Use of q-Virasoro algebra at root of unity limit for 2d-4d connection

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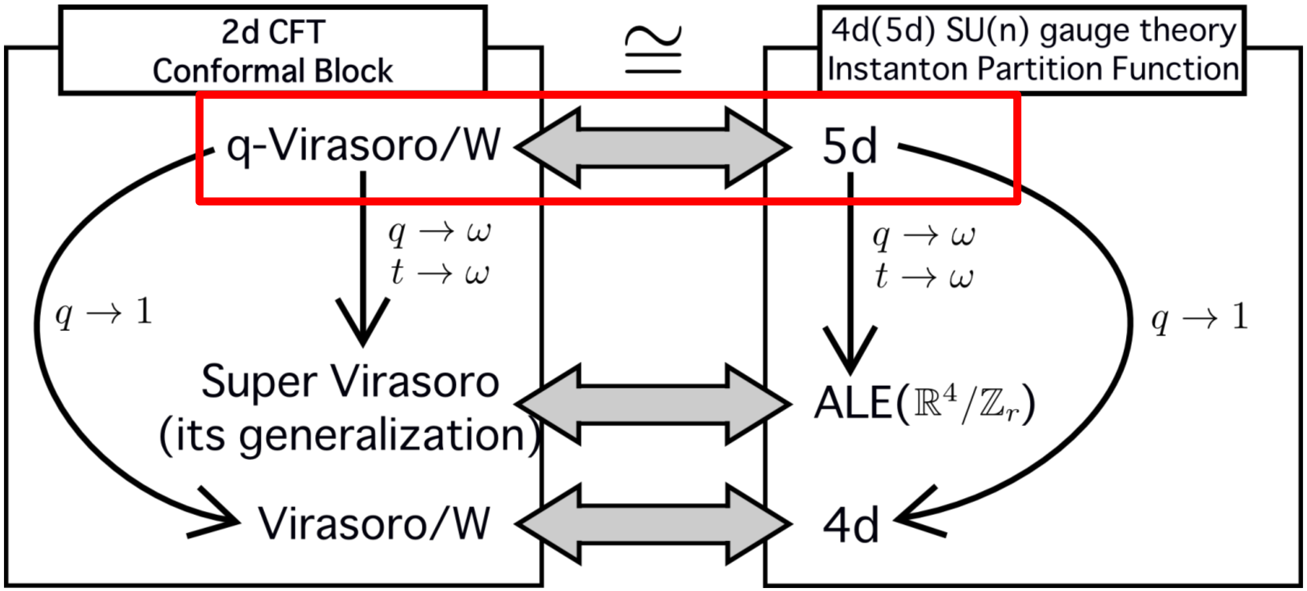
1. introduction

We regard q-Vir/W block–“5d” gauge theory correspondence as a parent one.



Awata-Yamada,...

Belavin-Feigin, Bonelli-Maruyoshi-Tanzini...

Alday,-Gaiotto-Tachikawa, Wyllard

$$\omega = e^{\frac{2\pi i}{r}} \qquad N_f = 2n$$

Procedure proposed:

(1) assume the q-(or K lifted) version of (W)AGT conjecture
(2) find the limiting procedure $q \to \omega$ for q-Virasoro/W block
(3) apply the same limiting procedure to $Z^{5d}_{\text{inst.}}$, which automatically generates ALE instanton partition function

Contents

2. q-Virasoro algebra

$$q,\ t=q^{\beta},\ p=q/t$$

Shiraishi, Kubo, Awata, Odake '96, Frenkel, Reshetikhin '96 …

$$f(w/z)\mathcal{T}(z)\mathcal{T}(w)-f(z/w)\mathcal{T}(w)\mathcal{T}(z)=\frac{(1-q)(1-t^{-1})}{(1-p)}\Big[\delta(pz/w)-\delta(p^{-1}z/w)\Big],$$

$$f(z)=\exp\left(\sum_{n=1}^{\infty}\frac{1}{n}\frac{(1-q^n)(1-t^{-n})}{(1+p^n)}z^n\right),\qquad \delta(z)=\sum_{n\in\mathbb{Z}}z^n$$

