# Complexification approaches to the sign problem

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The action functional of QCD becomes complex at finite baryon chemical potential, which prevents the direct application of the importance sampling techenique to evaluation of the partition function — the sign problem in QCD. We briefly discuss two old and new attempts to solve it, complex Langevin simulation and sampling on the Lefschetz thimble.

## I. INTRODUCTION

Direct evaluation of the QCD partition function $Z(T)$ at finite temperature has been successful with importance sampling algorithms in the path integral representation, providing information of the QCD equation of state. At finite baryon chemical potential $\mu$, however, the Dirac operator becomes non-Hermite to make the action functional complex and we encounter the sign problem. One should remember that the partition function $Z(T, \mu)$ itself is still real postive even at finite $\mu$. In this workshop, we discussed two attempts towards the solution of the sign problem, in both of which the configuration space is necessarily complexified.

## II. COMPLEX LANGEVIN SIMULATION

The statistical sampling with the complex Langevin equation has a long history since 80's [1–3]. The algorithm is very simple. One just solves a stochastic evolution equation in a fictitious time $\theta$:

$$\frac{\partial \phi(x, \theta)}{\partial \theta} = -\frac{\delta S[\phi]}{\delta \phi(x, \theta)} + \eta(x, \theta), \tag{1}$$

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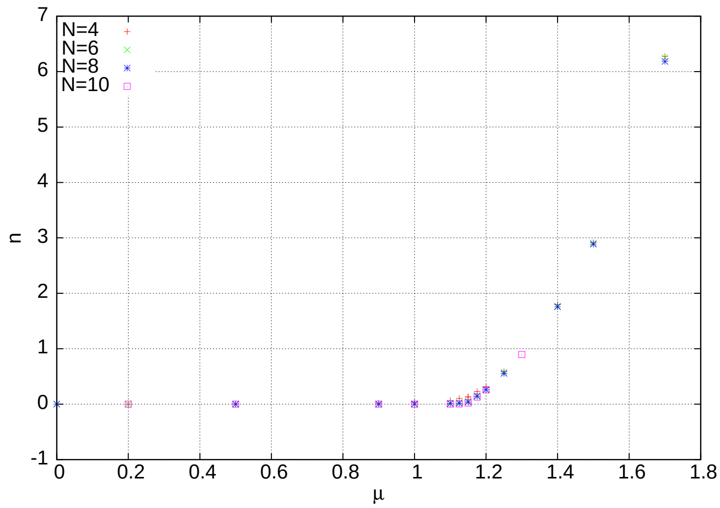


FIG. 1: Number density stays is independent of $\mu$ beign zero as far as $\mu < \mu_c$. The so-called Silver Blaze phenomenon is observed.

where $\phi$ is the scalar field and $\eta$ the Gaußian stochastic field with variance $\langle \eta(x,\theta)\eta(x',\theta')\rangle = 2\delta(x-x')\delta(\theta-\theta')$. The long-time average of an operator using this equation is known to coincide with the state average in equilibrium.

There is no apparent difficulty in generalizing Eq. (1) to the case of a complex action $S$. The price to pay is that, since the force $\delta S/\delta\phi$ is now complex, one needs to complexify as well the fields $\phi$, which was originally real. However, more delicate issue is about equilibration. Is equilibration achieved with the complex action? Is the equilibrium state correct? Hot discussions are revived recently regarding these points. [4, 5].

In the workshop we showed the numerical study of the U(1) $\lambda\phi^4$ theory in Ref. [6], whose action is

$$S = \int d^4x \left[\, |\partial_\nu \phi|^2 + (m^2 - \mu^2)|\phi|^2 + \mu(\phi^* \partial_4 \phi - \partial_4 \phi^* \phi) + \lambda |\phi|^4 \,\right] . \tag{2}$$

The field is $\phi = (\phi_1 + i\phi_2)/\sqrt{2}$ with $\phi_{1,2} \in \mathbb{R}$. The term proportional to the chemical potential $\mu$ makes the action complex, and therefore one needs to complexify the fields as $\phi_{1,2} \in \mathbb{C}$. We show the numerical results on the lattice with size $N^4$ ($N = 4, 6, 8, 10$) in Fig. 1. The constancy of the density (at zero) can be achieved only when the complex phase is properly treated[6]. The complex Langevin simulation seems very successful here. See also Ref. [7].

An application of the complex Langevin simulation to the chiral random matrix model was reported by T. Sano in this workshop.

## III. INTEGRATION ON THE LEFSCHETZ THIMBLE

Once one admits the necessity of complexification of the configuration space, one may recognize the freedom to deform the integration "path" in the functional space to improve the convergence of the integration in the spirit of the steepest descent. A simplest example,

$$\int_{-\infty}^{\infty} dx\, e^{i\kappa x^2} = \int_{\mathcal{C}} dz\, e^{i\kappa z^2} = e^{i\pi/4} \int_{-\infty}^{\infty} dt\, e^{-\kappa t^2} = e^{i\pi/4}\sqrt{\frac{\pi}{\kappa}} \ . \tag{3}$$

The counterpart to the deformed "path" is the Lefschetz thimble in the functional integration [8]. Denoting the complexified fields as $z$, we have the action $S[z]$ holomorphic in $z$. If one considers flows

$$\frac{dz}{dt} = \overline{\frac{\partial S[z]}{\partial z}} \tag{4}$$

from a critical point $\sigma$ where

$$\left.\frac{\partial S[z]}{\partial z}\right|_{\sigma} = 0 \ , \tag{5}$$

one finds that along the flow

$$\frac{\partial \mathrm{Re} S[z]}{\partial t} > 0 \ , \quad \frac{\partial \mathrm{Im} S[z]}{\partial t} = 0 \ . \tag{6}$$

The union of these flows are called Lefschetz thimble $\mathcal{J}_\sigma$ associated to the point $\sigma$. The Morse theory dictates that the functional integration can be represented the integration on the Lefschetz thimbles. Note that the imaginary part of $S[z]$ is constant on a thimble. Especially, since $S[0] = 0$, $\mathrm{Im}S[z] = 0$ on the Lefschetz thimble $\mathcal{J}_0$ associated with the origin $z = 0$. But there is no criterion known so far on the question which thimbles should be chosen as the integration path among several critical points. It seems natural to choose $\mathcal{J}_0$.

It is pointed out, however, that there still remains a residual sign problem steming from the Jacobian factor [9]. If we denote the coordinates on the Lefschetz thimble as $\xi$, then

$$\int d\phi\, e^{-S(\phi)} = \int_{\mathcal{J}_0} dz\, e^{-S(z)} = \int d\xi \left|\frac{dz}{d\xi}\right| e^{-S(\xi)} \ . \tag{7}$$

Even if $S[\xi]$ is kept real positive on the thimble $\mathcal{J}_0$, we have a complex phase of $\left|\frac{dz}{d\xi}\right|$ in general. Note that in contrast to the complex Langevin where the dimension of the configuration space is effectively doubled by complexification, here the dimension of the thimble is kept the same as the original although deformed.

We've just performed a pilot simulation with the hybrid Monte Calro algorithm for the $\lambda\phi^4$ theory, which hints that the residual phase may be very small and which is very encouraging. We are now developing the code to perform the simulation more efficiently. At the same time, analytic study of the sign problem on the Lefschetz thimble is underway in our group.

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