Velocity-Field Theory, Boltzmann's Transport Equation, Geometry and Emergent Time

Shoichi Ichinose

ichinose@u-shizuoka-ken.ac.jp
Laboratory of Physics, SFNS, University of Shizuoka

「場の理論と弦理論」 YITP workshop , Kyoto, Japan
ArXiv:1303.6616(hep-th), Aug. 20, 2013

Sec 1. Introduction: a.Boltzmann eq.

Boltzmann Equation, 1872

2nd Law of Thermodynamics

Dynamical Origin: Einstein Theory (Geometry of "dynamics") ?

- $\mathbf{u}(\mathbf{x},\ 't')$: Velocity distribution of Fluid Matter
- Size of fluid-particles: L Atomic (10^{-10}m)≪L≤ Optical Microscope (10^{-6})m
- Temporal development of Distribution Function $f('t', \mathbf{x}, \mathbf{v})$: probability of particle having velocity $\mathbf{v}$ at space $\mathbf{x}$ and time $'t'$

Sec 1. Introduction: b.Energy with Dissipation

Notion of Energy is obscure when Dissipation occurs.
Consider the movement of a particle under the influence of the friction force.
The emergent heat (energy) during the period [t_1, t_2] can **not** be written as.

$$\int_{x_1}^{x_2} F_{\text{friction}}\, dx = [E\{x(t), \dot{x}(t)\}]_{t_1}^{t_2} = E|_{t_2} - E|_{t_1}, \quad x_1 = x(t_1), x_2 = x(t_2) \tag{1}$$

where $x(t)$: Orbit (path) of Particle.

Sec 1. Introduction: c.Discrete Morse Flow

- Time should be re-considered, when dissipation occurs.
  $\rightarrow$ Step-Wise approach to time-development.
- Connection between step n and step $n-1$ is determined by the minimal energy principle.
- Time is "emergent" from the principle.
- Direction of flow (arrow of time) is built in from the beginning.

New approach to Statistical Fluctuation
Discrete Morse Flow Method(Kikuchi, '91)
Holography (AdS/CFT, '98)

Sec 2. Emergent Time and Diff. Eq. a.Energy Functional

1 dim viscous fluid, u(x): velocity field (distribution),
Energy Functional

$$I_n[u(x)] = \int dx \left\{ \frac{\sigma}{2\tilde{\rho}_0}(\frac{du}{dx})^2 + V(u) + u\frac{dV^1(x)}{dx} + \frac{1}{2h}(u-u_{n-1})^2 \right\} + I_n^0 \quad , \sigma \equiv 1,\ \tilde{\rho}_0 \equiv 1,\ n = 1, 2, \cdots$$

$$V(u) = \frac{m^2}{2}u^2 + \frac{\lambda}{4!}u^4 \quad , \quad u = u(x)\ ,\ u_{n-1} = u_{n-1}(x) \quad . \qquad (2)$$

$$\text{periodic bound. cond.} \quad u(x) = u(x+2l) \quad , \qquad (3)$$

Sec.2 Emer. T and Diff. Eq. : b.Variat. Principle

Variation $\delta I_n(u) = 0(u(x) \to u(x) + \delta u(x))$ gives Next step $u_n(x)$

$$\frac{1}{h}(u_n(x) - u_{n-1}(x)) = \frac{\sigma}{\tilde{\rho}_0}\frac{d^2 u_n}{dx^2} - \frac{\delta V(u_n)}{\delta u_n} - \frac{dV_n^1(x)}{dx} \quad , \qquad (4)$$

$$I_n[u_n] \quad \leq \quad I_n[u_{n-1}] \quad \text{but} \quad I_n[u_n] \; \leq \; I_{n-1}[u_{n-1}] \text{does NOT hold} \quad . \; (5)$$

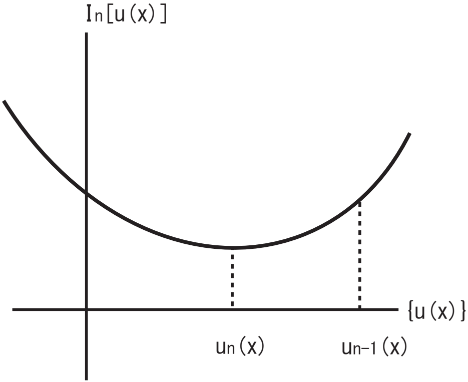
discrete time $t_n = nh = n\tau_0 \times (\frac{h}{\tau_0}),\ \tau_0 \equiv h\sqrt{\lambda\sigma}/m,\ t_0 \equiv 0. \qquad (6)$

