Instanton Counting for Classical Groups

Satoshi Nakamura (The University of Tokyo)
with Y. Matsuo (UT) and F.Okazawa (UT)

About this poster

Review

- There are integral representations of the instanton corrections of $4d$ $\mathcal{N}=2$ gauge theories.
- When $G=U(N)$, this integration was performed completely by evaluation of residues at simple poles.
- How about other (classical) gauge groups? We partly give a formula for $Sp(1)$ instantons. We checked this is true up to 6-instantons.

Motivations of the Reserch

$G=U(N)$	$G=SO(N), Sp(N)$	$G=E_{6,7,8}, F_4, G_2$
integral representation of instanton corrections		×
$\exists$good basis $\vec{Y}$	???	
act on AGT correspondence	?	
W-algebra of type G		
DDAHA for $\mathfrak{gl}(n,\mathbb{C})$	**DDAHA for $\cdots$?**	

We want to understand $U(N)$-**analogies** and algebraic structures of gauge theories.

Integral Representations of Instanton Corrections

Partition functions of $4d$ $\mathcal{N}=2$ SUSY gauge theories:

$$Z = Z_{\text{tree}} Z_{\text{1-loop}} Z_{\text{instanton}}$$

nonperturbative corrections from instantons (the main quantities of this poster)

$$Z_{\text{inst}} = \sum_{k=0}^{\infty} q^k \int_{\mathcal{M}_k} 1 \quad (\mathcal{M}_k : k\text{-instanton moduli space})$$

$$\equiv Z_k$$

a vector field $(e^{i\varepsilon_1}, e^{i\varepsilon_2}) \in U(1)^2 \curvearrowright \mathbb{C}^2 \simeq \mathbb{R}^4$

When the gauge group G is a classical group and the Ω-background is turned on, Z_k has an integral expression. [Nekrasov-Shadchin'04]

For example, the SYM with $G=U(N)$, ϕ_j comes from the instanton's internal group.

$$Z_k^{U(N)} = \frac{1}{k!}\left(\frac{\varepsilon}{2\pi i \varepsilon_1\varepsilon_2}\right)^k \int_{\mathbb{R}} \frac{d\phi_1\cdots d\phi_k}{\prod_{j=1}^k P(\phi_j-\varepsilon_+)P(\phi_j+\varepsilon_+)} \frac{\Delta(0)\Delta(\varepsilon)}{\Delta(\varepsilon_1)\Delta(\varepsilon_2)}$$

$$P(x) = \prod_{l=1}^{N}(x-a_l) \qquad (\varepsilon = 2\varepsilon_+ = \varepsilon_1+\varepsilon_2)$$

↑ the adjoint matter

$$\Delta(x) = \prod_{i<j}((\phi_i-\phi_j)^2 - x^2)$$

Remark

- *Why only classical?* — Because the ADHM construction is fully used.
- *Which integral contours are chosen?* — When $\varepsilon_{1,2}$ have $+i0$.

