Refined Holographic Entanglement Entropy for the AdS Solitons and AdS black Holes

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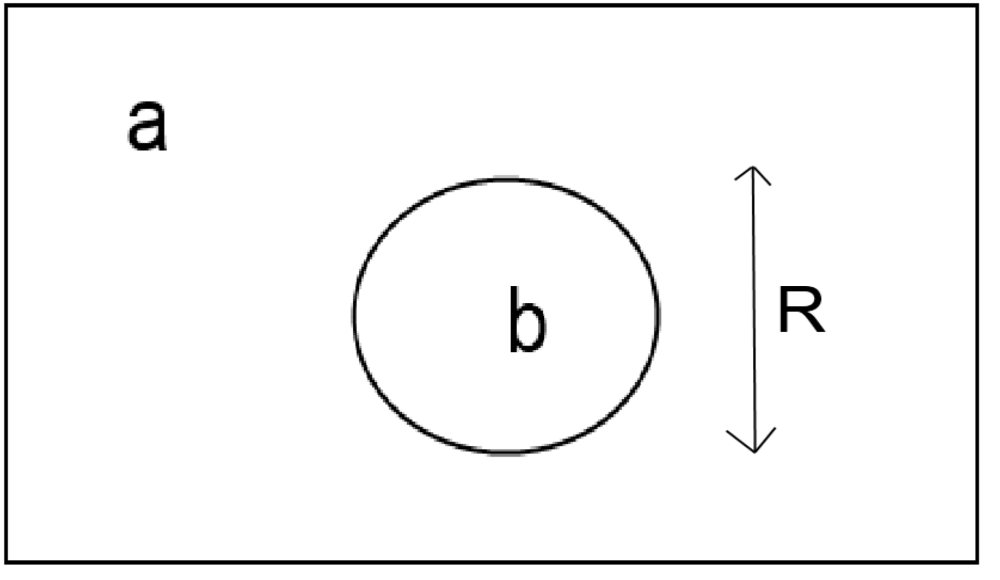
Entanglement Entropy

We divide the total system into two parts; region *a* and region *b*.

Entanglemet Entropy S_a is defined as von Neumann entropy with the reduced density matrix ρ_a which is obtained by tracing out area *"b"* from the total system density matrix ρ_{tot}.

$S_a = -Tr_a(\rho_a \, Log \, \rho_a)$ $\rho_a = Tr_b \rho_{tot}$ $\rho_{tot} = |\Psi\rangle\langle\Psi|$

EE counts the number of correlations between region *a* and region *b*

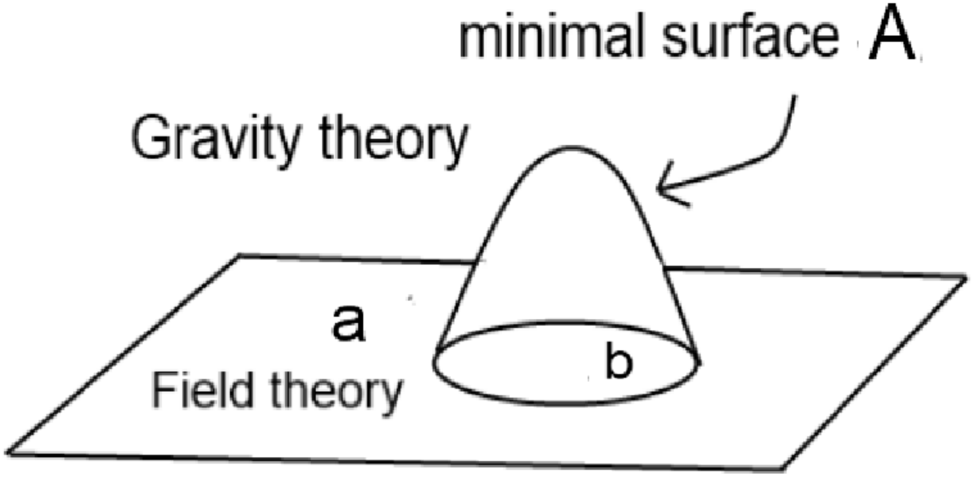


Holographic Entanglement Entropy

Holographic Entanglement Entropy *(Ryu and Takayanagi '06)*

$$S_a = A/(4G_N)$$

A is the area of the minimal surface in the bulk gravity background whose boundary is *a.*



Plan

- Entanglement Entropy
- UV-cutoff independent entanglement entropy $S_{uv\text{-}ind}$
- $S_{uv\text{-}ind}$ in (2+1)-dimensional gapped system.
- $S_{uv\text{-}ind}$ in (2+1)-dimensional finite temperature theory.
- Summary

