Z_N twisted orbifold models with magnetic flux[1]

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We propose new backgrounds of extra dimensions to lead to four-dimensional chiral models with three generations of matter fermions, that is T^2/Z_N twisted orbifolds with magnetic fluxes. We consider gauge theory on six-dimensional space-time, which contains the T^2/Z_N orbifold with magnetic flux, Scherk-Schwarz phases and Wilson line phases. We classify all the possible Scherk-Schwarz and Wilson line phases on T^2/Z_N orbifolds with magnetic fluxes. The behavior of zero modes is studied. We derive the number of zero modes for each eigenvalue of the Z_N twist, showing explicitly examples of wave functions. We also investigate Kaluza-Klein mode functions and mass spectra.

1 Gauge field theory on $M^4 \times T^2$ with magnetic flux

Let us study the behavior of gauge and matter fields on six-dimensional space-time, which contains four-dimensional Minkowski space-time M^4 and extra two-dimensional torus T^2. We denote coordinates on M^4 by x^μ ($\mu = 0, 1, 2, 3$) and we use the complex coordinate z on T^2. We consider a theory containing the torus with magnetic flux. Then, one can obtain an attractive feature that *the effect of the magnetic flux makes degenerate solutions of chiral fermions in four-dimensional space-time generated from one fermion in higher-dimensional space-time, and the number of solutions correspond to the magnitude of magnetic flux* in Ref. [2].

First of all, we consider the Lagrangian density based on a $U(1)$ gauge theory on $M^4 \times T^2$ such as

$$\mathcal{L}_{6\mathrm{D}} = -\frac{1}{4}F^{MN}F_{MN} + i\bar{\Psi}_+\Gamma^M D_M \Psi_+, \tag{1}$$

where $M, N = \mu(= 0, 1, 2, 3), z, \bar{z}$ and $D_M = \partial_M - iqA_M(x, z)$ with a $U(1)$ charge q. Here, $\Psi_\pm$ are six-dimensional Weyl fermions, and are obtained by projection operators $\frac{1\pm\Gamma_7}{2}$ such as

$$\Psi_+(x, z) = \sum_n \left(\psi_{4R,n}(x) \otimes \begin{pmatrix} \psi_{+,n}(z) \\ 0 \end{pmatrix} + \psi_{4L,n}(x) \otimes \begin{pmatrix} 0 \\ \psi_{-,n}(z) \end{pmatrix}\right), \tag{2}$$

where n means the label of mass eigenstates. Ψ is a six-dimensional Dirac fermion, $\psi_{4R/L,n}$ are four-dimensional chiral fermions with four components, and $\psi_{\pm,n}$ are elements of two-dimensional Weyl fermions.

When we require the Lagrangian density $\mathcal{L}_{6\mathrm{D}}$ (1) to be single-valued, i.e.,

$$\mathcal{L}_{6\mathrm{D}}(A(x, z), \Psi_+(x, z)) = \mathcal{L}_{6\mathrm{D}}(A(x, z + 1), \Psi_+(x, z + 1)) = \mathcal{L}_{6\mathrm{D}}(A(x, z + \tau), \Psi_+(x, z + \tau)), \tag{3}$$

[1] This talk is based on our recently work [1]

this field $\Psi_+(x,z)$ should satisfy the pseudo periodic boundary conditions

$$\psi_{\pm,n}(z+1)=e^{iq\chi_1(z)+2\pi i\alpha_1}\psi_{\pm,n}(z),\quad \psi_{\pm,n}(z+\tau)=e^{iq\chi_\tau(z)+2\pi i\alpha_\tau}\psi_{\pm,n}(z). \tag{4}$$

The consistency of the contractible loops, e.g., $z\to z+1\to z+1+\tau\to z+\tau\to z$, requires the magnetic flux quantization condition,

$$\frac{qb}{2\pi}\equiv M\in\mathbb{Z}. \tag{5}$$

Moreover, we focus on massless zero-mode solutions $\psi_{\pm,0}(z)$ on T^2 with magnetic flux. It turns out that there appear degenerate solutions from one fermion to satisfy certain boundary conditions and equations of motion due to the existence of magnetic flux as below.