- q-deformed Heisenberg algebra

$$[\alpha_n,\alpha_m]=-\frac{1}{n}\frac{(1-q^n)(1-t^{-n})}{(1+p^n)}\delta_{n+m,0},\qquad (n\neq 0),$$

$$[\alpha_n,Q]=\delta_{n,0},$$

$$\mathcal{T}(z)=:\exp\left(\sum_{n\neq 0}\alpha_n z^{-n}\right):p^{1/2}q^{\sqrt{\beta}\alpha_0}+:\exp\left(-\sum_{n\neq 0}\alpha_n(pz)^{-n}\right):p^{-1/2}q^{-\sqrt{\beta}\alpha_0},$$

$$\Rightarrow\quad \mathcal{T}(z)=2+h^2\left(z^2L(z)+\frac{Q_E^2}{4}\right)+O(h^4)\qquad Q_E=\sqrt{\beta}-1/\sqrt{\beta}$$

$$q=e^{-h}\to 1,$$

$L(z)$: Virasoro operator

q-deformed free boson

- q-boson fields $$\widetilde{\varphi}^{(\pm)}(z) = \beta^{\pm 1/2} Q + 2\beta^{\pm 1/2}\alpha_0 \log z + \sum_{n\neq 0} \frac{(1+p^{-n})}{(1-\xi_{\pm}{}^{n})}\alpha_n z^{-n}$$

$$\xi_+ = q,\ \xi_- = t$$

$$\langle \widetilde{\varphi}^{(\pm)}(z_1)\widetilde{\varphi}^{(\pm)}(z_2)\rangle = \log\left[z_1^{2\beta^{\pm 1}} \left(1-\frac{z_2}{z_1}\right) \frac{(p^{\pm 1} z_2/z_1;\xi_{\pm})_\infty}{(\xi_{\mp} z_2/z_1;\xi_{\pm})_\infty}\right],$$
$$\langle \widetilde{\varphi}^{(\pm)}(z_1)\widetilde{\varphi}^{(\mp)}(z_2)\rangle = \log(z_1-z_2) + \log(z_1 - p^{\mp 1} z_2).$$

$$(x;\xi)_\infty = \prod_{k=0}^{\infty}(1-x\,\xi^k).$$

- introduce deformed screening current:

$$S_{\pm}(z) =: e^{\pm\widetilde{\varphi}^{(\pm)}(z)} :$$

$$Q^{+}_{[a,b]} = \int_a^b \mathrm{d}_q z\, S_+(z),$$

$$Q^{-}_{[a,b]} = \int_a^b \mathrm{d}_t z\, S_-(z),$$

Jackson integral: $$\int_0^a \mathrm{d}_{\xi_\pm} z\, f(z) = a(1-\xi_\pm)\sum_{k=0}^{\infty} f(a\xi_\pm^k)\xi_\pm^k$$

q→-1, t→-1 limit $\omega = -1$

Realized by

$$q = -\mathrm{e}^{-(1/\sqrt{\beta})h}, \quad t = -\mathrm{e}^{-\sqrt{\beta}h}, \quad p = q/t = \mathrm{e}^{Q_E h}. \quad h \to +0$$

$$t = q^{\beta} \quad \Rightarrow \quad \beta = \frac{k_- + 1/2}{k_+ + 1/2} = \frac{2k_- + 1}{2k_+ + 1}, \quad k_\pm : \text{non-negative integer}$$

- decompose the q-boson fields into even and odd parts,

$$\widetilde{\varphi}^{(\pm)}(z) = \widetilde{\varphi}^{(\pm)}_{\mathrm{even}}(z) + \widetilde{\varphi}^{(\pm)}_{\mathrm{odd}}(z)$$

$$\widetilde{\varphi}^{(\pm)}_{\mathrm{even}}(z) := \beta^{\pm 1/2} Q + \beta^{\pm 1/2}\alpha_0 \log(z^2) + \sum_{n\neq 0} \frac{1+p^{-2n}}{1-\xi_\pm{}^{2n}} \alpha_{2n} z^{-2n},$$

$$\widetilde{\varphi}^{(\pm)}_{\mathrm{odd}}(z) := \sum_{n\in\mathbb{Z}} \frac{1+p^{-2n-1}}{1-\xi_\pm{}^{2n+1}} \alpha_{2n+1} z^{-2n-1}$$

