Massive fermion in (4+1) dimensional Anti de-Sitter black hole background

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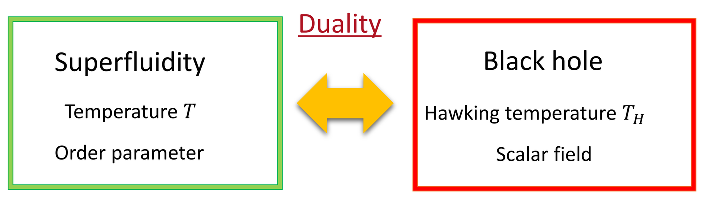
Introduction

Gauge/gravity duality

- The gauge/gravity duality is the theory that a strongly coupled gauge theory in *d* dimensions corresponds to a classical gravity theory in (*d+1*) dimensions.
- For example, the dual theory of superfluidity in (3+1) dimensions is Anti-de Sitter (AdS) black hole with Abelian Higgs hair in (4+1) dimensions.



Background

- Black hole with Abelian Higgs hair

$$S=\int d^5x\sqrt{-g}\Big[\frac{1}{16\pi G}(R-2\Lambda)+\Big(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}-(D_\mu\zeta)^*D^\mu\zeta-m^2\zeta^*\zeta\Big)\Big] \qquad \mu,\nu=0,1,2,3,4$$

- (4+1)dimensional black hole metric of polar coordinates

$$ds^2=-f(r)a^2(r)dt^2+\frac{1}{f(r)}dr^2+\frac{r^2}{4}(d\theta^2+d\varphi^2+d\psi^2+2\cos\theta d\varphi d\psi)$$

where $0\le\theta\le\pi, 0\le\varphi\le 2\pi, 0\le\psi\le 4\pi$

- The ansatz and boundary condition

ansatz $A_\mu dx^\mu=\phi(r)dt, \quad \zeta=\zeta(r)$

The behavior near horizon $r=r_0$

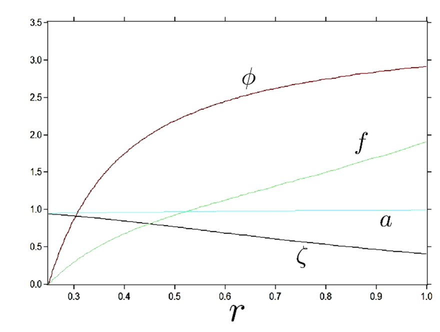
$$f=f_1^h(r-r_0)+\cdots,\quad a=a_0^h+a_1^h(r-r_0)+\cdots$$
$$\zeta=\zeta_0^h+\zeta_1^h(r-r_0)+\cdots,\ \phi=\phi_1^h(r-r_0)+\cdots$$

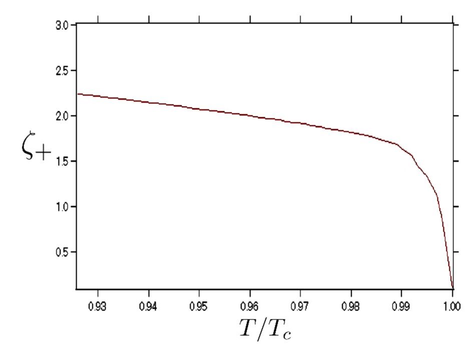
The behavior on the AdS boundary $r>>1$

$$f=1+\frac{r^2}{L^2}+\cdots,\quad a=1+\frac{a_2^b}{r^2}+\cdots$$
$$\zeta=\frac{\zeta_+}{r^3}+\cdots,\quad \phi=\mu+\frac{Q}{r^2}+\cdots$$

L : The AdS radius

The numerical solutions





ζ_+ : Scalar field on the AdS boundary corresponding to "condensate"

T : Temperature defined as $T=\frac{2L^2r_0^2+1}{2\pi r_0}$

r_0 : The radius of the black hole

T_c : Critical temperature

The scheme

The action of the fermion

- The action of the fermion:

$\hat{a}$: Tangent space indices
μ : Curved space indices

$$S_{\text{fermion}}=\int d^5x\sqrt{-g}\bar{\Psi}(\gamma^\mu D_\mu-m_f+g_c\zeta)\Psi$$

g_c : Coupling constant m_f : Mass of fermion $\gamma^{\hat{a}}$: Gamma matrices

$$D_\mu\Psi=(\partial_\mu+\frac{1}{8}\omega_{\mu\hat{a}\hat{b}}[\gamma^{\hat{a}},\gamma^{\hat{b}}]-A_\mu)\Psi$$