Noting $u(x, t_n) \equiv u_n(x),\ t_n = t_{n-1} + h$, as $h \to 0$,

$$\frac{\partial u(x,t)}{\partial t} = \frac{\sigma}{\tilde{\rho}_0}\frac{\partial^2 u(x,t)}{\partial x^2} - \frac{\delta V(u(x,t))}{\delta u(x,t)} - \frac{\partial V^1(x,t)}{\partial x}, \text{ 1 D diff. eq. } (7)$$

Sec.2 Emer. T and Diff. Eq. : b.Variat. Principle

Figure: The energy functional $I_n[u(x)]$, (2), of the velocity-field $u(x)$.



Sec.2 Emer T and Diff. Eq.: c.Burger's Eq.

Noting $u(x) - u_{n-1}(x)$ in (2) should be $u(x + hu_{n-1}) - u_{n-1}(x)$, (4) is corrected as $(I_n \to \tilde{I}_n)$

$$\frac{1}{h}(u_n(x) - u_{n-1}(x)) + u_{n-1}(x)\frac{du_n(x)}{dx} = \frac{\sigma}{\tilde{\rho}_0}\frac{d^2u_n}{dx^2} - \frac{\delta V(u_n)}{\delta u_n} - \frac{dV_n^1(x)}{dx}$$

Continuous time limit $(h \to 0)$ gives Burgers's equation (1D Navier-Stokes eq.)

$$\frac{\partial u(x,t)}{\partial t} + u(x,t)\frac{\partial u(x,t)}{\partial x} = \frac{\sigma}{\tilde{\rho}_0}\frac{\partial^2 u(x,t)}{\partial x^2} - \frac{\delta V(u(x,t))}{\delta u(x,t)} - \frac{\partial V^1(x,t)}{\partial x} .$$

Eq. (9), for $m = 0$, is inv. under global Weyl transformation.

$$V^1(x,t) \to \mathrm{e}^{-2\varepsilon}V^1(\mathrm{e}^{\varepsilon}x, \mathrm{e}^{2\varepsilon}t),\ u(x,t) \to \mathrm{e}^{-\varepsilon}u(\mathrm{e}^{\varepsilon}x, \mathrm{e}^{2\varepsilon}t),$$
$$\partial_x \to \mathrm{e}^{-\varepsilon}\partial_x,\ \partial_t \to \mathrm{e}^{-2\varepsilon}\partial_t,\ t \to \mathrm{e}^{2\varepsilon}t,\ x \to \mathrm{e}^{\varepsilon}x \qquad (10)$$

Sec 3. Statistical Fluctuation Effect: a. Uncertainty

Large # of Particles→Statistical Average
Inevitable uncertainty of Present Approach

1. The finite time-increment gives uncertainty to the minimal solution $u_n(x)$.
2. The existence of the characteristic particle size gives uncertainty to the minimal solution
3. The system energy generally changes step by step.

Claim: the fluctuation comes not from the quantum effect but from the statistics caused by above points.

Sec 3. Stat. Fluct. Effect: b.Path-Integral

The statistics is taken into account by **newly** defining the n-th energy functional $\Gamma[u(x); u_{n-1}(x)]$ using the **path-integral**.

$$\mathrm{e}^{-\frac{1}{\alpha}\Gamma[u(x);u_{n-1}(x)]} = \int \mathcal{D}u(x)\mathrm{e}^{-\frac{1}{\alpha}\tilde{I}_n[u(x)]} \qquad (11)$$

Let us evaluate it perturbatively around the minimal path $u_n(x)$.

$$u(x) = u_n(x) + \sqrt{\alpha}q(x),\ |\sqrt{\alpha}q| \ll |u_n|,\ \left.\frac{\delta \tilde{I}_n[u]}{\delta u}\right|_{u=u_n} = 0 \qquad (12)$$

new expansion parameter α is introduced. $[\alpha]=[I_n]=\mathrm{ML}^2\mathrm{T}^{-2}$