- $h \to +0$ limit $\quad \widetilde{\varphi}^{(\pm)}_{\mathrm{even}}(z) = \beta^{\pm 1/2}\phi(w) + O(h), \quad \widetilde{\varphi}^{(\pm)}_{\mathrm{odd}}(z) = \varphi(w) + O(h),$

$$w = z^2$$

$$\phi(w) = Q + a_0 \log w - \sum_{n\neq 0} \frac{a_n}{n} w^{-n},$$

$$[a_n, a_m] = n\delta_{n+m,0}, \quad [a_n, Q] = \delta_{n,0},$$

$$\varphi(w) = \sum_{n\in\mathbb{Z}} \frac{\tilde{a}_{n+1/2}}{n+1/2} w^{-n-1/2}.$$

$$[\tilde{a}_{n+1/2}, \tilde{a}_{-m-1/2}] = (n+1/2)\delta_{n,m}.$$

$$\langle \phi(w_1)\phi(w_2)\rangle = \log(w_1 - w_2), \quad \langle \varphi(w_1)\varphi(w_2)\rangle = \log\left(\frac{\sqrt{w_1}-\sqrt{w_2}}{\sqrt{w_1}+\sqrt{w_2}}\right).$$

- the screening currents in the $q \to -1$ limit:

$$\lim_{q\to -1} S_+(z) =: \mathrm{e}^{\sqrt{\beta}\phi(w)} \mathrm{e}^{\varphi(w)} :, \qquad \lim_{q\to -1} S_-(z) =: \mathrm{e}^{-(1/\sqrt{\beta})\phi(w)} \mathrm{e}^{-\varphi(w)} :$$

- We can construct two fermions,

$$\psi(w) \equiv \frac{\mathrm{i}}{2\sqrt{2w}}\Big(: \mathrm{e}^{\varphi(w)} : - : \mathrm{e}^{-\varphi(w)} :\Big), \quad \widehat{\psi}(w) \equiv \frac{1}{2\sqrt{2w}}\Big(: \mathrm{e}^{\varphi(w)} : + : \mathrm{e}^{-\varphi(w)} :\Big).$$

$$\langle 0|\psi(w_1)\psi(w_2)|0\rangle = \frac{1}{w_1 - w_2}, \qquad \langle 0|\widehat{\psi}(w_1)\widehat{\psi}(w_2)|0\rangle = \frac{1}{2(w_1-w_2)}\left(\sqrt{\frac{w_1}{w_2}} + \sqrt{\frac{w_2}{w_1}}\right),$$

$$\langle 0|\psi(w_1)\widehat{\psi}(w_2)|0\rangle = 0$$

- Jackson integral in the limit

$$\lim_{h\to 0}(1+q)\int_0^a \mathrm{d}_q z\, f(z) = \int_0^{a^2} g(w)\mathrm{d}w \qquad g(w) := \frac{f(z)-f(-z)}{2z}.$$

$$\frac{\mathrm{i}}{\sqrt{2}}(1+q)Q^{+}_{[a,b]} \longrightarrow Q^{(+)}_{[a^2,b^2]} \equiv \int_{a^2}^{b^2} \mathrm{d}w\, \psi(w) : \mathrm{e}^{\sqrt{\beta}\phi(w)} :$$

screening charge for the superconformal block

Kitazawa, Ishibashi, Kato, Kobayashi , Matsuo, Odake ’88
Alvarez-Gaume, Zaugg ’92

super Virasoro algebra

fermionic current for both NS & R

$$G(w) = \psi(w)\partial\phi(w) + Q_E\,\partial\psi(w),$$ appears

Through OPE,

$$T(w) = \frac{1}{2} : \left(\partial\phi(w)\right)^2 : + \frac{Q_E}{2}\partial^2\phi(w) - \frac{1}{2} : \psi(w)\partial\psi(w) :,$$

is generated.

$T(w)$ and $G(w)$ forming $\mathcal{N} = 1$ superconformal algebra.

