Beyond the ladder analysis of chiral and color symmetry breaking using the non-perturbative renormalization group

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The dynamical chiral symmetry breaking and the color superconductivity in finite density QCD are analyzed using the non-perturbative renormalization group (NPRG) approach. We show that an approximation almost respecting the gauge independence can be realized in the framework of NPRG, and also report the result of analyzing the "ladder approximated" color superconductivity. The non-ladder extended approximation at finite density encounters a singularity of the β function for the 4-fermi coupling constant as long as the normal regulator functions are used.

Introduction.—The dynamical chiral symmetry breaking (DχSB) has been analyzed by non-perturbative approaches such as the lattice simulation, the Schwinger-Dyson (SD) equation and so on. The lattice simulation is a most powerful tool to analyze QCD, but the simulation in the dense QCD is essentially difficult due to its sign problem. On the other hand, the SD approach does not suffer from the sign problem. Unfortunately, however, it is difficult to solve the SD equation beyond the ladder approximation, which has the strong gauge dependence of the physical quantities. Including the corrections of the non-ladder diagrams, which are crucial to recover the gauge independence, is difficult in the SD approach. In contrast to them, the non-perturbative renormalization group (NPRG) approach does not have the sign problem and may include the non-ladder corrections using the systematic approximation.

In this article, we show that in the NPRG approach we can define an approximation almost respecting the gauge independence. (The detailed discussion can been found in [1].) Also we report the result of analyzing the color superconductivity (CS), which is theoretically expected in dense QCD.

Effective action.—In order to evaluate the DχSB and the CS in QCD, we use the Wetterich-type flow equation [2] as a formulation of NPRG. The flow equation is a functional differential equation of the effective action $\Gamma_\Lambda[\Phi]$, which is defined by suppressing the corrections of the quantum fluctuation with the momentum lower than the scale Λ, that is the infrared cutoff. Therefore, solving the flow equation toward the infrared limit, we obtain the full effective action, which is corrected by the full quantum fluctuations without the infrared cutoff, from a bare action of theories such as QCD. Here we skip the details of the flow equation (see reviews [3]).

The Wetterich-type flow equation is an exact equation giving the full effective action, however we cannot exactly solve it. For an approximation, we project the full operator space of the effective action of QCD onto the subspace of the following effective action:

$$\Gamma_\Lambda[\Phi] = \int_x \left\{ \frac{Z_A}{4} F^a_{\mu\nu} F^{\mu\nu}_a + \frac{1}{2\xi} \left(\partial_\mu A_\mu\right)^2 + \bar{\psi}\left(Z_\psi \partial\!\!\!/ + i Z_1 \bar{g}_s A\!\!\!/\right)\psi - V(\psi, \bar{\psi}; \Lambda) \right\}, \tag{1}$$

where the ghost sector is not displayed for simplicity. The operator subspace consists of the operators of the bare QCD action and the multi-Fermi operators $V(\psi, \bar{\psi}; \Lambda)$, which we call the fermion potential. The β function for the gauge coupling constant agrees with the result of the one-loop perturbation theory because we ignore the higher dimensional operators including the gluon fields. Here we concentrate on evaluating the β function for the fermion potential, which plays the most important role for the DχSB. Note that we work with the only fermion operators without relying on the bosonization as adopted in [4–6].

Lowering the cutoff scale, the gauge interaction induces the infinite number of multi-Fermi operators. Especially the 4-Fermi operator, whose β function is diagrammatically represented in Fig. 1, brings about the DχSB at an intermediate scale as the Nambu–Jona-Lasinio model does. All the possible multi-Fermi operators cannot be evaluated, and we extract a class of the multi-Fermi operators relevant to the DχSB. They are the scalar multi-Fermi operators represented by powers of $\sigma(\equiv \bar{\psi}\psi)$, and hence we evaluate the fermion potential as a function of σ, $V(\sigma; t)$.