Sec 3. Stat. Fluct. Effect: c.Not $\hbar$ But α

Claim: α should be small and should be chosen as

1) dimension is consistent
2) proportional to the small scale parameter which characterizes the relevant physical phenomena (ex. the mean-free path of the fluid particle). *NOT* include Planck constant, $\hbar$, because fluctuation does not come from the quantum effect
3) the precise value should be best-fitted with the experimental data

Sec 3. Stat. Fluct. Effect: d.Background Field

The background-field method gives, at the Gaussian(quadratic, 1-loop) approximation,

$$e^{-\frac{1}{\alpha}\Gamma[u_n(x);u_{n-1}(x)]} = e^{-\frac{1}{\alpha}\tilde{I}_n[u_n(x)]} \times (\det D)^{-1/2},$$
$$D \equiv -\frac{\sigma(=1)}{\tilde{\rho}_0(=1)}\frac{d^2}{dx^2} + \lambda {u_n}^2 + m^2 + \frac{1}{h} - \frac{du_{n-1}}{dx},$$
$$(\det D)^{-1/2} = \exp\left\{\frac{1}{2}\mathrm{Tr}\int_0^\infty \frac{e^{-\tau D}}{\tau}d\tau + \mathrm{const}\right\}, \tag{13}$$

($[\tau]$=$[D^{-1}]$=L/M.)

Sec 3. Stat. Fluct. Effect: e.Renormalizability

Taking the infrared cut-off parameter $\mu \equiv \sqrt{\sigma}/l$ and the ultraviolet cut-off parameter $\Lambda \equiv h^{-1}$ the mass parameter m^2 shifts under the influence of the fluctuation.

$$m^2 \quad \rightarrow \quad m^2 + \frac{\alpha}{\sqrt{\pi\epsilon\mu}}\epsilon\lambda = m^2 + \alpha\lambda\sqrt{\frac{l\tilde{\rho}_0}{\pi\sigma\sqrt{\sigma}}} \quad , \qquad (14)$$

When the functional (2) (effectively) works well, all effects of the statistical fluctuation reduces to the simple shift of the original parameters. This corresponds to the renormalizability condition in the field theory.

Sec 4. Boltzmann's Transport Equation: a.Step-Wise Approach

The step-wise development equation (8) with $V_n^1 = 0$, is written as

$$\frac{1}{h}(u_n(x) - u_{n-1}(x)) = \frac{d^2 u_n}{dx^2} - m^2 u_n - \frac{\lambda}{3!}{u_n}^3 - u_{n-1}\frac{du_n}{dx}$$

$$\text{or} \quad u_{n-1}(x) = \frac{u_n(x) - h\{\frac{d^2 u_n}{dx^2} - m^2 u_n - \frac{\lambda}{3!}{u_n}^3\}}{1 - h\frac{du_n}{dx}} \quad . \qquad (15)$$

The equilibrium state $u^\infty(x)$, after sufficient recursive computation ($n \gg 1$), satisfies

$$\frac{d^2 u^\infty}{dx^2} - m^2 u^\infty - \frac{\lambda}{3!}u^{\infty 3} - u^\infty \frac{d u^\infty}{dx} = 0 \quad , \qquad (16)$$

Sec 4. Boltzmann's Trans. Eq.: b.Distribution

The probability for the particle in the interval $x \sim x + dx$ and $v \sim v + dv$, at the step n, is given by

$$\frac{1}{\bar{N}_n} f_n(x, v) dx dv \quad , \quad f_n(x, v)\text{: distribution function} \tag{17}$$

Then the n-th *distribution* $f_n(x, v)$ and the equilibrium distribution $f^\infty(x, v)$ can be introduced as

$$u^\infty(x) = \frac{1}{\rho_\infty(x)} \int v f^\infty(x, v) dv, \ \ u_n(x) = \frac{1}{\rho_n(x)} \int v f_n(x, v) dv,$$
$$u_n(x) \to u^\infty(x) \text{ and } f_n(x, v) \to f^\infty(x, v) \text{ as } n \to \infty, \tag{18}$$

where $u^\infty(x)$ is the equilibrium velocity distribution. $\rho_n(x)$ is the particle number density. The continuity equation is given by

$$\frac{1}{h}(\rho_n(x) - \rho_{n-1}(x)) + \frac{d}{dx}(\rho_n(x) u_n(x)) = 0 \quad . \tag{19}$$

