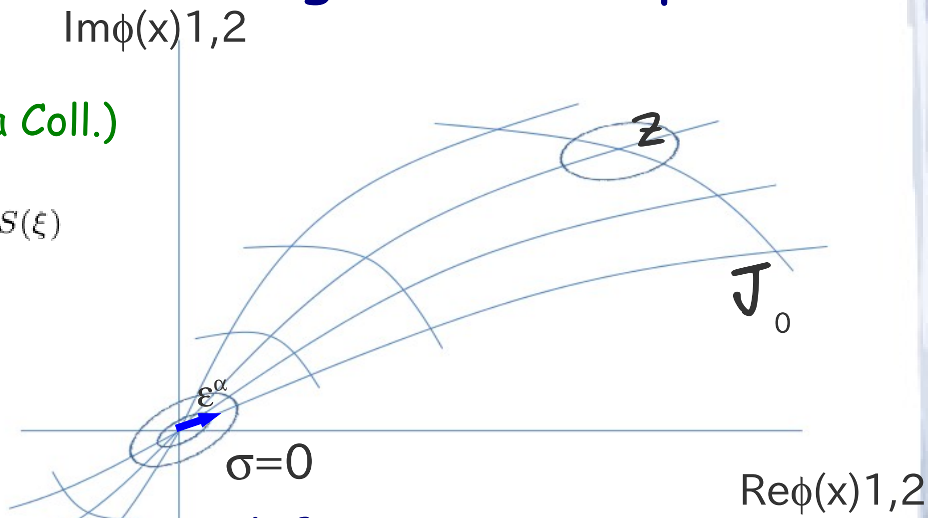

- Lefschetz thimble $\mathbf{J}_\sigma$ : a union of downward flows from σ
- Morse theory shows: $\int_{\mathbb{R}^n} dx_1 \cdots dx_n g(x) e^{f(x)} = \int_{\sum n_\sigma \mathcal{J}_\sigma} g(z) e^{f(z)}$
- $\mathbf{J}_0$ from z=0 is a natural choice for integration contour, on which ImS=0!! – No sign problem!?

# Path integral on Lefschetz thimble

- Coordinates on $J_0$
  - Thimble is $\mathbf{R}^n$ dimensional, the same as the original
  - For any point $z$ on $J_0$, there is a unique flow by $\tau$ starting at $z_0=\varepsilon^\alpha V^a(0)$ near 0: natural coordinates $(\tau,\varepsilon^\alpha)$
  - We need Jacobian det(V(z)), which is in general complex: residual sign problem

Cristforetti et al (Aurora Coll.)

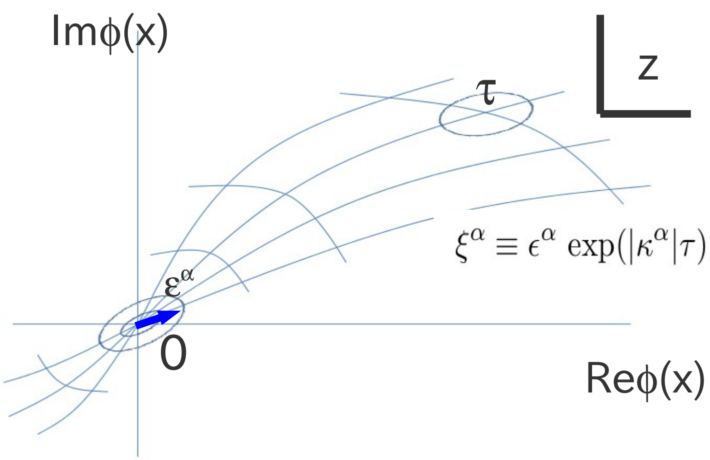
$$\int d\phi\, e^{-S(\phi)} = \int_{\mathcal{J}_0} dz\, e^{-S(z)} = \int d\xi \left|\frac{dz}{d\xi}\right| e^{-S(\xi)}$$