$\omega_{\mu\hat{a}\hat{b}}$: Spin connection defined as $\omega_{\mu\hat{a}\hat{b}}=\frac{1}{2}e^\nu_{\hat{a}}\nabla_\mu e_{\nu\hat{b}}$

$e^{\hat{a}}_\mu$: Vielbein defined as $g_{\mu\nu}=e^{\hat{a}}_\mu e^{\hat{b}}_\nu\eta_{\hat{a}\hat{b}}$

- We numerically solve the Dirac equation for the fixed background.

The gamma matrices and the vielbein

- The gamma matrices in tangent space

Define : $\{\gamma^{\hat{a}},\gamma^{\hat{b}}\}=2\eta^{\hat{a}\hat{b}}$

$$\gamma^{\hat{0}}=i\begin{pmatrix}I&0\\0&-I\end{pmatrix},\quad \gamma^{\hat{1}}=i\begin{pmatrix}0&\sigma^1\\-\sigma^1&0\end{pmatrix},\quad \gamma^{\hat{2}}=i\begin{pmatrix}0&\sigma^2\\-\sigma^2&0\end{pmatrix},\quad \gamma^{\hat{3}}=i\begin{pmatrix}0&\sigma^3\\-\sigma^3&0\end{pmatrix},\quad \gamma^{\hat{5}}=\begin{pmatrix}0&I\\I&0\end{pmatrix}$$

- The vielbein

$$\{e^{\hat{a}}_\mu\}=\begin{pmatrix}a\sqrt{f}&0&0&0&0\\0&\frac{1}{\sqrt{f}}\sin\frac{\theta}{2}\cos\frac{\psi-\varphi}{2}&\frac{r}{2}\cos\frac{\theta}{2}\cos\frac{\psi-\varphi}{2}&-\frac{r}{2}\sin\frac{\theta}{2}\sin\frac{\psi-\varphi}{2}&\frac{r}{2}\sin\frac{\theta}{2}\sin\frac{\psi-\varphi}{2}\\0&\frac{1}{\sqrt{f}}\sin\frac{\theta}{2}\sin\frac{\psi-\varphi}{2}&\frac{r}{2}\cos\frac{\theta}{2}\sin\frac{\psi-\varphi}{2}&\frac{r}{2}\sin\frac{\theta}{2}\cos\frac{\psi-\varphi}{2}&-\frac{r}{2}\sin\frac{\theta}{2}\cos\frac{\psi-\varphi}{2}\\0&\frac{1}{\sqrt{f}}\cos\frac{\theta}{2}\cos\frac{\psi+\varphi}{2}&-\frac{r}{2}\sin\frac{\theta}{2}\cos\frac{\psi+\varphi}{2}&-\frac{r}{2}\cos\frac{\theta}{2}\sin\frac{\psi+\varphi}{2}&-\frac{r}{2}\cos\frac{\theta}{2}\sin\frac{\psi+\varphi}{2}\\0&\frac{1}{\sqrt{f}}\cos\frac{\theta}{2}\sin\frac{\psi+\varphi}{2}&-\frac{r}{2}\sin\frac{\theta}{2}\sin\frac{\psi+\varphi}{2}&\frac{r}{2}\cos\frac{\theta}{2}\cos\frac{\psi+\varphi}{2}&\frac{r}{2}\cos\frac{\theta}{2}\cos\frac{\psi+\varphi}{2}\end{pmatrix}$$

Hamiltonian

- Dirac Hamiltonian within the ansatz $\Psi(t,\boldsymbol{x})=e^{iEt}\Psi(\boldsymbol{x})$

$$H\Psi=\begin{pmatrix}m_fa\sqrt{f}+\phi a\sqrt{f}+g_c\zeta&\bar{\boldsymbol{\sigma}}\cdot\boldsymbol{p}\\\bar{\boldsymbol{\sigma}}^\dagger\cdot\boldsymbol{p}&-m_fa\sqrt{f}+\phi a\sqrt{f}+g_c\zeta\end{pmatrix}\Psi=E\Psi$$

$\boldsymbol{p}$: Momentum in the background $\quad \bar{\boldsymbol{\sigma}}=(\sigma^1,\sigma^2,\sigma^3,-iI)$

- The numerical method
  - We solve the Dirac equation as the eigenproblem.
  - Diagonalization method

$$\det(\boldsymbol{A}-\epsilon\boldsymbol{B})=0 \quad \text{with} \quad A_{ij}=\int d^3x\Xi_i^\dagger H\Xi_j,\quad B_{ij}=\int d^3x\Xi_i^\dagger\Xi_j$$

- Thus, we need to find the complete basis Ξ_i.