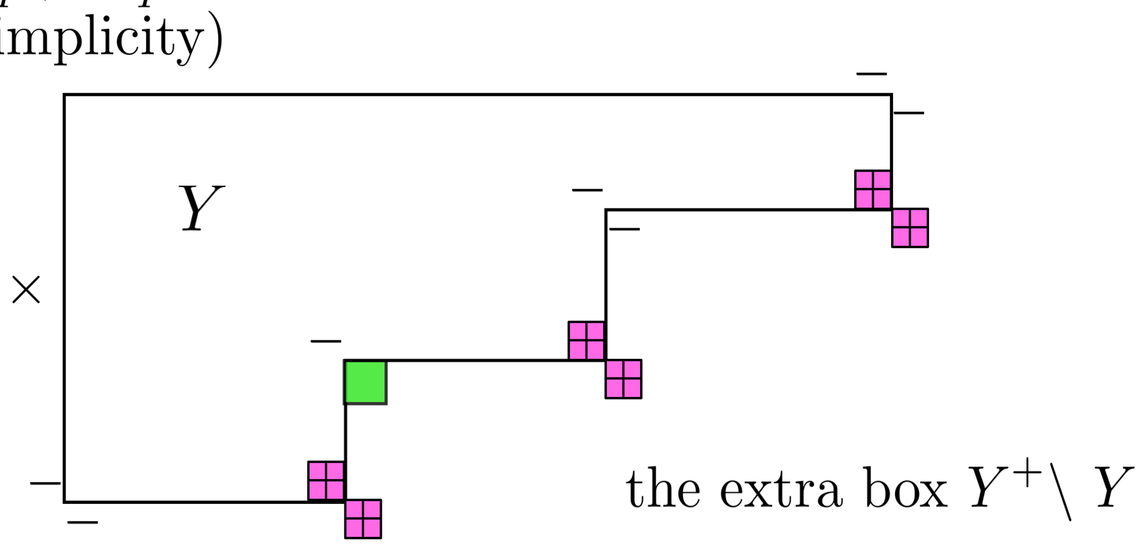
More on the SYM with $G=U(N)$ [Nekrasov'02] [Nakajima-Yoshioka'03]

1. This integral ends up with the sum of all the residues in the upper of ϕ_j's planes. Such poles are labeled by a vector $\vec{Y}=(Y_1, Y_2\cdots, Y_N)$ of Young diagrams with its total number of boxes $|\vec{Y}| = \sum_i |Y_i| = k$.

   Remark
   There are other poles which are also in the integration cycle.
   But the sum over all the permutations of one such pole vanishes.

2. Aside from such a combinational problem, one can get a recursion relation (the ratio $Z_{\vec{Y}^+}/Z_{\vec{Y}}$):
   ($G=U(1)$ only for simplicity)

$$\frac{Z_{Y^+}}{Z_Y} = \frac{\varepsilon}{\varepsilon_1\varepsilon_2} \times$$



where the diagram means the product over all the ■ and $-$,

provided ■ at a position (i,j) $\quad ((\hat{m}-i)\varepsilon_1 + (\hat{n}-1)\varepsilon_2)^{+1}$

$-$ at a position (i,j) $\quad ((\hat{m}-i)\varepsilon_1 + (\hat{n}-1)\varepsilon_2)^{-1}$

($=(\hat{m},\hat{n})$)

3. There is a formula;

$$Z_{k,\vec{Y}}^{U(N)} = \prod_{l,p=1}^{N} \prod_{(i,j)\in Y_l} \frac{1}{a_l - a_p + (\lambda_j^{(l)} - i + 1)\varepsilon_1 + (j - (\lambda^{(p)T})_i)\varepsilon_2}$$

$$\times \prod_{(i,j)\in Y_p} \frac{1}{a_l - a_p + (i - \lambda_j^{(p)})\varepsilon_1 + ((\lambda^{(l)T})_i - j + 1)\varepsilon_2}$$

(where $Y_l = (\lambda_1^{(l)} \geq \lambda_2^{(l)} \geq \cdots)$)

This gives the same recursion relations.

Problems for Other Classical Gauge Groups

We want the $U(N)$-analogies for the other (classical) cases.
There is no explicit formula like above. Even worse, there are more complicated combinational problems so that $\vec{Y}$-counterparts are unclear.

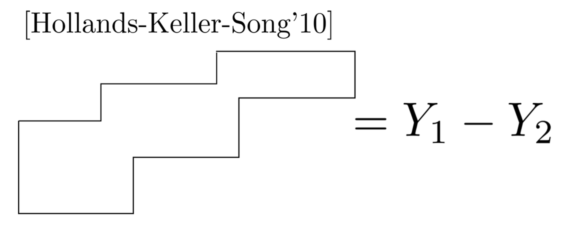
Remark
$\vec{Y}$-counterparts may contribute to understandings of the actions of W-algebras to the instanton moduli spaces, or the AGT correspondence. (like [Schiffmann-Vasserot'12] and [Kanno-Matsuo-Zhang'13])

We focus on $Sp(1) \simeq SU(2)$ SYM and recursion relations of $Z^{Sp(1)}$.

