Weak Solution of the Non-Perturbative Renormalization Group Equation Encountering the First Order Phase Transition.

Ken-Ichi Aoki, Yasuhiro Fujii, Shin-Ichiro Kumamoto, and Daisuke Sato
Institute for Theoretical Physics, Kanazawa University

Abstract

We propose quite a new method of analyzing the dynamical chiral symmetry breaking. Starting with the non-perturbative renormalization group equation for the Wilsonian fermion potential, we define the weak solution of it in order to mathematically authorize solutions with singularity. The weak solution successfully predicts the physically correct vacuum, chiral condensates, dynamical mass, through its auto-convexizing power for the effective potential. Thus it works perfectly even for the first order phase transition in the finite density Nambu-Jona-Lasinio model.

1 Introduction

We analyze the dynamical chiral symmetry breaking by solving non-perturbative renormalization group equations (NPRGEs) of the Wilsonian effective potential $V_{\mathrm{W}}(x,t)$ and the mass function $M(x,t) \equiv \frac{\partial V_{\mathrm{W}}(x,t)}{\partial x}$, where x and t are the bilinear fermion operator $\bar{\psi}\psi$ and the renormalization scale $\log(\Lambda_0/\Lambda)$ respectively. These NPRGEs are nonlinear partial differential equations (PDEs). In case that the dynamical chiral symmetry breaking occurs, these PDEs encounter some singularities at $t = t_{\mathrm{c}}$ even though the initial functions at $t = 0$ are continuous and smooth. Therefore, we can not go beyond t_{c}, and there is no way to calculate infrared physical quantities such as the chiral condensates or the dynamical mass.

Various methods have been used to bypass these singularities, e.g., the bare mass[10], auxiliary fields[3, 4, 9], *etc.* Here we propose a new direct method to solve the NPRGEs as PDEs [14, 15]. Such singular evolutions are unacceptable as classical solutions of the PDEs, but it is known that we can treat such solutions as the weak solutions of the PDEs. Taking the finite density Nambu-Jona-Lasinio model, we construct the weak solutions by using the method of characteristics.

2 Partial differential equations and the method of characteristics

The NPRGEs of $V_{\mathrm{W}}(x,t)$ and $M(x,t)$ in the local potential approximation are

$$\frac{\partial V_{\mathrm{W}}(x,t)}{\partial t} + f(M,t) = 0, \tag{1}$$

$$\frac{\partial M(x,t)}{\partial t} + \frac{\partial f(M(x,t),t)}{\partial x} = 0. \tag{2}$$

Here

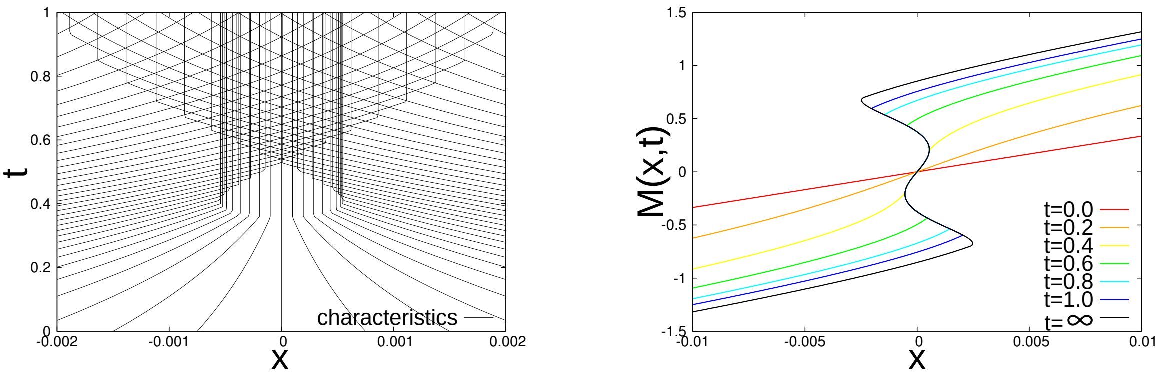
$$f(M,t) = -\frac{e^{-3t}}{\pi^2}\Big[\theta\left(e^{-2t} + M^2 - \mu^2\right)\sqrt{e^{-2t} + M^2} + \theta\left(-e^{-2t} - M^2 + \mu^2\right)\mu\Big], \tag{3}$$

where μ is the chemical potential. The initial conditions are $V_{\mathrm{W}}(x,0) = 2\pi^2 g x^2$ and $M(x,0) = 4\pi^2 g x$, where g is the coupling constant of the NJL 4-fermi interaction. The equation (1) can be viewed as the Hamilton-Jacobi type equation well-known in the analytical mechanics, where t, x, $V_{\mathrm{W}}(x,t)$, $M(x,t)$ and $f(M,t)$ correspond to the time, the coordinate, the the action, the momentum and the time-dependent Hamiltonian respectively. The equation (2) is derived from the equation (1) and it should be noted that it takes the form of the conservation law, where $M(x,t)$ and $f(M,t)$ correspond to the charge density and the current flux.