The basis set

1. The symmetry

- The symmetry of the metric in (4+1) dimensions.

SO(4)≅SU(2)$_R$ × SU(2)$_L$

- The metric written by 1-form $\tau^L_\alpha,\tau^R_\alpha$

$$ds^2=-f(r)a^2(r)dt^2+\frac{1}{f(r)}dr^2+\frac{r^2}{4}((\tau_1^{R,L})^2+(\tau_2^{R,L})^2+(\tau_3^{R,L})^2)$$

$$\tau_1^R=-\sin\psi d\theta+\cos\psi\sin\theta d\phi$$
$$\tau_2^R=\cos\psi d\theta+\sin\psi\sin\theta d\phi$$
$$\tau_3^R=d\psi+\cos\theta d\phi$$
$$\tau_1^L=\sin\phi d\theta-\cos\phi\sin\theta d\psi$$
$$\tau_2^L=\cos\phi d\theta+\sin\phi\sin\theta d\psi$$
$$\tau_3^L=d\phi+\cos\theta d\psi$$

- The Killing vectors

$$\xi^R_x=-\sin\psi\partial_\theta+\frac{\cos\psi}{\sin\theta}\partial_\varphi-\cot\theta\cos\psi\partial_\psi$$
$$\xi^R_y=\cos\psi\partial_\theta+\frac{\sin\psi}{\sin\theta}\partial_\varphi-\cot\theta\sin\psi\partial_\psi$$
$$\xi^R_z=\partial_\psi$$

$$\xi^L_x=\cos\varphi\partial_\theta+\frac{\sin\varphi}{\sin\theta}\partial_\psi-\cot\theta\sin\varphi\partial_\varphi$$
$$\xi^L_y=-\sin\varphi\partial_\theta+\frac{\cos\varphi}{\sin\theta}\partial_\psi-\cot\theta\cos\varphi\partial_\varphi$$
$$\xi^L_z=\partial_\varphi$$

- We can define two angular momenta : $L^R_\alpha=i\xi^R_\alpha, L^L_\alpha=i\xi^L_\alpha$
- The commutation relations

$$[L^L_\alpha,L^L_\beta]=i\epsilon_{\alpha\beta\gamma}L^L_\gamma$$
$$[L^R_\alpha,L^R_\beta]=-i\epsilon_{\alpha\beta\gamma}L^R_\gamma$$
$$[L^L_\alpha,L^R_\beta]=0.$$

2. Wigner D function

- The special case $L^2=(L^R)^2=(L^L)^2$

Eigenvalues of $(L^R)^2$ and $(L^L)^2$ are the same

- The eigenfunction of L^2,L^R_z,L^L_z:
  - Wigner D function $D^l_{M,K}(\varphi,\theta,\psi)$

$$L^2D^l_{M,K}=l(l+1)D^l_{M,K}$$
$$L^L_zD^l_{M,K}=MD^l_{M,K}$$
$$L^R_zD^l_{M,K}=KD^l_{K,M}$$

- The eigenstate : $|l,M,K;G,M_{G_1},M_{G_2}\rangle$

G,M_{G_1},M_{G_2} defined by $\boldsymbol{G}=\boldsymbol{l}+\frac{\boldsymbol{\sigma}}{2}$

