

Analysis of the Dynamical Chiral Symmetry Breaking in QCD at Finite Temperature and Density using the Non-Perturbative Renormalization Group

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We analyze the Nambu–Jona-Lasinio model which is a chiral effective model of QCD by using the non-perturbative renormalization group at finite temperature and finite density. We discuss the chiral phase diagram in the model. We include the large- N non-leading contribution in the beta function and discuss its effects for the phase boundary.

1 Introduction

Understanding the non-perturbative Quantum Chromodynamics (QCD) is one of the important subjects in elementary particle physics. The phase diagram of QCD at finite temperature and finite density has been studied intensively. There are various non-perturbative approaches to QCD, such as the lattice QCD, the mean-field approximation (MFA) and the Schwinger-Dyson equations (SDEs). However, these methods have serious problems respectively. The lattice QCD is a powerful method because of the first-principle calculation. However, it is difficult to maintain the chiral symmetry on the lattice field. At finite density, the QCD action has a complex phase due to the quark chemical potential. Owing to this problem, called the sign problem, the statistical errors of simulation cannot be controlled easily. The MFA or the SDEs with the ladder approximation have been used in various types of models. However, these methods have difficulties in the further improvement of the approximation without which the strong gauge dependences cannot be cured.

We study the non-perturbative properties using the non-perturbative renormalization group (NPRG). The NPRG method gives us not only the equivalent results to MFA and SDEs in the lowest order approximation [3], but also the systematic method for improving approximation. We don't confront the sign problem at finite density. However, the NPRG breaks gauge symmetry due to inclusion of the momentum cutoff and also it has the gauge dependences as MFA and SDEs. These problems can be treated by systematic improvement of approximation. It reported that the gauge dependence has almost been wiped away by including the non-ladder diagrams to the beta functions [4].

We apply the NPRG to the Nambu–Jona-Lasinio (NJL) model [9] at finite temperature and finite density. The NJL model is the effective theory with four-fermi interactions and describes the $S\chi$ SB of QCD. We compare the results improved by the NPRG method with the mean field approximation.

2 Non-perturbative Renormalization Group

We briefly explain the basic idea of NPRG [2] in quantum field theory. We divide the degrees of freedom of quantum field $\phi(p)$ into the higher modes with $|p| > \Lambda$ and the lower modes with $|p| < \Lambda$ in the Euclidean space. Then we define the effective action $S_{\text{eff}}[\phi; \Lambda]$, called the Wilsonian effective action, by integrating out only the higher modes in the path integral

$$Z = \int^{\Lambda_0} \mathcal{D}\phi e^{-S_0} = \int^{\Lambda} \mathcal{D}\phi_{<} \int_{\Lambda}^{\Lambda_0} \mathcal{D}\phi_{>} e^{-S_0[\phi_{<} + \phi_{>}]} = \int^{\Lambda} \mathcal{D}\phi_{<} e^{-S_{\text{eff}}[\phi_{<}; \Lambda]}, \quad (1)$$

where S_0 is the initial (bare) action at the initial cutoff Λ_0 . The NPRG equation describes the dependence of the Wilsonian effective action on the cutoff Λ ,

$$\frac{\partial}{\partial \Lambda} S_{\text{eff}}[\phi; \Lambda] = \beta[S_{\text{eff}}; \Lambda]. \quad (2)$$

The right hand side of this equation is called the beta function. It is evaluated as the infinitesimal change of the Wilsonian effective action by infinitesimally lowering the cutoff Λ .

There are various formulations of NPRG equation [5, 6, 7]. In this paper, we adopt the Wetterich flow equation [7] which is a differential equation for the Legendre effective action with IR cutoff,

$$\partial_\Lambda \Gamma_\Lambda[\Phi] = \frac{1}{2} \text{STr} \left\{ \left[\frac{\overrightarrow{\delta}}{\delta \Phi} \Gamma_\Lambda[\Phi] \frac{\overleftarrow{\delta}}{\delta \Phi} + R_\Lambda \right]^{-1} \cdot (\partial_\Lambda R_\Lambda) \right\}, \quad (3)$$

where R_Λ is the cutoff profile function which divides the higher and the lower modes of quantum field. This equation is exact. It describes the development of the effective action starting from the bare action S_0 to the Legendre effective action $\Gamma_{\Lambda=0}$.