We obtain the ordinary differential equations (ODEs) equivalent to (1) and (2) by the method of characteristics,

$$\begin{aligned}\frac{dx}{dt} &= \frac{\partial f}{\partial M},\\ \frac{dM}{dt} &= -\frac{\partial f}{\partial x} = 0,\\ \frac{dV_{\mathrm{W}}}{dt} &= M\frac{\partial f}{\partial M} - f.\end{aligned} \tag{4}$$

The ODEs of $x(t)$ and $M(x,t)$ correspond to the canonical equations of Hamilton in the analogy of analytical mechanics. Their solution $x(t)$ are called characteristics which are also contours of $M(x,t)$ in this simple case (Fig. 1 (a)). There are regions where three or five contours simultaneously passes at a point, which represent a multi-leaf structure that $M(x,t)$ seems to have the "multivalued" solution after t_{c} (Fig. 1(b)).



(a) (b)

Figure 1: $g = 1.7g_{\mathrm{c}},\ \mu = 0.7$. (a) Characteristics. (b) Evolution of mass function.

3 Weak solution of conservation law

The mass function $M(x,t)$ must be a single-valued function because it is the physical quantity defining the effective action at scale t. Instead of throwing away the NPRGE description after t_{c}, we introduce the weak solution of the PDE (2) [11, 14, 15]. We will make a patchwork of the leaves to define a single-valued function $M(x,t)$, but with discontinuities, so that it might be the weak solution.

The integral form of the PDE (2) is

$$\int_0^\infty dt \int_{-\infty}^\infty dx \left[\frac{\partial M}{\partial t} + \frac{\partial f(M,t)}{\partial x} \right] \varphi(x,t) = 0, \tag{5}$$

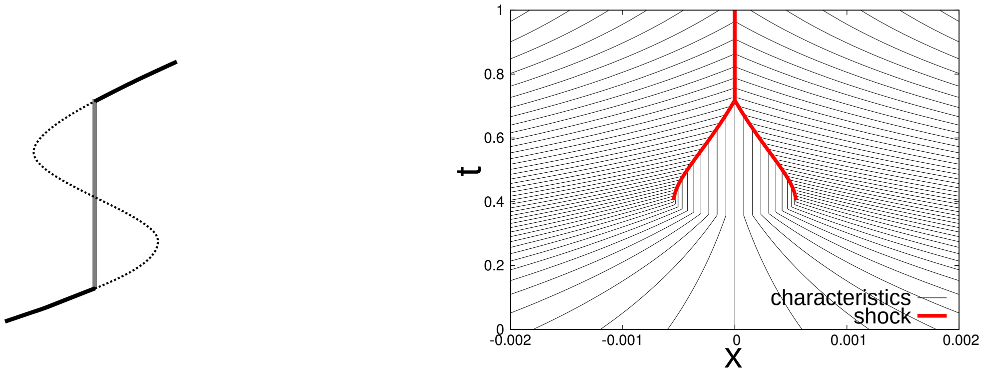
where $\varphi(x,t)$ is an arbitrary test function that is continuously differentiable and vanishes at $x = \pm\infty$ and $t = +\infty$. We integrate it by parts and obtain

$$\int_0^\infty dt \int_{-\infty}^\infty dx \left[M\frac{\partial \varphi}{\partial t} + f(M;t)\frac{\partial \varphi}{\partial x} \right] + \int_{-\infty}^\infty dx\ M(x,0)\varphi(x,0) = 0. \tag{6}$$

In contrast to the equation (5), the equation (6) makes sense even for the discontinuous $M(x,t)$. The weak solution of the PDE (2) is defined as to satisfy the equation (6) for any smooth and bounded test function $\varphi(x,t)$. The weak solution satisfies the original PDE (2) except for the points of discontinuities. The position of discontinuity $x = D(t)$, which is called the shock, is controlled by the Rankine-Hugoniot (RH) condition,

$$\frac{dD(t)}{dt}[M_+ - M_-] = f(M_+,t) - f(M_-,t), \tag{7}$$

where M_+ and M_- are right and left limits at the position of discontinuity respectively. The graphical interpretation of the RH condition for $M(x,t)$ is that the discontinuity must cut off lobes of equal area as shown in Fig. 2(a), where the solid lines show the weak solution[12]. In this way we uniquely determine the shock $D(t)$ which is showed in Fig. 2(b), where two shocks appears pairwisely and they move towards the origin to be merged finally.



(a) (b)

Figure 2: (a) Equal area rule. (b) Characteristics and shock of mass function.

4 Weak solution results for physical quantities

We show the results in the finite density NJL model where the first order phase transition occurs. Snapshots in the course of renormalization are shown in Fig. 3, where the mass function $M(t,x)$, the Wilsonian effective potential $V_{\mathrm{W}}(x,t)$ and the Legendre effective potential $V_{\mathrm{L}}(x,t)$ for $\langle\bar{\psi}\psi\rangle$ are plotted. The five-fold structure of $M(x,t)$ appears at the second row of Fig. 3, which means a pair of shocks are generated. At the third row, the mass function is five-fold even at the origin, which corresponds to the three-fold local minima in the Legendre effective potential. The time when the two shocks are merged with each other at the origin is exactly the first order phase transition point where the free energy of three local minima coincide. Finally at the fourth row, the chiral symmetry is dynamically broken with the unphysical metastable symmetric phase at the origin.