- central charge: $c = \frac{3}{2}\hat{c}$

$$\hat{c} = 1 - 2Q_E^2 = 1 - \frac{8(k_- - k_+)^2}{(2k_+ + 1)(2k_- + 1)} = 1 - \frac{2(m' - m)^2}{mm'}$$

$$m = 2k_+ + 1, \quad m' = 2k_- + 1$$

getting the one for the minimal model (odd integers only).

q-vertex operator

Defined by $V_\alpha(z) =: \mathrm{e}^{\Phi_\alpha(z)} :,$

$$\Phi_\alpha(z) = \alpha\, Q + 2\alpha\, \alpha_0 \log z + \sum_{n\neq 0} \frac{q^{-n}(1-q^{2\sqrt{\beta}\alpha|n|})}{(1-q^{-|n|})(1-t^{-n})} \alpha_n z^{-n}.$$

We restrict the parameter α to take values corresponding to those of the primary fields of the minimal theories in the NS sector:

$$\alpha = \alpha_{r,s} = -\left(\frac{1-r}{2}\right)\frac{1}{\sqrt{\beta}} + \left(\frac{1-s}{2}\right)\sqrt{\beta} \qquad r-s \in 2\mathbb{Z}$$

In the $q \to -1$ limit,

$$\lim_{q\to -1} V_{\alpha_{r,s}}(z) =: \mathrm{e}^{\alpha_{r,s}\phi(w)} : \quad \text{for } L_{r,s} \text{ even} \qquad L_{r,s} \equiv (2k_+ + 1)\sqrt{\beta}\alpha_{r,s}$$

which is exactly equal to the Coulomb gas representation of the bosonic primary field in the NS sector with scaling dimension,

$$\Delta_{\alpha_{r,s}} = \frac{1}{2}\alpha_{r,s}(\alpha_{r,s} - Q_E) = -\frac{1}{8}Q_E^2 + \frac{1}{8}\left(-\frac{r}{\sqrt{\beta}} + s\sqrt{\beta}\right)^2.$$

3. conformal block: integral representation

$\phi(z)$: $\mathfrak{h}$-valued chiral boson, $\mathfrak{h}$: Cartan subalgebra of SU(n)

$$\langle \phi_a(z)\phi_b(w)\rangle = (e_a, e_b)\log(z-w), \quad \phi_a(z) = \langle e_a, \phi(z)\rangle,$$
$$e_a \in \mathfrak{h}^* : \ \text{a simple root of } A_{n-1}, \quad a = 1, \cdots, n-1$$

- vertex operator: $V_\alpha(z) =: \mathrm{e}^{\langle \alpha, \phi(z)\rangle} : \quad (\alpha \in \mathfrak{h}^*)$
- screening charge: $\mathcal{Q}_a = \int_0^\Lambda \mathrm{d}z\, V_{\sqrt{\beta}e_a}(z), \qquad \widetilde{\mathcal{Q}}_a = \int_1^\infty \mathrm{d}z\, V_{\sqrt{\beta}e_a}(z)$
- the block:

$$\mathcal{F}(c, \Delta_I, \Delta_i|\Lambda) = \left\langle V_{(1/\sqrt{\beta})\alpha_1}(0) V_{(1/\sqrt{\beta})\alpha_2}(\Lambda) V_{(1/\sqrt{\beta})\alpha_3}(1) V_{(1/\sqrt{\beta})\alpha_4}(\infty) \prod_{a=1}^{n-1} \mathcal{Q}_a{}^{N_a} \widetilde{\mathcal{Q}}_a{}^{\widetilde{N}_a} \right\rangle,$$

rep. theoretic part of 4 point function (model independent)

Λ : cross ratio

$$\mathcal{F} = Z_S(\Lambda) = \left\langle\left\langle F(x,y|\Lambda)\right\rangle_+\right\rangle_-$$

$$F(x,y|\Lambda) := \prod_{a=1}^{n-1}\left\{\prod_{I=1}^{N_a}(1-\Lambda x_I^{(a)})^{v_{a-}}\prod_{J=1}^{\widetilde{N}_a}(1-\Lambda y_J^{(a)})^{v_{a+}}\right\}\prod_{a=1}^{n-1}\prod_{b=1}^{n-1}\prod_{I=1}^{N_a}\prod_{J=1}^{\widetilde{N}_b}(1-\Lambda x_I^{(a)}y_J^{(b)})^{\beta C_{ab}},$$