The equation (3) is a functional differential equation and we cannot solve it exactly. We have to make some approximation. First, the effective action is expanded into power series of derivative of fields,

$$\Gamma_\Lambda[\phi] = \int d^4x \left[V_\Lambda(\phi) + \frac{1}{2} Z_\Lambda(\phi) (\partial_\mu \phi)^2 + \frac{1}{2} Y_\Lambda(\phi) (\partial^2 \phi)^2 + \dots \right], \quad (4)$$

where V_Λ is the effective potential generated and Z_Λ and Y_Λ are the field renormalization factors. This method is called the derivative expansion. Next, we ignore all the connections to terms with derivatives. Then the effective action is represented by the effective potential V_Λ . This approximation, called the local potential approximation (LPA) [8], allows us to evaluate the effective action only with the zero momentum mode of fields. We reduce Eq. (3) to be a partial differential equation for the effective potential V_Λ . Furthermore if the effective potential is spanned by the polynomials of fields, we get infinitely coupled ordinary differential equations for the expansion coefficients (the coupling constants).

3 Nambu–Jona-Lasinio model

The Lagrangian of the NJL model [9] with one flavor and one color is given by

$$\mathcal{L}_{\text{NJL}} = \bar{\psi} i \not{\partial} \psi + \frac{G_0}{2} \{ (\bar{\psi} \psi)^2 + (\bar{\psi} i \gamma_5 \psi)^2 \}. \quad (5)$$

This Lagrangian is invariant under the chiral U(1) transformation: $\psi \rightarrow e^{i\gamma_5 \theta} \psi$. The four-fermi coupling constant G corresponds to the fluctuation of the chiral order parameter: $\langle (\bar{\psi} \psi)^2 \rangle$, therefore, we may conclude the $S_\chi\text{SB}$ by divergence of the four-fermi coupling constant at a finite energy scale.

The NJL model at finite temperature and finite density is defined by the following Euclidean bare action,

$$S_0 = \int_0^{1/T} d\tau \int d^3x \left[\bar{\psi} \not{\partial} \psi + \mu \bar{\psi} \gamma_0 \psi - \frac{G_0}{2} \{ (\bar{\psi} \psi)^2 + (\bar{\psi} i \gamma_5 \psi)^2 \} \right]. \quad (6)$$

The four-fermi interaction G_0 generates the effective four-fermi coupling constant G_Λ by quantum corrections, that is, by driving the infrared cutoff scale to the low energy. The effective action in LPA is denoted by

$$\Gamma_\Lambda = \int_0^{1/T} d\tau \int d^3x \left[\bar{\psi} \not{\partial} \psi + \mu \bar{\psi} \gamma_0 \psi - \frac{G_\Lambda}{2} \{ (\bar{\psi} \psi)^2 + (\bar{\psi} i \gamma_5 \psi)^2 \} \right]. \quad (7)$$

We calculate the generated four-fermi interactions through the diagrams in Fig. 1. The first diagram in the dashed box in Fig. 1 is the so-called large- N leading term. If we adopt only this term in the beta function of the four-fermi coupling constant, we get the equivalent results to the MFA or the ladder SDEs. Note that we can add the large- N non-leading terms without any difficulty.

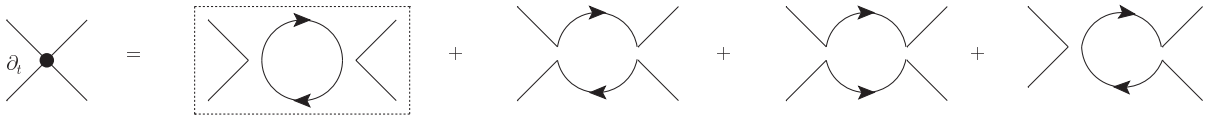


Figure 1: Feynman diagrams contributing to the beta function for the four-fermi interactions.

In this study, we use the 3- d optimized cutoff function [10] in Eq. (3),

$$R_\Lambda(\mathbf{p}) = \mathbf{p} \left(\frac{\Lambda}{|\mathbf{p}|} - 1 \right) \theta\left(1 - \frac{\mathbf{p}^2}{\Lambda^2}\right) = \mathbf{p} r(\mathbf{p}/\Lambda).$$

At finite temperature, the time direction momentum is discretized and the integration of it changes to the Matsubara summation, therefore, the 3- d cutoff function is appropriate to write down the renormalization group equations in a simple form.