It is astonishing that our method of weak solution uniquely determines their singularity structures and the resultant Legendre effective potential is always convexized. This means the dynamical mass and the chiral condensates are uniquely calculated, and perfectly correct in the sense that even in case there are multi local minima, the lowest free energy minimum is always chosen automatically. This feature is quite a new finding and shows powerfulness of the purely fermionic non-perturbative renormalization group and its weak solution[15]. This analysis has been applied to QCD, even with finite density or non-ladder, and proved to work perfectly to give physical quantities without any ambiguity[13].

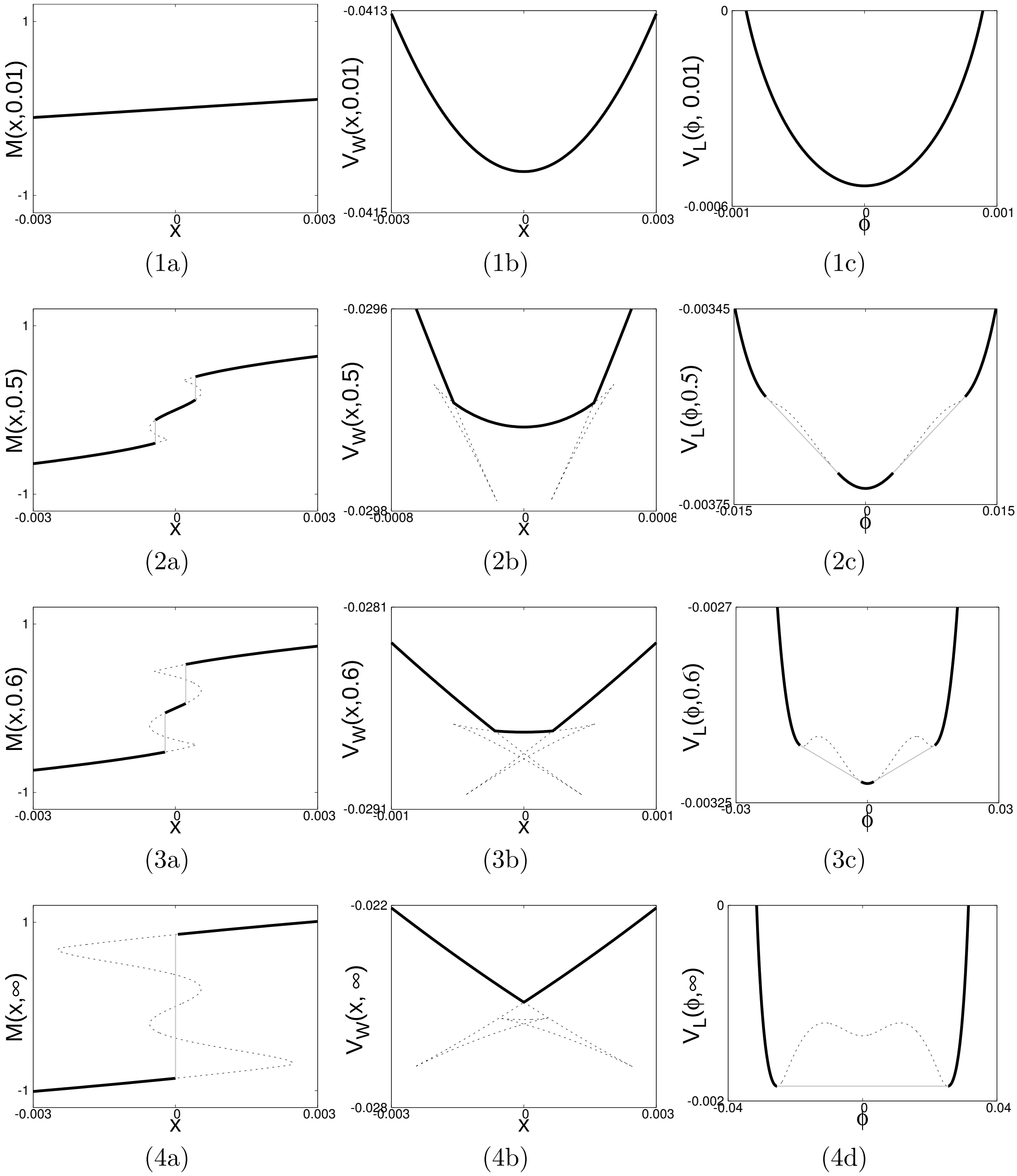


Figure 3: Evolution of physical quantities by weak solution (NJL $g = 1.7g_{\rm c}$, $\mu = 0.7$, $t = 0.01$, 0.5, 0.6, ∞). (a) Mass function. (b) Wilsonian fermion potential. (c) Legendre effective potential.

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