$$\langle f(x)\rangle_+ = \prod_{a=1}^{n-1}\prod_{I=1}^{N_a}\int_0^1 dx_I^{(a)}\prod_{I=1}^{N_a}(x_I^{(a)})^{u_{a+}}(1-x_I^{(a)})^{v_{a+}}\prod_{1\le I<J\le N}(x_I^{(a)}-x_J^{(a)})^{2\beta}f(x)$$

$$\langle f(y)\rangle_- = \prod_{a=1}^{n-1}\prod_{I=1}^{\widetilde{N}_a}\int_0^1 dx_I^{(a)}\prod_{I=1}^{\widetilde{N}}(y_I^{(a)})^{u_{a-}}(1-y_I^{(a)})^{v_{a-}}\prod_{1\le I<J\le \widetilde{N}}(y_I^{(a)}-y_J^{(a)})^{2\beta}f(y)$$

$$v_{a+} = (\alpha_2, e_a),\quad v_{a-} = (\alpha_3, e_a)$$
$$u_{a+} = (\alpha_1, e_a),\quad u_{a-} = (\alpha_4, e_a)$$

C_{ab} : Cartan matrix

For $n = 2$, AGT at lower orders successfully checked due to Kadell formula

Itoyama, Oota '10

For general n, some progress on the generalization

Zhang, Matsuo '12

q-lift(deformation)

Z_S

$\downarrow$

Mironov, Morozov, Shakirov, Smirnov '11

$$Z_S^{(q)} = \left\langle\left\langle \prod_{a=1}^{n-1} \left\{ \prod_{I=1}^{N_a} \prod_{i=0}^{v_{a-}-1} (1-\Lambda x_I^{(a)} q^i) \prod_{J=1}^{\widetilde{N}_a} \prod_{j=0}^{v_{a+}-1} (1-\Lambda y_J^{(a)} q^j) \right\} \times \right.\right.$$

$$\left.\left. \times \prod_{a,b=1}^{n-1} \prod_{\ell=0}^{\beta-1} \prod_{I=1}^{N_a} \prod_{J=1}^{\widetilde{N}_a} (1-\Lambda x_I^{(a)} y_J^{(b)} q^\ell)^{C_{ab}} \right\rangle_{N+,q} \right\rangle_{\widetilde{N}-,q}$$

Here

$$\Big\langle f(x) \Big\rangle_{N\pm,q} = \left(\prod_{I=1}^{N} \int_0^1 d_q x_I \right) \prod_{I=1}^{N} x_I^{u_{a\pm}} \prod_{i=1}^{v_{a\pm}-1} (1-x_I q^i) \prod_{1\leq I\neq J\leq N} \prod_{i=1}^{\beta-1} (x_I - q^i x_J) f(x)$$

Kaneko '96, Warnaar '05…

superconformal block

$$Z_S^{(q)} = \sum_{k=0}^{\infty} \Lambda^k \sum_{\substack{\vec{Y} \\ |\vec{Y}|=k}} \left\langle \prod_{a=1}^{n} M_{Y_a} \left(-r_k^{(a)} - \frac{[v_{a+}]'_{q^k}}{[\beta]_{q^k}} \right) \right\rangle_{N+,q} \times$$

$$\times \left\langle \prod_{a=1}^{n} M_{Y_a} \left(\tilde{r}_k^{(a)} + \frac{[v_{a-}]'_{q^k}}{[\beta]_{q^k}} \right) \right\rangle_{\tilde{N}-,q}$$

conjectured equality

M_{Y_α} : Macdonald polynomial

$$= \quad Z_{\text{inst.}}^{5d} = \sum_{k=0}^{\infty} \widetilde{\Lambda}^k \sum_{\substack{\vec{Y} \\ |\vec{Y}|=k}} Z_{\vec{Y}}^{5d} \quad \Rightarrow \quad \text{ALE instanton}$$

- Both sides in the limit are identified and the expansions generated with no difficulty
- Once the dictionary is found, the conjecture at q-lifted case provides the equality between the block & ALE instanton partition function.