3. The form of Dirac spinor

- The plane-wave basis $\{\Xi_i\}=\{u_i\},\{v_i\}$

$$u_a=N_k\begin{pmatrix}i\omega^+_{E_k}\frac{J_{2G}(kr)}{r}|0\rangle_R\\\omega^-_{E_k}\frac{J_{2G+1}(kr)}{r}|2\rangle_R\end{pmatrix}\qquad v_a=N_k\begin{pmatrix}i\omega^+_{E_k}\frac{J_{2G+1}(kr)}{r}|2\rangle_L\\-\omega^-_{E_k}\frac{J_{2G}(kr)}{r}|0\rangle_L\end{pmatrix}$$

$$u_b=N_k\begin{pmatrix}i\omega^+_{E_k}\frac{J_{2G+2}(kr)}{r}|1\rangle_R\\-\omega^-_{E_k}\frac{J_{2G+1}(kr)}{r}|3\rangle_R\end{pmatrix}\qquad v_b=N_k\begin{pmatrix}i\omega^+_{E_k}\frac{J_{2G+1}(kr)}{r}|3\rangle_L\\\omega^-_{E_k}\frac{J_{2G+2}(kr)}{r}|1\rangle_L\end{pmatrix}$$

$|0\rangle_{R,L}\sim|3\rangle_{R,L}$: Angular basis
$J_i(r)$: Bessel function
k : Eigenvalue of momentum

N_k : Normalization constant
$\omega^+_{E_k>0},\omega^-_{E_k<0}=\mathrm{sgn}(E_k),$
$\omega^-_{E_k>0},\omega^+_{E_k<0}=\frac{k}{E_k+m}$

4. The angular basis

- SU(2) Clebsch-Gordan coefficient $C^{GM_{G_1}}_{lM\frac{1}{2}\sigma^3},C^{GM_{G_2}}_{lK\frac{1}{2}\sigma^3}$

The angular basis

$$|i\rangle_L=\sum_{M\sigma^3}C^{GM_{G_1}}_{lM\frac{1}{2}\sigma^3}|lMK\rangle|\sigma^3\rangle$$
$$|i\rangle_R=\sum_{K\sigma^3}C^{GM_{G_2}}_{lK\frac{1}{2}\sigma^3}|lMK\rangle|\sigma^3\rangle$$

$$|0\rangle_L=\sqrt{\frac{G+M_{G_1}}{2G}}D^{G-\frac{1}{2}}_{M_{G_1}-\frac{1}{2},M_{G_2}}\begin{pmatrix}1\\0\end{pmatrix}+\sqrt{\frac{G-M_{G_1}}{2G}}D^{G-\frac{1}{2}}_{M_{G_1}+\frac{1}{2},M_{G_2}}\begin{pmatrix}0\\1\end{pmatrix}$$
$$|1\rangle_L=-\sqrt{\frac{G-M_{G_1}+1}{2G+2}}D^{G+\frac{1}{2}}_{M_{G_1}-\frac{1}{2},M_{G_2}}\begin{pmatrix}1\\0\end{pmatrix}+\sqrt{\frac{G+M_{G_1}+1}{2G+2}}D^{G+\frac{1}{2}}_{M_{G_1}+\frac{1}{2},M_{G_2}}\begin{pmatrix}0\\1\end{pmatrix}$$
$$|2\rangle_L=\sqrt{\frac{G+M_{G_2}+\frac{1}{2}}{2G}}D^{G}_{M_{G_1},M_{G_2}+\frac{1}{2}}\begin{pmatrix}1\\0\end{pmatrix}-\sqrt{\frac{G-M_{G_2}+\frac{1}{2}}{2G}}D^{G}_{M_{G_1},M_{G_2}-\frac{1}{2}}\begin{pmatrix}0\\1\end{pmatrix}$$
$$|3\rangle_L=-\sqrt{\frac{G-M_{G_2}+\frac{1}{2}}{2G+2}}D^{G}_{M_{G_1},M_{G_2}+\frac{1}{2}}\begin{pmatrix}1\\0\end{pmatrix}-\sqrt{\frac{G+M_{G_2}+\frac{1}{2}}{2G+2}}D^{G}_{M_{G_1},M_{G_2}-\frac{1}{2}}\begin{pmatrix}0\\1\end{pmatrix}.$$
$$|0\rangle_R=\sqrt{\frac{G-M_{G_2}}{2G}}D^{G-\frac{1}{2}}_{M_{G_1},M_{G_2}+\frac{1}{2}}\begin{pmatrix}1\\0\end{pmatrix}+\sqrt{\frac{G+M_{G_2}}{2G}}D^{G-\frac{1}{2}}_{M_{G_1},M_{G_2}-\frac{1}{2}}\begin{pmatrix}0\\1\end{pmatrix}$$
$$|1\rangle_R=\sqrt{\frac{G+M_{G_2}+1}{2G+2}}D^{G+\frac{1}{2}}_{M_{G_1},M_{G_2}+\frac{1}{2}}\begin{pmatrix}1\\0\end{pmatrix}-\sqrt{\frac{G-M_{G_2}+1}{2G+2}}D^{G+\frac{1}{2}}_{M_{G_1},M_{G_2}+\frac{1}{2}}\begin{pmatrix}0\\1\end{pmatrix}$$
$$|2\rangle_R=-\sqrt{\frac{G-M_{G_1}+\frac{1}{2}}{2G}}D^{G}_{M_{G_1}-\frac{1}{2},M_{G_2}}\begin{pmatrix}1\\0\end{pmatrix}+\sqrt{\frac{G+M_{G_1}+\frac{1}{2}}{2G}}D^{G}_{M_{G_1}+\frac{1}{2},M_{G_2}}\begin{pmatrix}0\\1\end{pmatrix}$$
$$|3\rangle_R=-\sqrt{\frac{G-M_{G_1}+\frac{1}{2}}{2G+2}}D^{G}_{M_{G_1}-\frac{1}{2},M_{G_2}}\begin{pmatrix}1\\0\end{pmatrix}-\sqrt{\frac{G+M_{G_1}+\frac{1}{2}}{2G+2}}D^{G}_{M_{G_1}-\frac{1}{2},M_{G_2}}\begin{pmatrix}0\\1\end{pmatrix}.$$