Furthermore we need to limit the infinite number of interactions to constitute the β function. At first, we select the infinite number of the ladder type diagrams. The ladder diagrams of the the 4-Fermi β function correspond to the ones surrounded by the dashed line in Fig. 1. Then, the ladder-approximated flow equation [4] is given by the following partial differential equation (PDE),

$$\partial_t V(\sigma; t) = -\eta_\psi \sigma \partial_\sigma V + \frac{\Lambda^4}{4\pi^2} \log\left[1 + \frac{1}{\Lambda^2}\left(\partial_\sigma V + \frac{(3+\xi) C_2 \pi \alpha_s}{\Lambda^2}\sigma\right)^2\right], \tag{2}$$

where the ξ is the gauge-fixing parameter of the covariant gauge, C_2 is the second Casimir invariant of the quark representation in $SU(3)$ and $\alpha_s(\equiv Z_1^2 \bar{g}_s^2 / 4\pi Z_\psi^2 Z_A)$ is the running gauge coupling constant which obeys the one-loop perturbation theory. We introduce the infrared cutoff which stops running of the gauge coupling constant in order to take into account of the confinement [7]. The anomalous dimension of the quark field, η_ψ, is evaluated at $\sigma = 0$ using the momentum scale expansion as the sharp cutoff regulator function [8] is used here. We note that the ladder flow equation (2) gives the result equivalent to the improved ladder SD equation in the Landau gauge [4].

Solving the flow equation as PDE— Usually, in order to solve the flow equation, we expand the equation with respect to polynomials in the field operators, and define the coupled ordinary differential equations for the coupling constants of operators. The coupled equations, namely the RG equations, are numerically solvable, but the RG

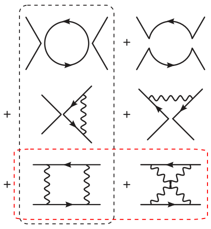

FIG. 1: β function for the 4-Fermi operator.

flow cannot go below a critical infrared scale Λ_c because the flow of the 4-Fermi coupling constant diverges at Λ_c. Actually, the infrared singularity is related to the DχSB by the correspondence between the 4-Fermi coupling constant and the susceptibility of the system, that is, the inverse mass of the composite channel meson.

Here we go beyond the critical scale Λ_c by solving the flow equation as a partial differential equation (PDE) without the bosonization and the field operator expansion. In the practical analysis, using the grid method, we numerically solve the PDE of the mass function $M(\sigma;t) \equiv \partial_\sigma V(\sigma;t)$, which is obtained by differentiating the flow equation (2). As realized by the name of the mass function, its value at $\sigma = 0$ is the effective quark mass.

The mass function is an odd function of σ because the operator subspace has the discrete chiral symmetry where the fermion potential is invariant under the γ_5 transformation: $\sigma \to -\sigma$ ($\psi \to \gamma_5\psi$, $\bar{\psi} \to \bar{\psi}\gamma_5$). Therefore the mass function at $\sigma = 0$ vanishes due to the chiral symmetry as long as it maintains the continuity. The numerical solution of the PDE of the mass function is shown in Fig. 2. The mass function at the ultraviolet scale,

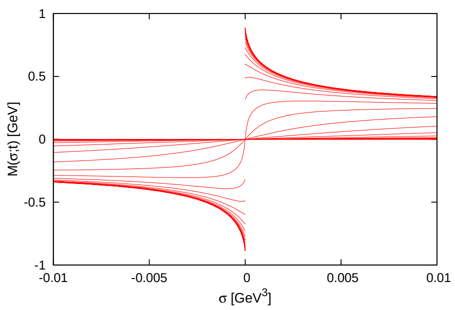

FIG. 2: Revolution of the mass function

namely its initial condition, vanishes since the bare QCD action has no multi-Fermi operators. Lowering the cutoff scale $\Lambda(t)$, the mass function grows up, but its value at the origin keeps vanishing above the critical scale Λ_c due to the chiral symmetry as noted above. At the scale Λ_c the slope of the mass function at the origin diverges. This divergence corresponds to the infrared singularity of the 4-Fermi coupling constant, which is the signal of the DχSB in the fermionic system. Below the critical scale, the function loses the analyticity at the origin , and it has the finite jump around the origin. Actually it is impossible to solve the PDE with such singular point. In the practical numerical computation, we drop the singular point by transforming σ to the logarithmic variable $x = \log \sigma$. A solution allowing singular points can be mathematically authorized as a global solution, which is called the weak solution [9].