The equations of zero modes $\psi_{\pm,0}(z)$ without any Wilson line phase are given by

$$\left(\partial_{\bar z}+\frac{\pi M}{2\mathrm{Im}\tau}z\right)\psi_{+,0}(z)=0,\quad \left(\partial_{z}-\frac{\pi M}{2\mathrm{Im}\tau}\bar z\right)\psi_{-,0}(z)=0. \tag{6}$$

From the conditions (4), the zero-mode solutions of $\psi_{\pm,0}(z;a_w)$ are found to be of the form

$$\psi_{+,0}(z)=\mathcal{N}e^{i\pi Mz\frac{\mathrm{Im}z}{\mathrm{Im}\tau}}\cdot\vartheta\begin{bmatrix}\frac{j+\alpha_1}{M}\\ -\alpha_\tau\end{bmatrix}(Mz,M\tau)\equiv\psi_{+,0}^{(j+\alpha_1,\alpha_\tau)}(z)\qquad \text{for } M>0, \tag{7}$$

$$\psi_{-,0}(z)=\mathcal{N}e^{i\pi M\bar z\frac{\mathrm{Im}\bar z}{\mathrm{Im}\bar\tau}}\cdot\vartheta\begin{bmatrix}\frac{j+\alpha_1}{M}\\ -\alpha_\tau\end{bmatrix}(M\bar z,M\bar\tau)\equiv\psi_{-,0}^{(j+\alpha_1,\alpha_\tau)}(z)\qquad \text{for } M<0, \tag{8}$$

where $j=0,1,\cdots,|M|-1$, $\mathcal{N}$ is the normalization factor, and the ϑ function is defined by

$$\vartheta\begin{bmatrix}a\\ b\end{bmatrix}(c\nu,c\tau)=\sum_{l=-\infty}^{\infty}e^{i\pi(a+l)^2c\tau}e^{2\pi i(a+l)(c\nu+b)}. \tag{9}$$

We would like to note the two features that for $M>0$ ($M<0$), only $\psi_{+,0}$ ($\psi_{-,0}$) has solutions, and that the number of solutions is given by $|M|$. Thus, we can obtain a $|M|$-generation chiral theory in four-dimensional space-time from eq.(1).

2 Twisted orbifolds with magnetic flux

In this section, we study the $U(1)$ gauge theory on twisted orbifolds T^2/Z_N with magnetic flux, and investigate the degeneracy of zero-mode solutions and the allowed values of the Scherk-Schwarz phases α_1 and α_τ in the Wilson line phase $a_w=0$.

2.1 Field theory on T^2/Z_N orbifold

A two-dimensional twisted orbifold T^2/Z_N is defined by dividing a one-dimensional complex plane by lattice shifts t_1, t_τ and a Z_N discrete rotation (twist) s such as

$$t_1 : z \to z+1, \quad t_\tau : z \to z+\tau, \quad s : z \to \omega z, \tag{10}$$

with $\omega \equiv e^{2\pi i/N}$.

Let us consider the following Lagrangian density on six-dimensional space-time with the orbifold T^2/Z_N,

$$\mathcal{L}_{\rm 6D}^{\rm Weyl} \equiv i\bar{\Psi}_{T^2/Z_N+}(x,z)\Gamma^M(\partial_M - iqA_M(x,z))\Psi_{T^2/Z_N+}(x,z), \tag{11}$$

where $\Psi_{T^2/Z_N+}(x,z)$ is a six-dimensional Weyl fermion on $M^4 \times T^2/Z_3$. In a way similar to eq.(2), we can expand the Weyl fermion $\Psi_{T^2/Z_N+}(x,z)$ on $M^4 \times T^2/Z_N$ such as

$$\Psi_{T^2/Z_N+}(x,z) = \sum_n \left(\psi_{4R,n}(x) \otimes \begin{pmatrix} \psi_{T^2/Z_N+,n}(z) \\ 0 \end{pmatrix} + \psi_{4L,n}(x) \otimes \begin{pmatrix} 0 \\ \psi_{T^2/Z_N-,n}(z) \end{pmatrix} \right). \tag{12}$$