4 Results

The RGEs we solve are the following three simultaneous differential equations,

$$\begin{aligned} \partial_t \tilde{g} &= 2\tilde{g} - \frac{1}{3}(4I_0 - I_1), \\ \partial_t \tilde{T} &= \tilde{T}, \\ \partial_t \tilde{\mu} &= \tilde{\mu}, \end{aligned} \quad (8)$$

where $\partial_t = -\Lambda \partial_\Lambda$, $1/\tilde{g} = g = G_\Lambda \Lambda^2 / 4\pi^2$, $\tilde{T} = T/\Lambda$ and $\tilde{\mu} = \mu/\Lambda$. The threshold functions $I_0(\tilde{T}, \tilde{\mu})$ and $I_1(\tilde{T}, \tilde{\mu})$ are given by the large- N leading diagram and the non-leading diagrams respectively in Fig. 1, and are expressed as

$$I_0 = \left[\left(\frac{1}{2} - n_+ \right) + \left(\frac{1}{2} - n_- \right) + \frac{\partial}{\partial \omega} (n_+ + n_-) \right] \Big|_{\omega \rightarrow 1}, \quad (9)$$

$$I_1 = \left[\frac{1}{(1 + \tilde{\mu})^2} \left(\frac{1}{2} - n_+ \right) + \frac{1}{(1 - \tilde{\mu})^2} \left(\frac{1}{2} - n_- \right) + \frac{1}{1 + \tilde{\mu}} \frac{\partial}{\partial \omega} n_+ + \frac{1}{1 - \tilde{\mu}} \frac{\partial}{\partial \omega} n_- \right] \Big|_{\omega \rightarrow 1}, \quad (10)$$

where n_\pm are the Fermi-Dirac distribution functions, $n_\pm = (e^{(\omega \pm \tilde{\mu})/\tilde{T}} + 1)^{-1}$. The large- N non-leading effect I_1 contributes towards the restoration of chiral symmetry. Especially, this effect is large at low temperature and high density. If temperature vanishes, the threshold function I_1

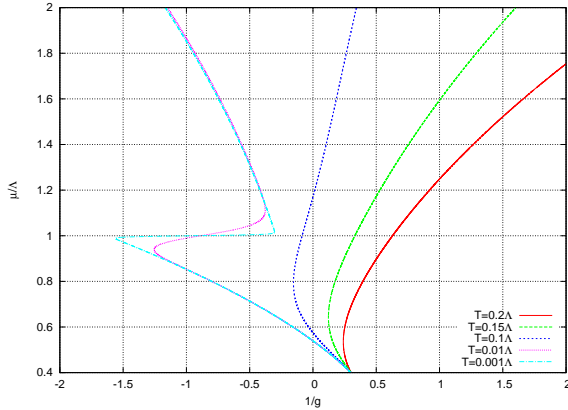


Figure 2: The RG flows of \tilde{g} of the large- N leading on the $\tilde{g} - \tilde{\mu}$ plane at $\mu/\Lambda_0 = 0.3$ and various temperatures. We set the initial inverse four-fermi coupling to 0.3.

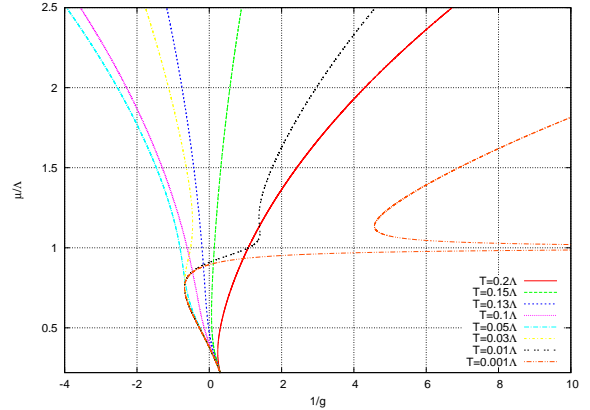


Figure 3: The RG flows of \tilde{g} of the large- N non-leading on the $\tilde{g} - \tilde{\mu}$ plane at $\mu/\Lambda_0 = 0.22$ and various temperatures. We set the initial inverse four-fermi coupling to 0.3.

diverges at $\tilde{\mu} = 1(\mu = \Lambda)$ when the cutoff reaches the fermi surface.

We numerically solve the RGEs (8). We may conclude the $S\chi$ SB when \tilde{g} passes the origin. We show the RG flows of \tilde{g} of the large- N leading and the non-leading in Fig. 2 and 3 respectively. We draw RG flows after \tilde{g} passes the origin. This is justified by the "Weak solution" method [11] in case of the large- N leading. Infact, the inverse four-fermi coupling constant \tilde{g} corresponds to the mass squared of meson fields introduced as the auxiliary fields. In other words, \tilde{g} corresponds to the curvature of the effective potential of meson field at the origin, and thus, the negative of \tilde{g} means that the curvature of the effective potential at the origin becomes negative. In this case, the effective potential has a global minimum at non-zero expectation value of meson field, then the $S\chi$ SB occurs. If we directly solve the RGE for the four-fermi coupling constant, the RG flows stop on the way because the four-fermi coupling constant diverges at finite t . However, as shown in Fig. 2, some flows go to the negative region, and then comes back to the positive region afterwards. We may regard such turn-over flows as symmetric phase in case of the large- N leading. In case of the large- N non-leading, it is difficult to justify of this interpretation. However, we adopt the some criterion here.