Beyond "the ladder"—The ladder approximation suffers from strong gauge dependence of the physical quantities. As for the β function for the 4-Fermi coupling constant, the contributions of the pair of the box and the crossed box diagrams surrounded by a red dashed line in Fig. 1 are crucial for the gauge independence. So we attempt to include such paired contributions of non-ladder diagrams at all orders. For this purpose, we introduce the corrected vertex, which consists of the ladder element and the crossed ladder element as shown in Fig. 3 [5]. Here the ingoing (outgoing) external line denotes a quark (antiquark) field, and the internal quark line denotes the dressed propagator, which consists of the infinite number of the ladder (large-N leading) interactions of the multi-Fermi operators as shown in Fig. 4.

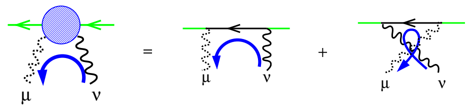

FIG. 3: Corrected vertex.

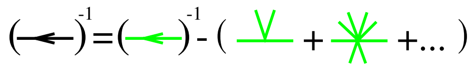

FIG. 4: Dressed inverse propagator of the fermion.

Now the non-ladder extended flow equation for the fermion potential are represented by the infinite number of the ladder form diagrams using the corrected vertex, and can be summed up by the logarithmic functions as follows,

$$\begin{aligned}\partial_t V(\sigma;t) =& \frac{\Lambda^4}{4\pi^2}\log\left[1+\frac{B^2}{\Lambda^2}\right] + \frac{\Lambda^4}{4\pi^2}\log\left[1+\xi\frac{MG}{\Lambda^2+M^2}\right] \\ &+ \frac{\Lambda^4}{8\pi^2}\log\left[\frac{\Lambda^2+B^2}{\Lambda^2+M^2}+\frac{3\Lambda^2G^2}{(\Lambda^2+M^2)^2}\right], \qquad (3)\end{aligned}$$

where $B = M + 2\Lambda^{-2}C_2\alpha_s\sigma$ and $G = 2\Lambda^{-2}C_2\alpha_s\sigma$. Here the commutator contributions of the generator of the $SU(3)_c$ is ignored, that is, the interactions are evaluated by Abelian factors only.

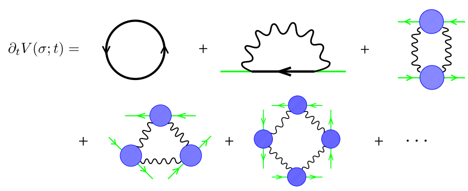

FIG. 5: Non-ladder extended flow equation

Numerical results—Evaluating the PDEs (2) or (3), we obtain the dynamical mass, defined by $m_{\rm dyn.} = \lim_{\sigma\to+0}\lim_{t\to\infty} M(\sigma;t)$, the chiral order parameter. Moreover, we also obtain the chiral condensates $\langle\bar{\psi}\psi\rangle_{\rm 1GeV}$ (renormalized at 1 GeV) by introducing the bare mass term as its source term.

Figs. 6 and 7 show the gauge dependence of the chiral condensates and the dynamical mass, respectively. These results show that the chiral condensates obtained by the non-ladder flow equation (3) are much more stable against the gauge-fixing parameter than the ones obtained by the ladder approximated flow equation (2).

However the dynamical mass looks different. Indeed the dynamical mass is an off-shell quantity that is non-observable, although the chiral condensates are the observable. Therefore we claim that the non-ladder extended flow equation respects the gauge invariance for the physically observable quantities. Here it should be noted that, in the Landau gauge ($\xi = 0$), the almost gauge independent non-ladder result of the chiral condensates *coincides with* the gauge dependent ladder one.

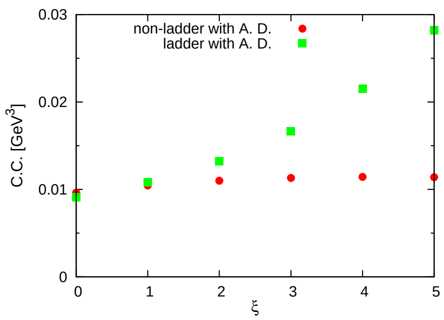

FIG. 6: Gauge dependence of the chiral condensates $\langle\bar{\psi}\psi\rangle_{1\mathrm{GeV}}$ (C.C.).