Then, the boundary conditions for $\Psi_{T^2/Z_N+}(x,z)$ are replaced by those for $\psi_{T^2/Z_N\pm,n}(z)$, i.e.,

$$\begin{aligned} &\psi_{T^2/Z_N\pm,n}(z+1) = U_1(z)\psi_{T^2/Z_N\pm,n}(z), \quad \psi_{T^2/Z_N\pm,n}(z+\tau) = U_\tau(z)\psi_{T^2/Z_N\pm,n}(z), \\ &\psi_{T^2/Z_N+,n}(\omega z) = V(z)\psi_{T^2/Z_N+,n}(z), \quad \psi_{T^2/Z_N-,n}(\omega z) = \omega V(z)\psi_{T^2/Z_N-,n}(z), \end{aligned} \tag{13}$$

where $U_1(z) = e^{iq\chi_1(z)+2\pi i\alpha_1}$, $U_\tau(z) = e^{iq\chi_\tau(z)+2\pi i\alpha_\tau}$ and $V(z) = e^{2\pi i\beta}$. Here, it is worthwhile to note that the wave functions $\psi_{T^2/Z_N\pm,n}(z)$ on the orbifold T^2/Z_N can be constructed from certain linear combinations of $\psi_{\pm,n}(z)$ on the torus T^2. This is because the orbifold T^2/Z_N is obtained by dividing the torus T^2 by the Z_N discrete rotation.

Moreover, let us investigate the boundary conditions for general lattice shifts $m+n\tau$ ($m,n \in \mathbb{Z}$) and Z_N twists ω^k ($k \in \mathbb{Z}$). To this end, we define the transformation function $U_{m+n\tau}(z)$ through the relation

$$\Psi_{T^2/Z_N}(x, z+m+n\tau) = U_{m+n\tau}(z)\Psi_{T^2/Z_N}(x,z). \tag{14}$$

Then, we obtain

$$\Psi_{T^2/Z_N}(x, \omega^k(z+m+n\tau)) = U_{\omega^k(m+n\tau)}(\omega^k z)\Psi_{T^2/Z_N}(x,\omega^k z), \tag{15}$$

because $\omega^k(m+n\tau)$ for ${}^\forall k,m,n \in \mathbb{Z}$ can be equivalently expressed as a lattice shift $m'+n'\tau$ for ${}^\exists m',n' \in \mathbb{Z}$.

2.1.1 Scherk-Schwarz phases with magnetic flux

Next, let us investigate the Scherk-Schwarz phases with magnetic flux. For example, from the condition (15) for $N = 3$, it follows that we find

$$U_1(z) = U_\tau(\omega z), \quad U_\tau(z) = U_{\omega\tau}(\omega z). \tag{16}$$

Thus, we obtain

$$e^{2\pi i\alpha_1} = e^{2\pi i\alpha_\tau}, \quad e^{2\pi i\alpha_\tau} = e^{-2\pi i(\alpha_1+\alpha_\tau)+i\pi M}, \tag{17}$$

i.e.,

$$\begin{aligned} (\alpha_1, \alpha_\tau) &= (0,0),\ (1/3,1/3),\ (2/3,2/3) \qquad && \text{for} \ \ M = \text{even}, \\ (\alpha_1, \alpha_\tau) &= (1/6,1/6),\ (1/2,1/2),\ (5/6,5/6) \qquad && \text{for} \ \ M = \text{odd}. \end{aligned} \tag{18}$$

It is found that the variety of the Scherk-Schwarz phases still corresponds to the number of fixed points even with non-zero magnetic flux for $N = 2, 3, 4, 6$. However, it is remarkable that the non-zero magnetic flux with $M =$ odd affects the values of the Scherk-Schwarz phases for $N = 3, 6$, and especially does not permit them to vanish.

2.2 Z_N eigenstates of fermions

Here, we explain how to construct the wave functions $\psi_{T^2/Z_N\pm,n}(z)$ on T^2/Z_N from the wave functions $\psi_{\pm,n}(z)$ on T^2. To clearly distinguish wave functions on T^2 from those on T^2/Z_N, we rewrite the wave functions on T^2 as $\psi_{T^2\pm,n}(z)$. Since the wave functions $\psi_{T^2\pm,n}(z)$ should obey the desired boundary conditions (4) as well as the zero-mode wave functions $\psi_{T^2\pm,0}^{(j+\alpha_1,\alpha_\tau)}(z) = \psi_{\pm,0}^{(j+\alpha_1,\alpha_\tau)}(z)$, we rewrite them as $\psi_{T^2\pm,n}^{(j+\alpha_1,\alpha_\tau)}(z)$ $(j = 0, 1, \cdots, |M|-1)$, which are constructed by a way similar to the analysis of harmonic oscillator in the quantum mechanics (see section 4).