Sec 4. Boltzmann's Trans. Eq.: c.Equation

(15) is expressed, in terms of the distribution function, as

$$\frac{1}{h}\left[f_n(x+hu_{n-1}(x),v)-f_{n-1}(x,v)\right]=$$
$$\frac{\partial^2 f_n(x,v)}{\partial x^2}-m^2 f_n(x,v)-\frac{\lambda}{3!}f_n(x,v)u_n(x)^2 \quad ,$$
$$\text{where} \quad u_n(x)=\frac{1}{\rho_n(x)}\int v f_n(x,v)dv \quad , \qquad (20)$$

This is Boltzmann's transport equation. Physical quantities are

$$\text{Entropy}: \; S_n \equiv -k_B\int dv\int dx \; f_n(x,v)\ln f_n(x,v)$$
$$\text{Total particle } \#: \; \bar{N}_n=\int dx\;\rho_n(x)=\int dx\int dv\; f_n(x,v)$$
$$\text{Particle } \# \text{ density}: \; \rho_n(x)=\int dv\; f_n(x,v), \qquad (21)$$

Sec 4. Boltzmann's Trans. Eq.: d. Temperature

The momentum conservation at each point, x, requires

$$0 = \tilde{\rho}_n(x) \int dv(v - u_n(x)) f_n(x, v), \;\; u_n(x) = \frac{1}{\rho_n(x)} \int dv \; v \; f_n(x, v) \quad (22)$$

Some distributions are given by

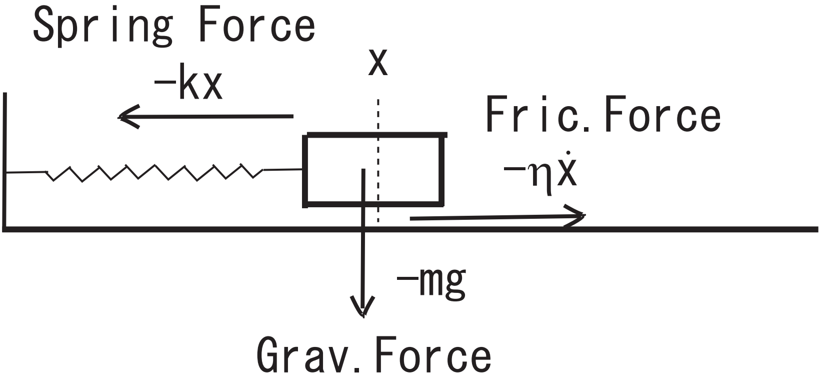
$$\text{Temperature}: \;\; \frac{1}{2} k_B \mathcal{T}_n(x) \equiv \frac{1}{\rho_n(x)} \int dv \frac{m_1}{2} (v - u_n(x))^2 f_n(x, v),$$

$$\text{Heat Current}: \;\; q_n(x) \equiv \int dv \frac{m_1}{2} (v - u_n(x))^3 f_n(x, v),$$

$$\text{Pressure}: \;\; P_n(x) \equiv m_1 \int dv (v - u_n(x))^2 f_n(x, v), \quad (23)$$

Sec 5. Trajectory Geometry: a.Figure

Figure: The harmonic oscillator with friction.



Sec 5. Traj. Geom.: b.Energy Funct. & Mini Princ.

The n-th energy function

$$K_n(x) = V(x) + \frac{\eta}{2h}(x - x_{n-1})^2 + \frac{m}{2h^2}(x - 2x_{n-1} + x_{n-2})^2 + K_n^0,$$

$$\text{Harmonic oscillator}: \quad V(x) = kx^2/2, \quad \text{Constant}: \quad K_n^0,$$

$$\text{friction coefficient}: \quad \eta, \quad \text{mass}: \quad m \qquad (24)$$

minimal principle : $\delta K_n = 0, \ x \to x + \delta x.$

$$\text{Disc-Time Evol}: \quad \left.\frac{\delta V}{\delta x}\right|_{x=x_n} + \frac{\eta}{h}(x_n - x_{n-1}) + \frac{m}{h^2}(x_n - 2x_{n-1} + x_{n-2}) = ($$

$$\text{Diff Eq of HO with Friction}: \quad \frac{dV(x)}{dx} + \eta\frac{dx}{dt} + m\frac{d^2x}{dt^2} = ($$

See Fig.2. This is a simple *dissipative* system.