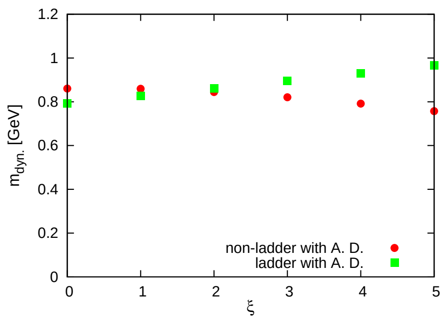

FIG. 7: Gauge dependence of the dynamical mass ($m_{\mathrm{dyn.}}$).

Color superconductivity—In extremely dense matters such as the interior of compact stars, the color superconductivity (CS) is theoretically predicted, where the color antisymmetric diquark pair behaves like the Cooper pair in the BCS theory due to its attractive channel of exchanging gluons. In the rest of this article, we show the result of analyzing the two flavor color superconductivity (2SC) with the massless quarks. So as to evaluate the 2SC in our framework, we extract the diquark-type scalar operator, $\Delta \equiv \bar{\psi}^C i\sigma_2\lambda_2\gamma_5\psi + \bar{\psi}i\sigma_2\lambda_2\gamma_5\psi^C$, where σ_2 and λ_2 are the antisymmetric elements of the Pauli matrices in the flavor space and the Gell-Mann matrices in the color space, respectively. Using the ladder approximation, we can write down the flow equation (PDE) of the fermion potential which is a function of two variables, σ and Δ. Fig. 8 shows the dependence on the density μ of the chiral condensates $\langle\bar{\psi}\psi\rangle_{1\mathrm{GeV}}$ and diquark condensates $\left|\langle\bar{\psi}^C i\gamma_5\sigma_2\lambda_2\psi\rangle_{1\mathrm{GeV}}\right|$ calculated by the PDE. Increasing the density, the chiral condensates vanishes at the critical density, $\mu_c = 0.43$ GeV, and the diquark condensates has a finite value at this density μ_c.

The ladder approximated result has strong gauge dependence, and therefore we extend the approximation to the non-ladder one. However the non-ladder β function for the 4-Fermi coupling constant has a singularity around $\mu \sim 0.3$ MeV. This singularity is induced by the non-ladder type diagrams in Fig. 1. The loop integrals of these diagrams has a singular factor, $1/(\Lambda^2 - \mu^2 + m^2)$, as

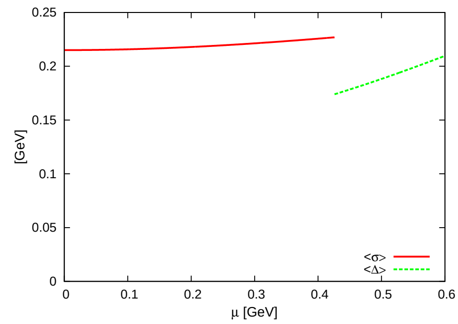

FIG. 8: Phase transition of the chiral symmetry and color superconductivity at finite density. $\langle\sigma\rangle$ and $\langle\Delta\rangle$ denote the chiral condensates and the diquark condensates, respectively.

long as the diquark condensates vanishes. This singularity should be distinguished from the infrared divergence of the 4-Fermi coupling constant induced by its quadratic term in the β function.

We guess the β function singularity of the non-ladder diagrams comes from the fact that the regulator function does not properly evaluate the fluctuations of the modes around the Fermi surface. Therefore, by adopting an appropriate regulator function respecting the Fermi surface structure, we may resolve the singularity problem.

Conclusion—We developed a non-ladder extended flow equation of the fermion potential in QCD, and it gives gauge independent results for the chiral condensates. We also analyzed the finite density QCD and showed the phase transition between the dynamical chiral symmetry breaking and the color superconductivity. Non-ladder extension of the finite density system has a new singularity which requires improvement of the regulator function.

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