In (2+1)-dimensional CFT , the entanglement entropy *S* has the following UV-divergent structure,

$$S \sim \frac{R}{\epsilon} - \mathrm{c} + \frac{\epsilon}{R} \cdots ,$$

where *c* is constant and invariant under redefining UV cutoff *ε* as

$$\epsilon \to a_0 \epsilon (1 + a_1 \epsilon + \cdots)$$

Thus *c* is the universal part of the entanglement entropy.

c is also obtained by defining the following UV-cutoff independent entanglement entropy

$$S_{\mathrm{UV-ind}} \equiv \left(R \frac{d}{dR} - 1\right) S = \mathbf{c}$$

In general, entanglement entropy S in (2+1) Lorentz invariant theory, has the following divergent structure and its UV-independent term $S_{uv\text{-}ind}$ depends on the size of the region R.

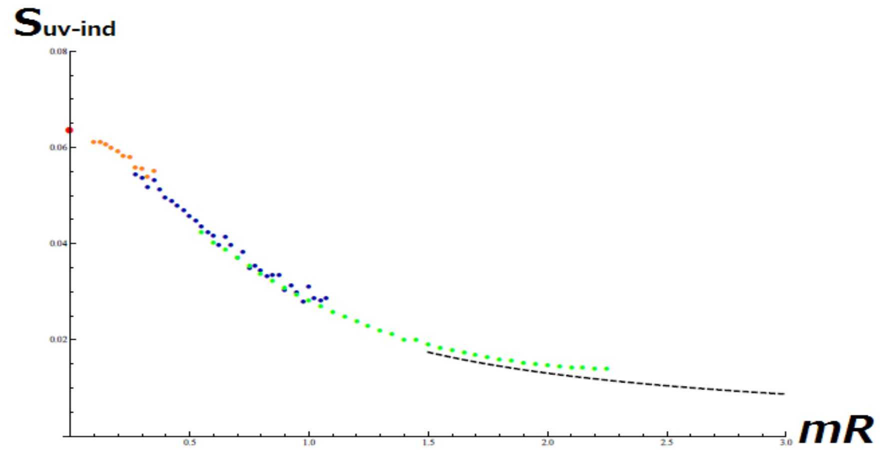
$$S \sim \frac{R}{\epsilon} - c(R) + \mathcal{O}(\epsilon) \qquad S_{\mathrm{UV-ind}} \equiv \left(R\frac{d}{dR} - 1\right) S = c(R)$$

$S_{uv\text{-}ind}$ is considered to count the degree of freedom at the scale R and shown that it monotonically decreases as R becomes large.

(Casini and Huerta '12) (Klebanov, Nishioka, Pufu, and Safdi '12) (Liu and Mezei'12)

Example : Free massive scalar theory

(arXiv 1202.2070, Liu and Mezei)



AdS5-Soliton

We consider entanglement entropy in gapped (2+1)-dimensional theory which is dual to AdS5-soliton space-time.

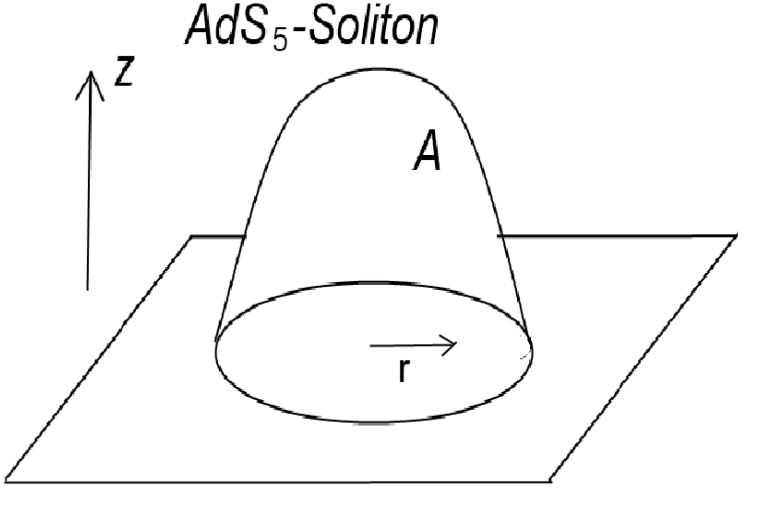
$$ds^2 = \frac{L_{AdS}^2}{z^2}\left(\frac{dz^2}{f(z)} + f(z)d\theta^2 - dt^2 + dr^2 + r^2 d\Omega\right) \qquad f(z) = 1 - \left(\frac{z}{z_0}\right)^4$$

r is a radial direction on the boundary and Lorentz symmetry is broken because of compactifying θ direction.