Sec 5. Traj. Geom.: c.Fluctuation from QM

Fluctuation of Path comes from uncertainty principle of quantum mechanics in this case. (1 degree of freedom. No statistical procedure.)
Classical value x_n : $x = x_n + \sqrt{\hbar}\, q$ where $\hbar$ is Planck constant.

$$\mathrm{e}^{-\frac{1}{\hbar}h\Gamma(x_n;x_{n-1},x_{n-2})} = \int dx\ \mathrm{e}^{-\frac{1}{\hbar}hK_n(x)} = \int dq\ \mathrm{e}^{-\frac{1}{\hbar}hK_n(x_n+\hbar q)},$$

$$\Gamma_n \equiv \Gamma(x_n; x_{n-1}, x_{n-2}) = K_n(x_n) + \frac{\hbar}{2h}\ln(k + \frac{\eta}{h} + \frac{m}{h^2}) \qquad (26)$$

The quantum effect does not depend on the step number n.

Sec 5. Traj. Geom.: d.Metric in Energy

$x_n - x_{n-1} \equiv \Delta x_n$ and $(x_n - 2x_{n-1} + x_{n-2})/h \equiv v_n - v_{n-1} \equiv \Delta v_n$
We find the **metric** in the energy at n-step.

$$K_n(x_n) = V(x_n) + \frac{\eta}{2h}(x_n - x_{n-1})^2 + \frac{m}{2h^2}(x_n - 2x_{n-1} + x_{n-2})^2 + K_n^0$$
$$= \frac{1}{h^2}\{V(x_n)(\Delta t)^2 + \frac{\eta h}{2}(\Delta x_n)^2 + \frac{mh^2}{2}(\Delta v_n)^2\} + K_n^0 \quad ,(27)$$

$$(\Delta s_n)^2 \equiv 2h^2 K_n(x_n) = 2V(x_n{}'/\sqrt{\eta h})(\Delta t)^2 + (\Delta x_n{}')^2 + (\Delta v_n')^2 \quad ,$$
$$x_n{}' \equiv \sqrt{\eta h}x_n \quad , \quad v_n{}' \equiv \sqrt{mh^2}v_n \quad ,(28)$$

$V(x_n{}'/\sqrt{\eta h}) = (k'/2)x_n{}'^2$, $k' \equiv k/\eta h$.
Energy line-element Δs^2 in the $(t, x_n{}', v_n{}')$ space.
$\rightarrow$ the geometrical basis for fixing the statistical ensemble.

Sec 5. Traj. Geom.: e. Choice of K_n^0

Taking the value K_n^0 as

$$K_n^0 = -V(x_n) - \frac{m}{2h^2}(x_n - 2x_{n-1} + x_{n-2})^2 + V(x_0) + \frac{m}{2h^2}(x_1 - x_0)^2, (29$$

the graphs of movement and energy change, for various viscosities, are shown in Fig.3-9.

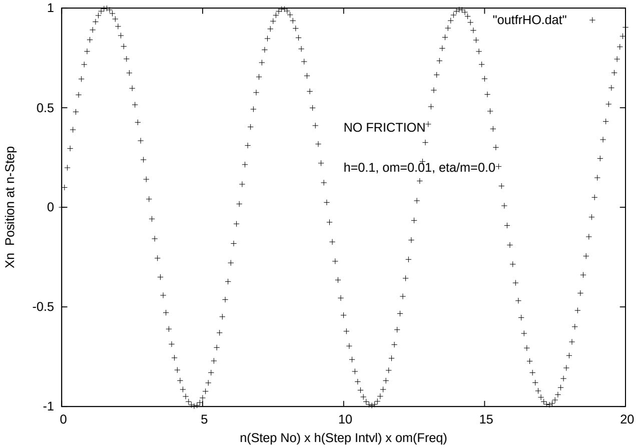
no friction case: the oscillator keeps the initial energy (Fig.4).

viscous case: the *energy changes step by step*, and finally reaches a constant *nonzero* value (Fig.6,Fig.7, Fig.9).

finally-remaining energy (constant) : dissipative one. Physically, the pressure and the temperature of the particle's "environment".