In addition to the torus boundary conditions (the first two conditions of eq.(13)), the wave functions $\psi_{T^2/Z_N\pm,n}(z)$ have to satisfy these orbifold boundary conditions (the last two conditions of eq.(13)) Then, we can construct $\psi_{T^2/Z_N\pm,n}(z)$ by the linear combinations of $\psi_{T^2\pm,n}^{(j+\alpha_1,\alpha_\tau)}(z)$,

$$\psi_{T^2/Z_N\pm,n}(z) = \mathcal{N}_{\pm,\omega^\ell}^{(j)} \sum_{k=0}^{N-1} \bar{\omega}^{\ell k} \psi_{T^2\pm,n}^{(j+\alpha_1,\alpha_\tau)}(\omega^k z) \equiv \psi_{T^2/Z_N\pm,n}^{(j+\alpha_1,\alpha_\tau)}(z)_{\omega^\ell}, \tag{19}$$

where j $(= 0, 1, \cdots, |M|-1)$ stand for the degeneracy with respect to the n-mode wave functions. The index of ω^ℓ $(\ell = 0, 1, \cdots, N-1)$ for $\psi_{T^2/Z_N\pm,n}^{(j+\alpha_1,\alpha_\tau)}(z)_{\omega^\ell}$ means Z_N eigenvalues on T^2/Z_N, and $\mathcal{N}_{\pm,\omega^\ell}^{(j)}$ are normalization factors, which depend on j, the chirality of $\psi_{T^2\pm,n}^{(j+\alpha_1,\alpha_\tau)}(z)$, and the Z_N eigenvalues ω^ℓ.

3 Zero-mode eigenstates on T^2/Z_N

Here, we focus on the zero-mode eigenstates for each Z_N eigenvalue, and study their number for each M. In particular, we will pay attention to the cases that the number of zero-mode eigenstates is given by around three, because we would like to construct a three generation model.

As discussed in section 2.2, the zero-mode eigenstates $\psi^{(j+\alpha_1,\alpha_\tau)}_{T^2/Z_N+,0}(z)_{\omega^\ell}$ with the Z_N eigenvalue ω^ℓ and $M>0$ will be given, in terms of the zero-mode functions $\psi^{(j+\alpha_1,\alpha_\tau)}_{T^2+,0}(z)$ on T^2, as

$$\psi^{(j+\alpha_1,\alpha_\tau)}_{T^2/Z_N+,0}(z)_{\omega^\ell} = \mathcal{N}^{(j)}_{+,\omega^\ell} \sum_{k=0}^{N-1} \bar{\omega}^{\ell k} \psi^{(j+\alpha_1,\alpha_\tau)}_{T^2+,0}(\omega^k z), \tag{20}$$

which obey the eigenvalue equations

$$\psi^{(j+\alpha_1,\alpha_\tau)}_{T^2/Z_N+,0}(\omega z)_{\omega^\ell} = \omega^\ell \psi^{(j+\alpha_1,\alpha_\tau)}_{T^2/Z_N+,0}(z)_{\omega^\ell}. \tag{21}$$

All of $\psi^{(j+\alpha_1,\alpha_\tau)}_{T^2/Z_N+,0}(z)_{\omega^\ell}$ with a fixed ℓ are not always linearly independent. To find the number of linearly independent zero-mode eigenstates $\psi^{(j+\alpha_1,\alpha_\tau)}_{T^2/Z_N+,0}(z)_{\omega^\ell}$, we need information on the relations between $\psi^{(j+\alpha_1,\alpha_\tau)}_{T^2+,0}(\omega^k z)$ and $\psi^{(j+\alpha_1,\alpha_\tau)}_{T^2+,0}(z)$. Since $\psi^{(j+\alpha_1,\alpha_\tau)}_{T^2+,0}(\omega^k z)$ for any j and k satisfies the same zero-mode equations and boundary conditions on T^2 as $\psi^{(i+\alpha_1,\alpha_\tau)}_{T^2+,0}(z)$, and since $\{\psi^{(i+\alpha_1,\alpha_\tau)}_{T^2+,0}(z)\}$ forms a complete set of the zero-mode eigenstates on T^2, $\psi^{(j+\alpha_1,\alpha_\tau)}_{T^2+,0}(\omega^k z)$ have to be expressed by some linear combination of $\psi^{(j+\alpha_1,\alpha_\tau)}_{T^2+,0}(z)$ such that

$$\psi^{(j+\alpha_1,\alpha_\tau)}_{T^2+,0}(\omega^k z) = \sum_{i=0}^{M-1} C^{ji}_k \psi^{(i+\alpha_1,\alpha_\tau)}_{T^2+,0}(z), \tag{22}$$

where C^{ji}_k are complex coefficients.