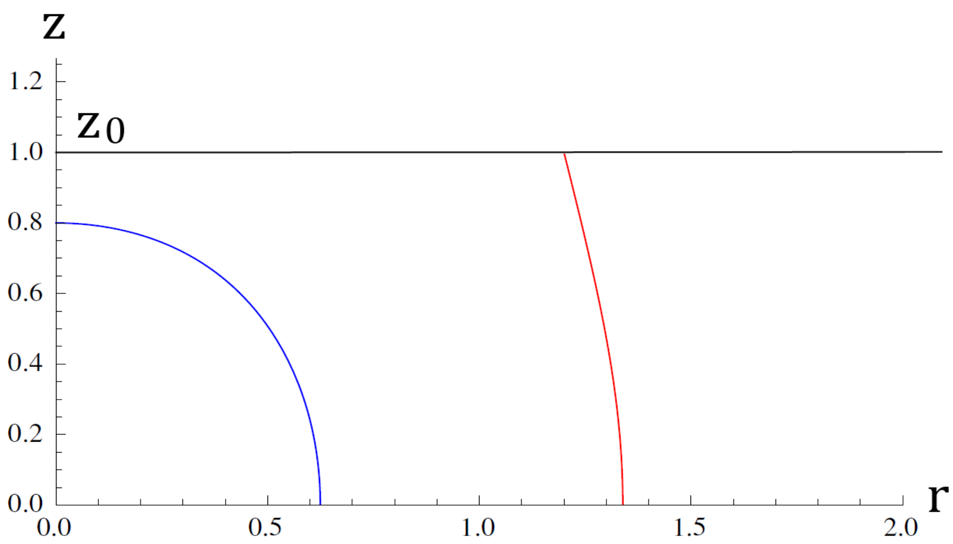
The area of minimal surface *A* ($\propto$ *S*) and the equation of motion for *r(z)* are given by

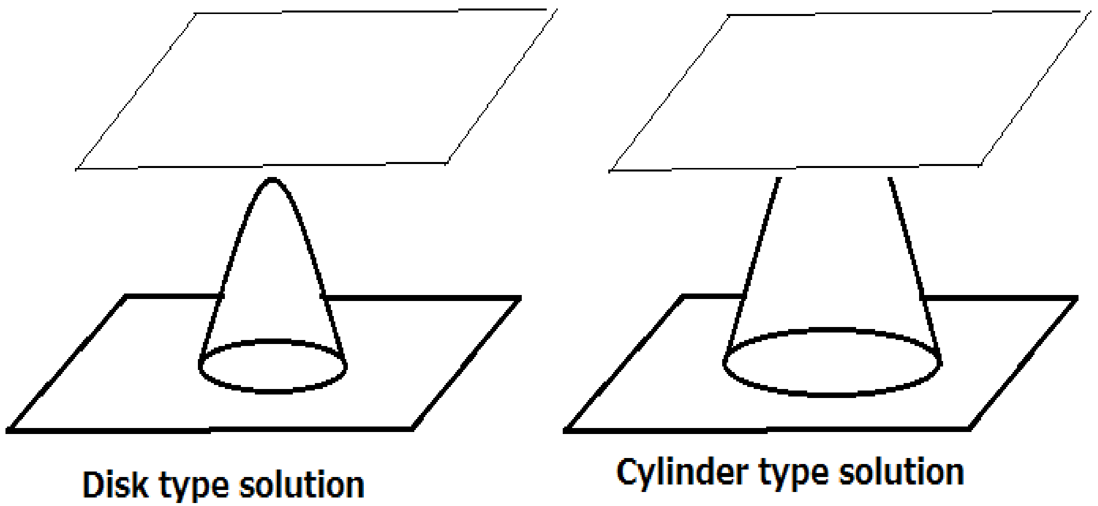
$$A = \int \sqrt{\det g_{\text{ind}}} = \int_{\epsilon}^{z_m} dz\, \frac{r^{d-3}}{z^{d-1}} \sqrt{1 + f\dot{r}^2}$$

$$6f^2 r\dot{r}^3 + 2z(1 - r\dot{r}\dot{f}) + f\left(2z\dot{r}^2 - r\left(-6\dot{r} + z\dot{f}\dot{r}^3 + 2z\ddot{r}\right)\right) = 0$$



There are two types of solutions; Disk topology solutions (blue line) and Cylinder topology solutions (red line)





Solution r(z) is expanded around UV (z~0) region as

$$r(z) = R - \frac{z^2}{4R} + a_4(R) z^4 + \frac{z^4}{32R^3} \log(\mu z) + \cdots$$

where μ is arbitrary number. $a_4(R)$ will be determined by solving full equation of motion numerically.