The phase diagrams of the large- N leading and the non-leading calculation are shown in Fig. 4. We can see the drastic difference of behavior of phase boundaries due to the large- N non-leading effects at low temperature and high density.

5 Summary and Discussion

We study the Nambu–Jona-Lasinio model by using the non-perturbative renormalization group at finite temperature and finite density. We discuss the difference of the RG flows and phase boundaries between the mean field approximation results and those including the large- N non-leading effects. We find that the large- N non-leading effects become large and contribute to the restoration of chiral symmetry at low temperature and high density.

However, this analysis tells us only second-order phase transition because we evaluate only the RG flow of the four-fermi coupling constant. In other words, we see the behavior of the curvature of the mesonic effective potential at the origin only, and therefore, we overlook the possibility of the first-order transitions. We should bosonize the four-fermi interactions [3, 12, 13]

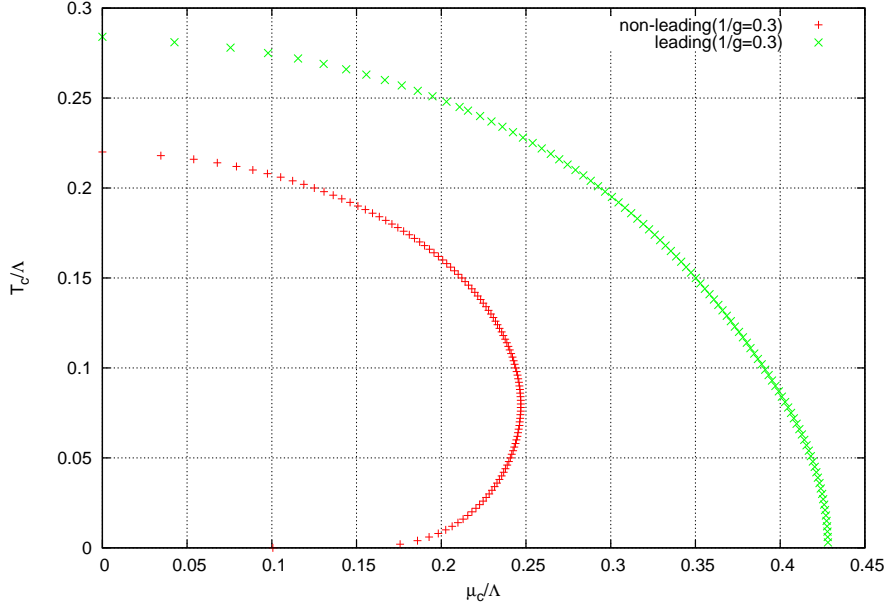


Figure 4: The chiral phase diagram in the large- N leading and non-leading calculation

or treat total mass function [11] in order to evaluate the order of phase transition.

References

- [1] K-I. Aoki, *Int. J. Mod. Phys. B* **14**, 1249 (2000).
- [2] K. G. Wilson and I. G. Kogut, *Phys. Rep.* **12** 75 (1974)
- [3] K-I Aoki, K. Morikawa, J-I. Sumi, H. Terao and M. Tomoyose, *Phys. Rev. D* **61**, 045008 (2000).
- [4] K-I. Aoki and D. Sato, *Prog. Theor. Exp. Phys.* **2013**, 043B04 (2013).
- [5] F. J. Wegner and A. Houghton, *Phys. Rev. A* **8**, 401 (1973).
- [6] J. Polchinski, *Nucl. Phys. B* **301**, 90 (1984).
- [7] C. Wetterich, *Phys Lett. B* **301**, 90 (1993).
- [8] A. Hazenfratz and P. Hazenfratz, *Nucl. Phys. B* **270**, 269 (1986).
- [9] Y. Nambu and G. Jona-Lasinio, *Phys Rev.* **122**, 345 (1961).
- [10] D. F. Litim, *Phys. Rev. D* **64**, 105007 (2001)
- [11] K-I. Aoki, S-I. Kumamoto, D. Sato, arXiv:1304.3289.
- [12] H. Gies and C. Wetterich, *Phys. Rev. D* **65**, 065001 (2002).
- [13] H. Gies and C. Wetterich, *Phys. Rev. D* **69**, 025001 (2004).