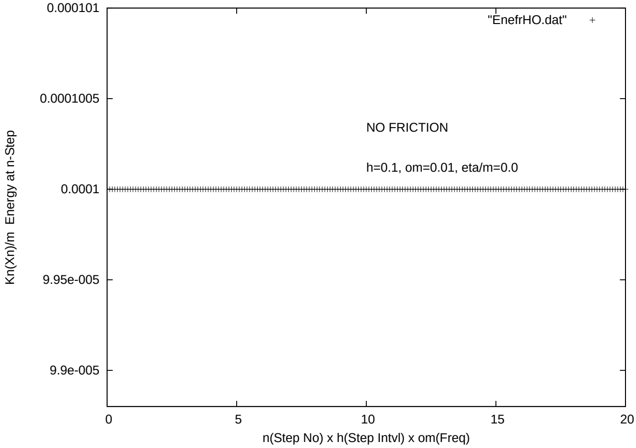
Sec 6. Mov & Ene Change: a.No Friction, Move

Figure: Harmonic oscillator with no friction, Movement



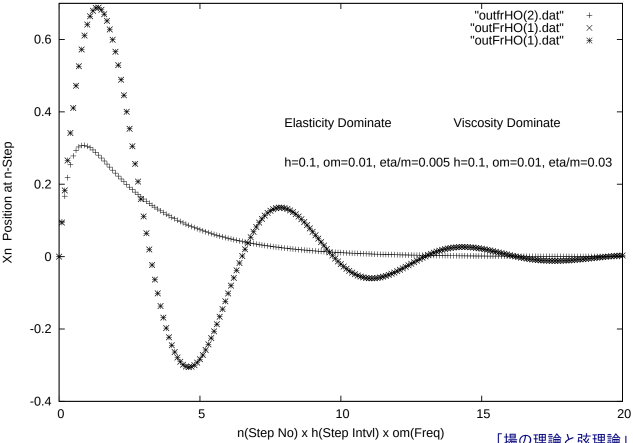
Sec 6. Mov & Ene Change: b.No Friction, Energy

Figure: Harmonic oscillator with no friction, Energy change



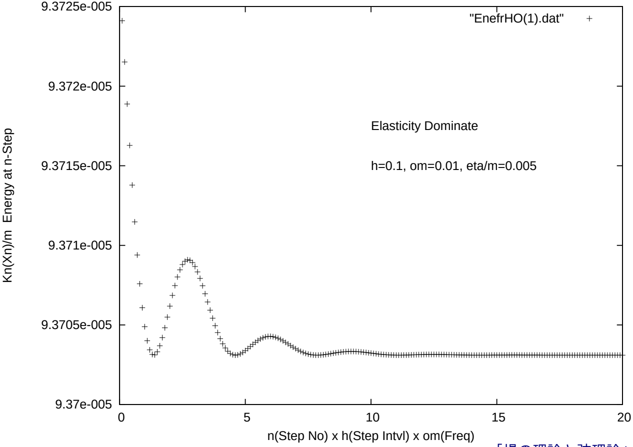
Sec 6. Mov & Ene Change: c.Friction, Move

Figure: Harmonic oscillator with friction, Movement, (1)Elasticity dominate and (2)Viscosity dominate



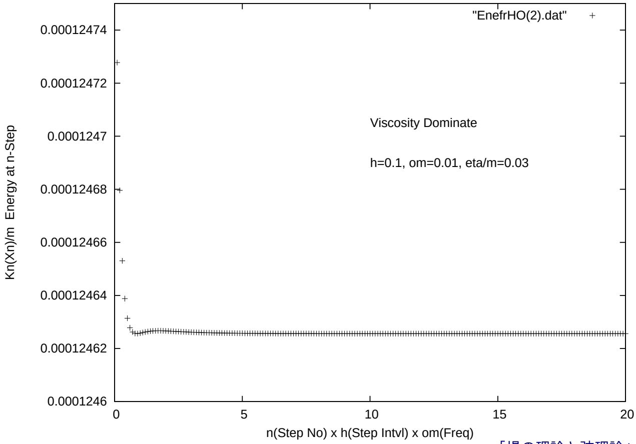
Sec 6. Mov & Ene Change: d.Elast. Dom., Ene