Instead of calculating the minimal surface *A* itself, we can calculate *dA/dR* by Hamilton-Jacobi method with UV-cutoff (*z=ε*) by following formula. *(Liu and Mezei '12)*

$$\frac{dA}{dR} = -\mathcal{H}(z_m)\frac{dz_m}{dR} - \Pi(\epsilon)\frac{dr(\epsilon)}{dR} = -\Pi(\epsilon)\frac{dr(\epsilon)}{dR}$$

$$\Pi := \frac{\delta\mathcal{L}}{\delta\dot{r}} = \frac{r\,f\dot{r}}{z^3\sqrt{1+f\dot{r}^2}} \qquad \mathcal{H} = \Pi\dot{r} - \mathcal{L} = -\frac{r}{z^3\sqrt{1+f\dot{r}^2}}$$

where $z=z_m$ is the maximal value of z

The first Hamiltonian term is zero in this case because of the boundary condition at $z=z_m$ as follows.

For disk topology $\quad \frac{dz_m}{dR} = \frac{dz_0}{dR} = 0$

For cylinder topology $\quad r(z_m) = 0 \text{ s.t. } \mathcal{H}(z_m) = 0$

By using the formula of the previous page and UV expansion of the solution $r(z)$, we can obtain

$$\frac{dA}{dR} = \frac{1}{2\epsilon^2} - \frac{1}{8R^2}\log(\mu\epsilon) - \frac{3}{32R^2} - 4Ra_4(R) + \mathcal{O}(\epsilon)$$

$a_4(R)$ will be obtained by solving the full equation of motions.

By redefining UV cutoff ε as $\epsilon \to a_0\epsilon(1 + a_1\epsilon + \cdots)$

, R-dependent finite terms are shifted.

UV divergent structure is different from that of previous cases. Thus, we have to consider the another UV-cutoff independent entanglement entropy $S_{uv\text{-}ind}$.

We can define the following UV-cutoff independent entanglement entropy.

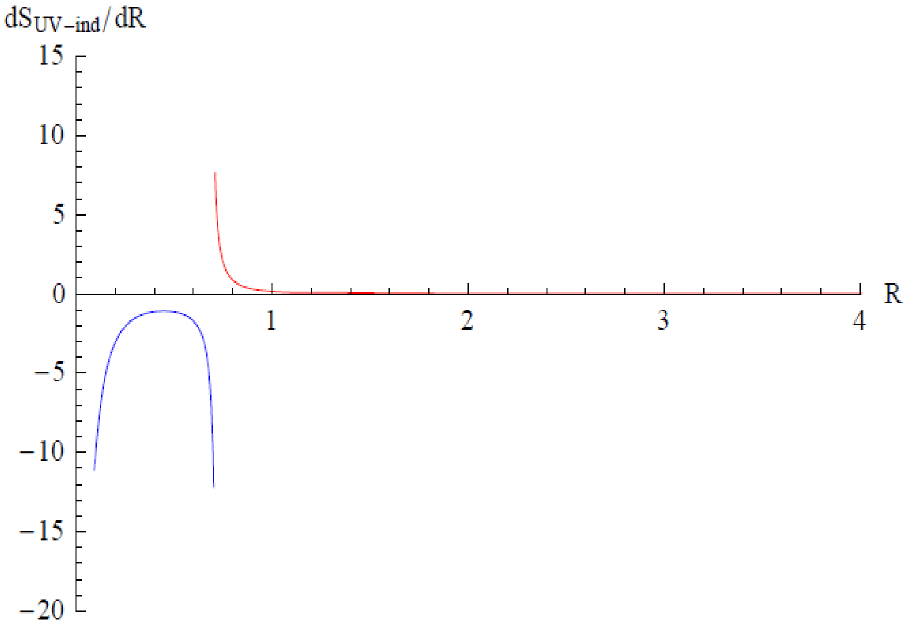
$$S_{\mathrm{UV-ind}} = \frac{1}{2}\,(R\frac{d}{dR} + 1)(R\frac{d}{dR} - 1)\,S$$

Then, we can obtain the RG-flow of $S_{uv\text{-}ind}$ as

$$\frac{dS_{\mathrm{UV-ind}}}{dR} = \frac{1}{2}\,(R\frac{d}{dR} + 2)\,R\,\frac{d}{dR}\frac{dS}{dR}$$

Figure shows the numerical result of RG-flow $dS_{uv\text{-}ind}/dR$.