  - {V(z)} needs to be parallel-transported from 0

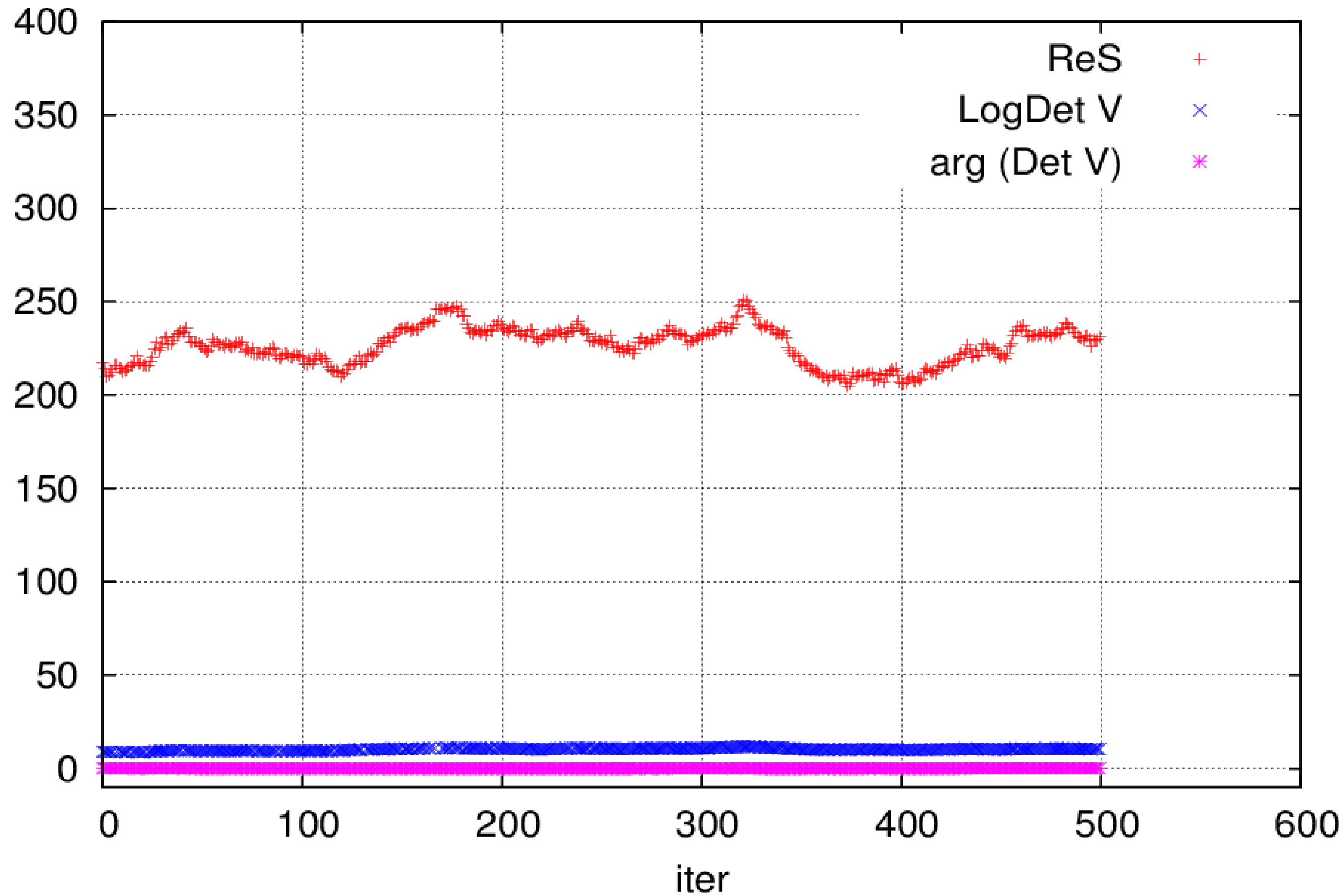
# HMC algorithm on Lefschetz thimble $J_0$

- Pick up an initial pt randomly: $$z = \epsilon^\alpha V_i^\alpha(0) + \int_0^\tau dt\, \bar\partial \bar S[\bar z(t)] \equiv z[\epsilon^\alpha, \tau]$$
- Use coordinates $\xi^\alpha(\varepsilon^\alpha,\tau)$ $$\dot\xi^\alpha = p^\alpha$$ $$\dot p^\alpha = -\frac{\partial S}{\partial \xi^\alpha} = -\frac{1}{2}\left\{\partial_i S[z] V_i^\alpha(\tau) + \bar\partial_i \bar S[\bar z] \bar V_i^\alpha(\tau)\right\} \exp(-|\kappa^\alpha|\tau)$$
- HMC needs ( Field conf "z" & tangent vecs "$V^\alpha$" )
- Prepare ( z , $\{V^\alpha\}$ ) at $(\varepsilon^\alpha,\tau)$ by solving flow eqn (parallel-transport of $\{V^\alpha(0)\}$)
- Jacobian det{V} should also be included
- Repeat HMC towards thermalization



# First trial look at HMC on $J_0$ of $\phi 4$

- $\kappa=\lambda=1$, $\mu=0.3$, N=4
- The HMC code runs!




- Trajectory length=0.08, step size=0.008 (very rough)

# HMC on Lefschetz thimble $J_0$ of $\phi 4$

- Time consuming: several min for one trajectory
  - Core i7 PC w/ C2070 GPU
- Residual sign problem seems numerically almost absent!
  - If exact!?, physical reasoning & proof should be possible (by Honda, Kato, Komatsu + us)
  - Unlike the Fresnel integral, $Z(\mu)$ is real positive and respects Charge Conjugation symmetry

# Discussion and outlook

- Complex Langevin
  - does sampling in enlarged complexified space
  - works beautifully in some cases
  - more realistic cases with fermions, phase transitions, to be examined
- Lefschetz thimble
  - functional version of steepest descent method
  - time consuming to handle thimble geometry
  - But sign problem may be almost absent!