e.g. SU(2):

$$\beta N = -a + m_2, \quad \beta \tilde{N} = a + m_3$$

$$\beta u_+ = m_1 - m_2 - (1-\beta), \quad \beta v_+ = -(m_1 + m_2)$$

$$\beta v_- = -(m_3 + m_4), \quad \beta u_- = m_4 - m_3 - (1-\beta)$$

$$m_i = m_i^{5\text{d}} + \frac{1}{2}(1-\beta)$$

4.

brief review of SU(n) instanton partition function on ALE:

Kronheimer, Nakajima ’90
Fucito, Morales, Poghossian ’04

- localization
- torus action generated by $(\epsilon_1, \epsilon_2, a_1, \cdots, a_n)$
- fixed points labeled by an n-tuple of Young diagrams $Y_\alpha, \ \alpha = 1, \cdots, n$
- weight of $(i,j) \in Y_\alpha; \quad a_\alpha + (i-1)\epsilon_1 + (j-1)\epsilon_2$
- $\boldsymbol{Z}_r$ orbifold action is

$$\epsilon_1 \to \epsilon_1 - \frac{2\pi \mathrm{i}}{r}, \quad \epsilon_2 \to \epsilon_2 + \frac{2\pi \mathrm{i}}{r}, \quad a_\alpha \to a_\alpha + q_\alpha \frac{2\pi \mathrm{i}}{r}.$$

- $\boldsymbol{Z}_r$ charge carried by the box $(i,j) \in Y_\alpha$

$$q_{\alpha,(i,j)} = q_\alpha - (i-1) + (j-1).$$

more on labeling of ALE instanton:

n_ℓ : the number of Young diagrams $\{Y_\alpha\}$ such that $\boldsymbol{Z}_r$ charge $q_\alpha = \ell$

k_ℓ : the total number of the boxes such that $\boldsymbol{Z}_r$ charge $q_{\alpha,(i,j)} = \ell$

$$k = \sum_{\ell=0}^{r-1} k_\ell, \quad n = \sum_{\ell=0}^{r-1} n_\ell$$

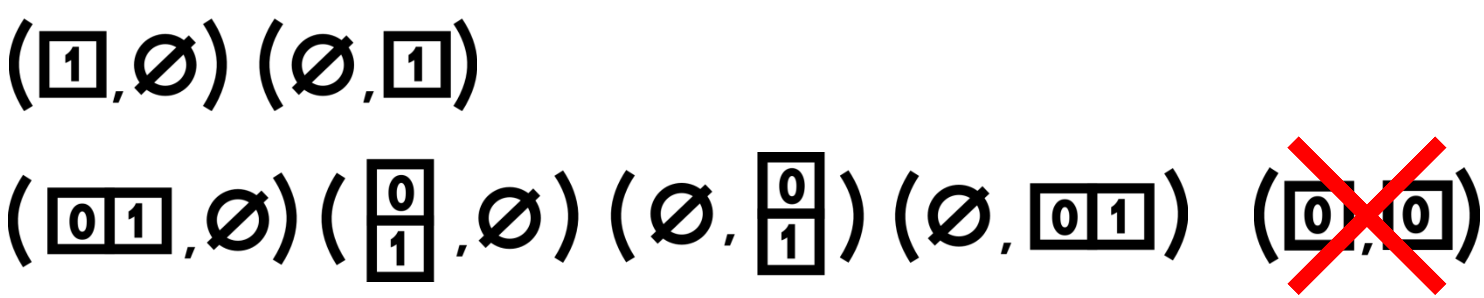
- condition of vanishing 1st Chern class

$$n_\ell - 2k_\ell + k_{\ell+1} + k_{\ell-1} = 0$$

- $n = 2,\ r = 2$ (SU(2) in $\boldsymbol{R}^4/\boldsymbol{Z}_2$) case

  - $(n_0, n_1) = (0, 2), \quad (k_0, k_1) = (k_0, k_0 + 1)$
  - $(n_0, n_1) = (2, 0), \quad (k_0, k_1) = (k_0, k_0)$

$$k_0 = 0, 1, 2, \cdots$$



limiting procedure from 5d instanton partition function:

$$Z^{\mathbb{R}^5} = \sum_{k=0}^{\infty} \widetilde{\Lambda}^k \sum_{|\vec{Y}|=k} \mathcal{A}_{\vec{Y}},$$

$$\mathcal{A}_{\vec{Y}} = \frac{\prod_{s=1}^{n} \prod_{k=1}^{n} f_{Y_s}^{q+}(m_k + a_s) f_{Y_s}^{q-}(m_{k+n} + a_s)}{\prod_{s,t=1}^{n} g_{Y_s Y_t}^{q}(a_t - a_s)},$$

$$g_{YW}^{q}(x) = \prod_{(i,j)\in Y} [x + \beta\ell_Y(i,j) + a_W(i,j) + \beta]_q [-x - \beta\ell_Y(i,j) - a_W(i,j) - 1]_q,$$

$$f_A^{q\pm}(x) = \prod_{(i,j)\in A} \left[\pm x \mp i\beta \pm j \mp \frac{1}{2}(1-\beta)\right]_q, \qquad [x]_q = \frac{1-q^x}{1-q}.$$

e.g. Awata, Kanno '08

This reduces to that on $\boldsymbol{R}^4/\boldsymbol{Z}_r$ by

$$q = \omega e^{h\epsilon_2}, \quad t = \omega e^{-h\epsilon_1}, \quad q^{\frac{a_\alpha}{\epsilon_2}} = \omega^{q_\alpha} e^{h a_\alpha}, \qquad h \to +0.$$

Kimura

This automatically generates the projection onto 4d ALE.

ALE instanton partition function:

- SU(2) & r =2

$$Z^{(2)}_{k+1/2} := \lim_{h\to 0} \frac{h^2}{2^2} \sum_{|A|+|B|=2k+1} \mathcal{A}^{(1,1)}_{AB}$$

$$Z^{(2)}_{k} := \lim_{h\to 0} \sum_{|A|+|B|=2k} \mathcal{A}^{(0,0)}_{AB}$$

$$\mathcal{A}^{(0,0)}_{(1)(1)} \to 0$$

$$\Rightarrow \quad Z^{\mathbb{R}^4/\mathbb{Z}_2}_{SU(2)} = \sum_{k=0}^{\infty} (\Lambda')^k \, Z^{(2)}_{k} + \sum_{k=0}^{\infty} (\Lambda')^{k+1/2} \, Z^{(2)}_{k+1/2}$$

- SU(2) & general r

$$Z^{(r)}_{k+\frac{q_a(r-q_a)}{r}} := \lim_{h\to 0} \Xi_{q_a} \sum_{\substack{|A|+|B| \\ = rk+q_a(r-q_a)}} \mathcal{A}^{(q_a,-q_a)}_{AB}$$

$$\Xi_0 = 1, \qquad \Xi_1 = h^2 \frac{1}{(1-\omega)(1-\omega^{-1})},$$

$$\Xi_i = h^{2i} \frac{\prod_{k=1}^{i-1} (1-\omega^k)^{2i-3k} (1-\omega^{-k})^{2i-3k}}{\prod_{l=1}^{i} (1-\omega^{i+l-1})(1-\omega^{-(i+l-1)})} \qquad 2 \le i \le \left\lfloor \frac{r}{2} \right\rfloor,$$

$$Z^{\mathbb{R}^4/\mathbb{Z}_r}_{SU(2)} = \sum_{q_a=0}^{\lfloor \frac{r}{2} \rfloor} \sum_{k=0}^{\infty} (\Lambda')^{k+\frac{q_a(r-q_a)}{r}} \, Z^{(r)}_{k+\frac{q_a(r-q_a)}{r}}$$