In small R region (disk solutions), $dS_{uv\text{-}ind}/dR$ is negative and $S_{uv\text{-}ind}$ decreases monotonically.



In large R region, (cylinder solutions), $dS_{uv\text{-}ind}/dR$ becomes positive and goes to zero as R becomes larger.

The positivity of $dS_{uv\text{-}ind}/dR$ in the large R region (red line) is still not clear.

AdS4 Black Hole

We consider the AdS4 black hole which is dual to the (2+1)-dimensional field theory with finite temperature T and chemical potential μ *(Hartnoll '11)*.

$$ds^2 = \frac{L_{\rm AdS}^2}{z^2}\left(-f(z)dt^2 + \frac{dz^2}{f(z)} + dr^2 + r^2 d\phi^2\right)$$

$$f(z) = 1 - (1 + \frac{z_+^2\mu^2}{2\gamma^2})(\frac{z}{z_+})^3 + \frac{z_+^2\mu^2}{2\gamma^2}(\frac{z}{z_+})^4 \qquad T = \frac{1}{4\pi z_+}(3 - \frac{z_+^2\mu^2}{2\gamma^2})$$

The area of minimal surface is given by

$$A = \int \sqrt{\det g_{\rm ind}} = \int_{\epsilon}^{z_m} dz\ \frac{r}{z^2}\sqrt{\frac{1}{f} + \dot{r}^2}$$

dA/dR is obtained in the same way as

$$\frac{dA}{dR} = \frac{1}{\epsilon} - 3Ra_3(R) + \mathcal{O}(\epsilon^2)$$

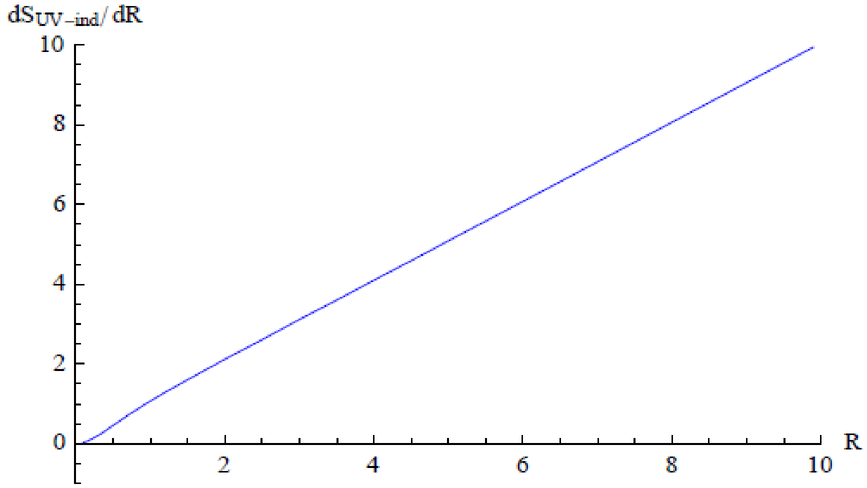
where *a₃(R)* is given by UV-expansion of r(z)

$$r(z) = R - \frac{z^2}{2R} + a_3(R)z^3 + \mathcal{O}(z^4)$$

Then, the RG-flow of UV-cutoff independent entanglement entropy is given by

$$\frac{dS^{(3)\,\mathrm{BH}}_{\mathrm{UV-ind}}}{dR} = R\partial_R \frac{dA}{dR} = R\partial_R \left(-3Ra_3(R)\right)$$

$dS_{uv\text{-}ind}/dR$ is always positive, implying that more and more states are thermally excited as we go to higher temperature regime (large R region).



In large R region, $S_{uv\text{-}ind}/dR$ becomes linear and $S_{uv\text{-}ind}$ obeys the volume law in large R region like the volume law of the thermal entropy.

Summary

- We calculate the entanglement entropy in (2+1)-dimensional gapped theory by using AdS_5-solitons and define the UV-cutoff independent entanglement entropy $S_{uv\text{-}ind}$.
- We calculate the RG-flow ($dS_{UV\text{-}ind}/dR$) and found that $dS_{uv\text{-}ind}/dR < 0$ in small R region (disk solutions), $dS_{uv\text{-}ind}/dR > 0$ at large R region (cylinder solutions). At very large R region, $dS_{uv\text{-}ind}/dR \rightarrow 0$.
- The reason of the positivity of *$dS_{uv\text{-}ind}/dR$ in* large *R region* is not clear.
- We also calculate *$S_{uv\text{-}ind}$* in (2+1)-dimensional finite temperature theory by using AdS_4-Black hole background and found that *$S_{uv\text{-}ind}$* obeys the volume law at large R-region.