Figure: Harmonic oscillator with friction, Energy change, Elasticity dominate



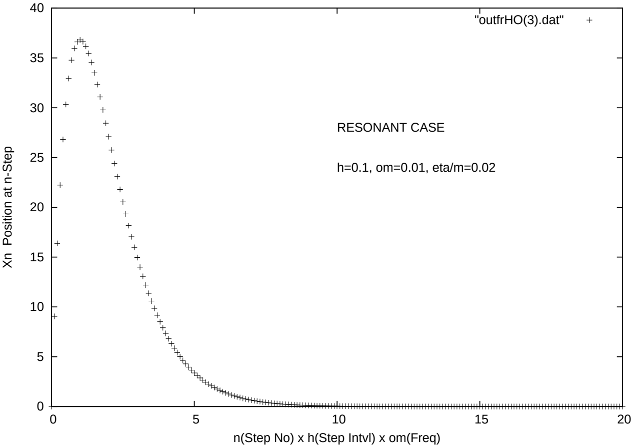
Sec 6. Mov & Ene Change: e. Visc. Dom., Ene

Figure: Harmonic oscillator with friction, Energy change, Viscosity dominate



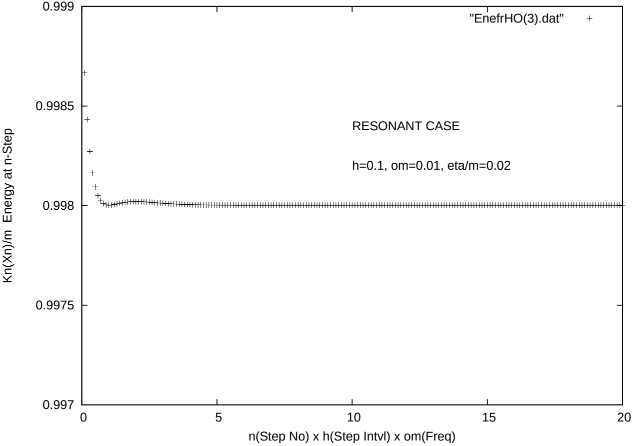
Sec 6. Mov & Ene Change: f.Reson, Move.

Figure: Harmonic oscillator with friction, Movement, Resonant



Sec 6. Mov & Ene Change: g.Reson, Ene

Figure: Harmonic oscillator with friction, Energy change, Resonant



Sec 7. Statistical Ensemble: a.Dirac-type Metric

Take **N 'copies'** of previous model. Consider the 'macro' system: **N$\gg$1**. They interact each other and exchange energy, but we assume the interaction is so moderate that every particle obeys the **common** field equation (25). They form a **statistical ensemble** caused by the arbitrariness of **initial condition**, Taking "**Dirac-type**" metric[SI,2010Apr].

$$(ds^2)_D \equiv 2V(X)dt^2 + dX^2 + dP^2 \quad - \text{on-path} \rightarrow (2V(y) + \dot{y}^2 + \dot{w}^2)dt^2,$$

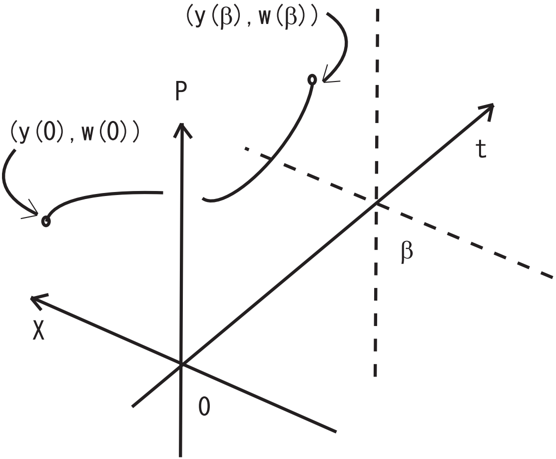
$$L_D = \int_0^\beta ds|_{on-path} = \int_0^\beta \sqrt{2V(y) + \dot{y}^2 + \dot{w}^2}dt,$$

$$d\mu = \mathrm{e}^{-\frac{1}{\alpha}L_D}\mathcal{D}y\mathcal{D}w \quad , \quad \mathrm{e}^{-\beta F} = \int \prod_n dy_n dw_n \mathrm{e}^{-\frac{1}{\alpha}L_D} \quad , \qquad (30)$$

α: a parameter with dimension of length ([α]=L). See Fig.10.