- SU(n) & general r

$$Z^{(r)}_{k+\frac{d}{r}} := \lim_{h\to 0} \Xi_{q_\alpha} \sum_{\substack{|\vec{Y}| \\ =rk+d}} \mathcal{A}^{(q_\alpha)}_{\vec{Y}}$$

$$q_\alpha = (q_1, q_2, \cdots, q_n),$$
$$q_n = -q_1 - \cdots - q_{n-1}$$

$$d = \sum_{\alpha=1}^{n-1} q_\alpha \left(r - \sum_{\alpha'=1}^{\alpha} q_{\alpha'} \right)$$

$$\Xi_{q_\alpha} = h^{2\sum_{\alpha=1}^{n-1} \alpha q_\alpha} \xi_{q_\alpha}(\omega) \xi_{q_\alpha}(\omega^{-1})$$

explicit form obtained

$$Z^{\mathbb{R}^4/\mathbb{Z}_r}_{SU(n)} = \sum_{q_\alpha} \sum_{k=0}^{\infty} (\Lambda')^{k+\frac{d}{r}} Z^{(r)}_{k+\frac{d}{r}}$$

q-lift (5d K-theoretic lift) is useful to 2d-4d connection

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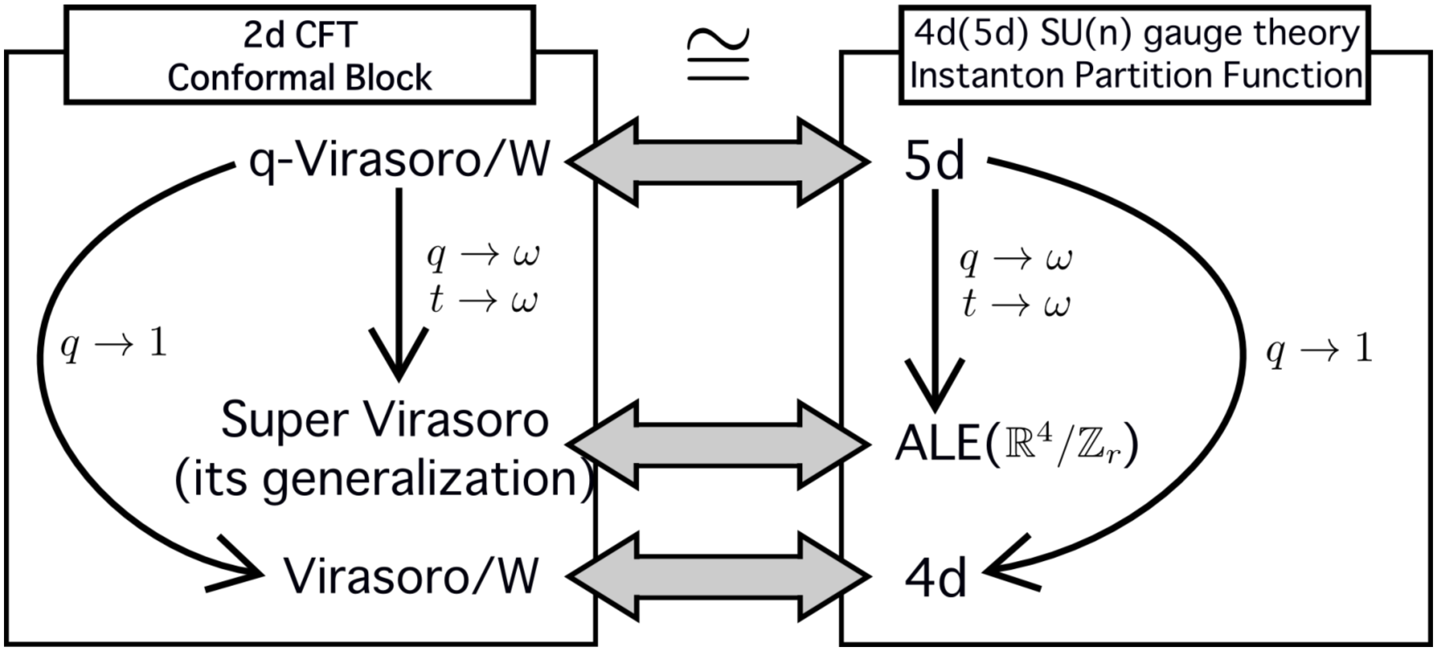

$$\omega = e^{\frac{2\pi i}{r}} \qquad N_f = 2n$$

Procedure proposed:

(1) assume the q-(or K lifted) version of (W)AGT conjecture

(2) find the limiting procedure $q \to \omega$ for q-Virasoro/W block

(3) apply the same limiting procedure to $Z^{5d}_{\mathrm{inst.}}$, which automatically generates ALE instanton partition function