Inserting eq.(22) into eq.(21), we obtain[2]

$$\psi^{(j+\alpha_1,\alpha_\tau)}_{T^2/Z_N+,0}(z)_{\omega^\ell} = \mathcal{N}^{(j)}_{+,\omega^\ell} \sum_{k=0}^{N-1}\sum_{i=0}^{M-1} \bar{\omega}^{\ell k} C^{ji}_k \psi^{(i+\alpha_1,\alpha_\tau)}_{T^2+,0}(z). \tag{23}$$

A result for T^2/Z_3 is given in Table 1. Those tables show the number of linearly independent Z_N eigenfunctions $\psi^{(j+\alpha_1,\alpha_\tau)}_{T^2/Z_N\pm,0}(z)_\eta$ for each combination of $\eta=\omega^\ell$ and $|M|$. For example, when we want to construct a three-generation model on T^2/Z_3, we may choose one of $(|M|,\eta) = (6,1),(8,1),(10,1),(8,\bar{\omega}),(10,\bar{\omega}),(12,\bar{\omega})$ with $\omega=e^{2\pi i/3}$ in Tables 1.

[2]The numerical values of C^{ji}_k will be confirmed by another approach of the operator formalism given in Ref.[3].

$\|M\|$		2	4	6	8	10	12	14
η	1	1	1	3	3	3	5	5
	ω	0	2	2	2	4	4	4
	$\bar{\omega}$	1	1	1	3	3	3	5

Table 1: The number of linearly independent zero-mode eigenstates $\psi_{T^2/Z_3\pm,0}(z)_\eta$ for $M=$ even and $(\alpha_1,\alpha_\tau)=(0,0)$ on T^2/Z_3.

4 Kaluza-Klein mode functions and mass spectra

In the previous section, we have considered zero-mode solutions on T^2/Z_N. It is also worthwhile to discuss Kaluza-Klein modes on T^2 and T^2/Z_N. Then, we can understand the Kaluza-Klein modes by a way similar to the analysis of harmonic oscillator in the quantum mechanics, as we will see below.

Let us study the masses of Kaluza-Klein modes on $M^4\times T^2/Z_N$. In the same way, the masses for $\psi^{(j+\alpha_1,\alpha_\tau)}_{T^2/Z_N\pm,n}(z)$ are given by

$$\begin{pmatrix} -4D_z^{(b)}D_{\bar{z}}^{(b)} & 0 \\ 0 & -4D_{\bar{z}}^{(b)}D_z^{(b)} \end{pmatrix}\begin{pmatrix} \psi^{(j+\alpha_1,\alpha_\tau)}_{T^2/Z_N+,n}(z) \\ \psi^{(j+\alpha_1,\alpha_\tau)}_{T^2/Z_N-,n}(z) \end{pmatrix} = m_n^2 \begin{pmatrix} \psi^{(j+\alpha_1,\alpha_\tau)}_{T^2/Z_N+,n}(z) \\ \psi^{(j+\alpha_1,\alpha_\tau)}_{T^2/Z_N-,n}(z) \end{pmatrix}, \tag{24}$$

where $D_z^{(b)}=\partial_z - iqA_z^{(b)}(z)$ and $D_{\bar{z}}^{(b)}=\partial_{\bar{z}} - iqA_{\bar{z}}^{(b)}(z)$. Here, we define the two-dimensional Laplace operator as $\Delta\equiv -2(D_z^{(b)}D_{\bar{z}}^{(b)}+D_{\bar{z}}^{(b)}D_z^{(b)})$, which satisfies the relations with $D^{(b)}_{z(\bar{z})}$

$$[\Delta, D_z^{(b)}]=\frac{4\pi M}{\mathcal{A}}D_z^{(b)},\quad [\Delta, D_{\bar{z}}^{(b)}]=-\frac{4\pi M}{\mathcal{A}}D_{\bar{z}}^{(b)},\quad [D_z^{(b)},D_{\bar{z}}^{(b)}]=\frac{\pi M}{\mathcal{A}}, \tag{25}$$

where $\mathcal{A}$ $(=\mathrm{Im}\tau\cdot 1)$ is the area of the torus. This algebra of operators for $\psi_{T^2\pm,n}(z)$ is similar to one in the one-dimensional harmonic oscillator in quantum mechanics.