Sec 7. Stat. Ensemble: b.Path(line) in 3D Bulk

Figure: The path of line in 3D bulk space (X,P,t).



Sec 7. Stat. Ensemble: c.Standard Metric

Taking "**Standard-type**" metric,

$$(ds^2)_S \equiv \frac{1}{dt^2}[(ds^2)_D]^2 \quad - \text{on-path} \rightarrow$$
$$(2V(y) + \dot{y}^2 + \dot{w}^2)^2 dt^2,$$
$$L_S = \int_0^\beta ds|_{on-path} = \int_0^\beta (2V(y) + \dot{y}^2 + \dot{w}^2) dt,$$
$$d\mu = \mathrm{e}^{-\frac{1}{\alpha}L_S}\mathcal{D}y\mathcal{D}w \quad , \quad \mathrm{e}^{-\beta F} = \int \prod_n dy_n dw_n \mathrm{e}^{-\frac{1}{\alpha}L_S}. \qquad (31)$$

Exactly the same expression as the free energy expression in the Feynman's textbook.

Sec 7. Stat. Ensemble: d.Surfaces in 3D Bulk

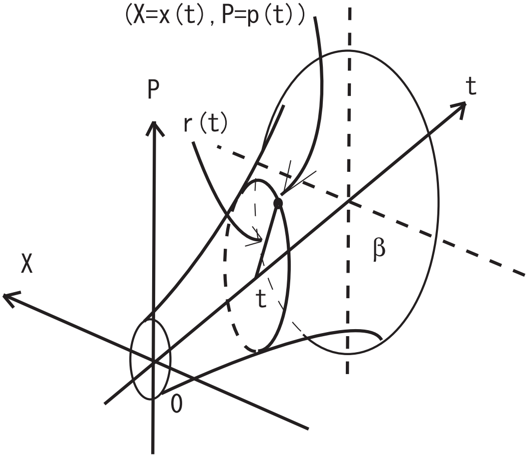
Another choice: **surfaces** instead of **lines**.

$$X^2 + P^2 = r^2(t) \quad , \quad 0 \le t \le \beta \tag{32}$$

We respect here the **isotropy** of the 2 dim phase space (X, P). See Fig.11.

Sec 7. Stat. Ensemble: e.Path(surface) in 3D Bulk

Figure: Two dimensional surface in 3D bulk space (X,P,t).



Sec 7. Stat. Ensemble: f.Induced Metric & Area

induced metric g_{ij} on the surface (32)

$$(ds^2)_D\big|_{\text{on-path}} = 2V(X)dt^2 + dX^2 + dP^2\big|_{\text{on-path}}$$
$$= \sum_{i,j=1}^{2} g_{ij}dX^i dX^j,\ (g_{ij}) = \begin{pmatrix} 1+\frac{2V}{r^2\dot{r}^2}X^2 & \frac{2V}{r^2\dot{r}^2}XP \\ \frac{2V}{r^2\dot{r}^2}PX & 1+\frac{2V}{r^2\dot{r}^2}P^2 \end{pmatrix} \quad (33)$$

where $(X^1, X^2) = (X, P)$. **Area** is given by

$$A = \int \sqrt{\det g_{ij}}\, d^2X = \int \sqrt{1+\frac{2V}{\dot{r}^2}}\, dXdP \quad , \quad (34)$$

Sec 7. Stat. Ensemble: g.Path-Integral Measure

Consider all possible surfaces. Statistical distribution is

$$\mathrm{e}^{-\beta F} = \int_0^\infty d\rho \int_{\substack{r(0)=\rho \\ r(\beta)=\rho}} \prod_t \mathcal{D}X(t)\mathcal{D}P(t)\mathrm{e}^{-\frac{1}{\alpha}A} \quad , \tag{35}$$

We have directly defined the distribution function $f(t, x, v)$ using geometry of the 3 dim bulk space.