Results

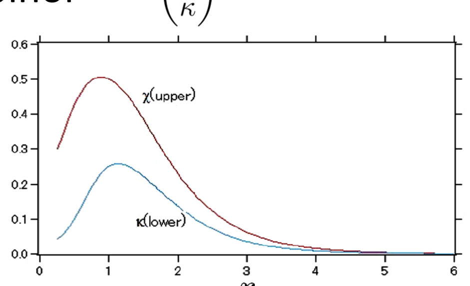
Numerical result 1

- Additional boundary condition for numerical calculation

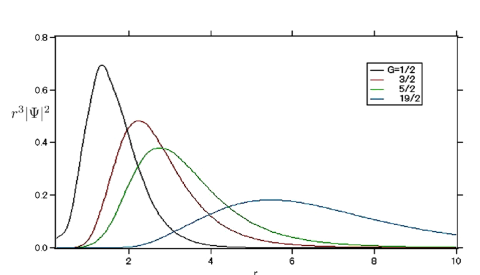
$$J_{2G+1}(k_iD)=0$$

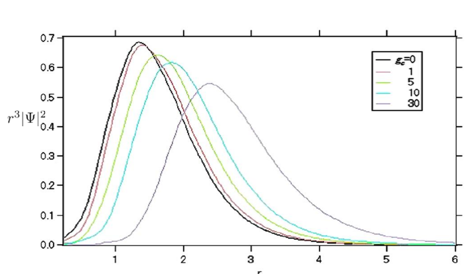
k_i : Discretized momentum $\quad D$: The radius of spherical box

- The components of Dirac spinor $\Psi=\begin{pmatrix}\chi\\\kappa\end{pmatrix}$ for $m_f=2,D=20,G=1/2,$ $r_0=0.25,g_c=5$



Numerical result 2





The density $r^3|\Psi|^2$ with some G and g_c.

Conclusions and future outlooks

- Our Yukawa coupling model

$$\mathcal{L}_{\mathrm{Y}}=\bar{\Psi}(\gamma^\mu\partial_\mu-m_f)\Psi+\frac{1}{2}\partial^\mu\zeta\partial_\mu\zeta-\frac{1}{2}m^2\zeta^2+gm\bar{\Psi}\zeta\Psi$$

can identify as a semi bosonized Gross-Neveu model

$$\mathcal{L}_{\mathrm{GN}}=\bar{\Psi}(\gamma^\mu\partial_\mu-m_f)\Psi+g^2(\bar{\Psi}\Psi)^2$$

in term of a bosonisation $\zeta\sim\bar{\Psi}\Psi$ and letting $m\to\infty$.
  - From this, we maybe discuss a mechanism of the condensation.
- Our metric is the static and spherical black hole.

We apply the basis to the rotating black hole in (4+1) dimensions.