For example, it is found that for $M>0$,

$$\begin{aligned} &\Delta=\frac{4\pi|M|}{\mathcal{A}}\left(\hat{N}_+ +\frac{1}{2}\right),\quad \hat{N}_+\equiv \hat{a}_+^\dagger\hat{a}_+, \\ &\hat{a}_+\equiv i\sqrt{\frac{\mathcal{A}}{\pi|M|}}D_{\bar{z}}^{(b)},\quad \hat{a}_+^\dagger\equiv i\sqrt{\frac{\mathcal{A}}{\pi|M|}}D_z^{(b)},\quad [\hat{a}_+,\hat{a}_+^\dagger]=1, \end{aligned}\tag{26}$$

with

$$|0\rangle_+\equiv \psi^{(j+\alpha_1,\alpha_\tau)}_{T^2+,0}(z). \tag{27}$$

Since $\psi^{(j+\alpha_1,\alpha_\tau)}_{T^2/Z_N\pm,0}(z)_\eta$ are made by linear combinations of $\psi^{(j+\alpha_1,\alpha_\tau)}_{T^2\pm,0}(z)$, the Kaluza-Klein modes on the orbifolds should be made by operating $(\hat{a}_+^\dagger)^n$ on $|0\rangle_{+,\eta}\equiv \psi^{(j+\alpha_1,\alpha_\tau)}_{T^2/Z_N+,0}(z)_\eta$. Here, we should notice that for $\hat{a}_+^\dagger$, $D^{(b)}_{\omega^k z}$ are operated on $\psi^{(j+\alpha_1,\alpha_\tau)}_{T^2+,0}(\omega^k z)$. We define $\hat{a}_{\omega^k+}$ and $\hat{a}^\dagger_{\omega^k+}$ as

$$\hat{a}_{\omega^k+} = \omega^k \hat{a}_+, \quad \hat{a}^\dagger_{\omega^k+} = \bar{\omega}^k \hat{a}^\dagger_+. \tag{28}$$

Actually, operating $a_+^\dagger$ on $\psi^{(j+\alpha_1,\alpha_\tau)}_{T^2/Z_N\pm,0}(z)_\eta$, we obtain the first Kaluza-Klein modes

$$\begin{aligned}\hat{a}_+^\dagger|0\rangle_{+,\eta} &= \mathcal{N}^{(j)}_{+,\eta}\sum_{k=0}^{N-1}\bar{\eta}^k \hat{a}_+^\dagger \psi^{(j+\alpha_1,\alpha_\tau)}_{T^2+,0}(\omega^k z) \\ &= \mathcal{N}^{(j)}_{+,\eta}\sum_{k=0}^{N-1}(\omega\bar{\eta})^k \hat{a}^\dagger_{\omega^k+}\psi^{(j+\alpha_1,\alpha_\tau)}_{T^2+,0}(\omega^k z) \equiv \psi^{(j+\alpha_1,\alpha_\tau)}_{T^2/Z_N+,1}(z)_{\bar{\omega}\eta}.\end{aligned} \tag{29}$$

Thus, the Z_N eigenstate with the eigenvalue η at the nth Kaluza-Klein modes is made by operating $a_+^\dagger$ on $\psi^{(j+\alpha_1,\alpha_\tau)}_{T^2/Z_N+,n-1}(z)_{\omega\eta}$, or by operating $a_-^\dagger$ on $\psi^{(j+\alpha_1,\alpha_\tau)}_{T^2/Z_N-,n-1}(z)_{\bar{\omega}\eta}$. The Kaluza-Klein mode functions are given by

$$\begin{aligned}\psi^{(j+\alpha_1,\alpha_\tau)}_{T^2/Z_N+,n}(z)_{\bar{\omega}^n\eta} &\equiv \frac{1}{\sqrt{n!}}(\hat{a}_+^\dagger)^n \psi^{(j+\alpha_1,\alpha_\tau)}_{T^2/Z_N+,0}(z)_\eta \\ &= \mathcal{N}^{(j)}_{+,\eta}\sum_{k=0}^{N-1}(\omega^n\bar{\eta})^k \psi^{(j+\alpha_1,\alpha_\tau)}_{T^2-,n}(\omega^k z), \\ \psi^{(j+\alpha_1,\alpha_\tau)}_{T^2/Z_N-,n}(z)_{\bar{\omega}^n\eta} &= \frac{2}{m_n}D^{(b)}_{\bar{z}}\psi^{(j+\alpha_1,\alpha_\tau)}_{T^2/Z_N+,n}(z)_{\bar{\omega}^n\eta} \qquad \text{for} \quad M>0,\end{aligned} \tag{30}$$

Then, the Kaluza-Klein modes $\psi^{(j+\alpha_1,\alpha_\tau)}_{T^2/Z_N\pm,n}(z)_\eta$ for ${}^\forall\eta$ possess the masses squared

$$m_n^2 = \frac{4\pi|M|}{\mathcal{A}}n \qquad \text{for} \quad n\in\{0,\mathbb{N}\}. \tag{31}$$

Here, let us show an illustrative example. Figure 1 shows the zero-mode eigenstates $\psi^{(j,0)}_{T^2/Z_3+,0}(z)_\eta$ $(j=0,1)$ for $M=2$ in Table 1 and its Kaluza-Klein modes. The important fact is how Kaluza-Klein modes grow up. In the orbifolds, they grow up as changing the Z_N eigenstates.

5 Conclusions and discussions

We have studied the $U(1)$ gauge theory on the T^2/Z_N orbifolds with magnetic fluxes, Scherk-Schwarz phases and Wilson line phases. We have shown all of the possible Scherk-Schwarz and Wilson line phases. It is remarkable that the allowed Scherk-Schwarz phases as well as Wilson line phases depend on the magnitude of magnetic flux for the T^2/Z_3 and T^2/Z_6 orbifolds, that is, whether M is even or odd. At any rate, the variety of possible Scherk-Schwarz and

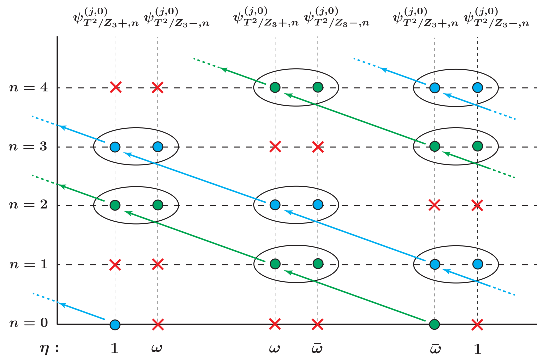

Figure 1: The mass spectra of $\psi^{(j,0)}_{T^2/Z_3\pm,n}(z)_\eta$ $(j=0,1)$ for $M=2$ in table 1. The red crosses mean the absence of zero-mode solutions and Kaluza-Klein modes, and the blue (green) filled circles correspond to a zero mode and its Kaluza-Klein modes. The blue (green) arrows mean that $\hat{a}^\dagger$ operates on nth modes $\psi^{(j,0)}_{T^2/Z_3+,n}(z)$ and the next modes $\psi^{(j,0)}_{T^2/Z_3+,n+1}(z)$ are made by it. The black ovals mean the pairs constructing mass terms.

Wilson line phases corresponds to the number of fixed points on each orbifold with any value of magnetic flux. Under these backgrounds, we have studied the behavior of zero modes. We have derived the number of zero modes for each eigenvalue of the Z_N twist. This result was obtained by showing explicitly and analytically wave functions for some examples and also by studying numerically Z_N-eigenfunctions for many models. The exactly same results will be derived by another approach for generic case [3]. The Kaluza-Klein modes were also investigated.

Our results show that one can derive models with three generations of matter fermions in various backgrounds, i.e., the T^2/Z_N orbifolds for $N=2,3,4,6$ with various magnetic fluxes and Scherk-Schwarz phases. Using these results, one could construct realistic three-generation models.

At any rate, our results can become a starting point for these studies. Also, our study is applicable to more general twisted orbifold models in higher-dimensional theory more than six-dimensional one, e.g., T^6/Z_N, $T^6/(Z_N \times Z'_N)$ and so on.

References

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