- $Z_k^{Sp(1)} = Z_k^{SU(2)}$ for lower levels. This avoids the combinational problems and gives an appropriate factor to each residue.
- Aside from such problems, recursion relations can be constructed for "generalized Young diagrams".

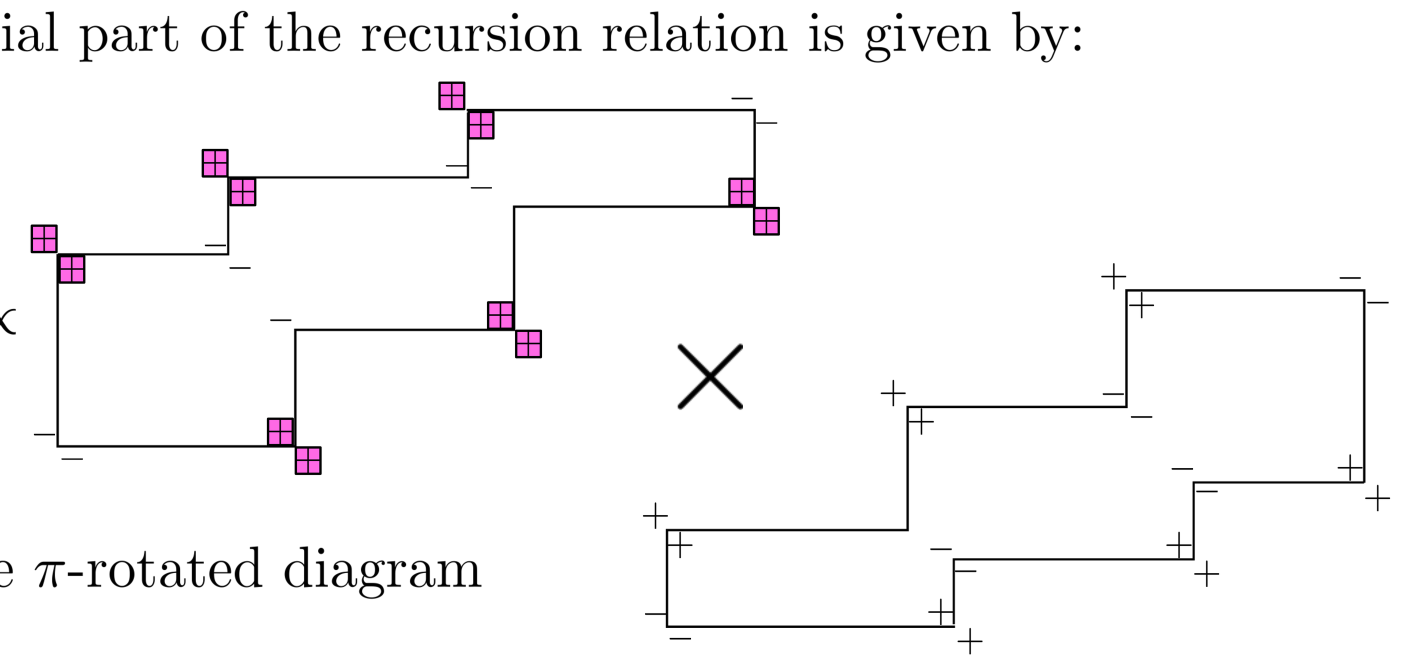
Solve the relations and get the residues with the appropriate factors.

[Hollands-Keller-Song'10]



The crutial part of the recursion relation is given by:

$$\frac{Z_{Y^+}}{Z_Y} \propto$$



Comparing this to the $U(N)$-case, we obtain a formula for generalized Young diagrams of $Sp(1)$-instantons. (See appendix.)

Future Works

- When a generalized Young diagram overlaps with its rotation, the formula has to be revised.
- Poles are not expected ones from the view of the W-algebra. So, we have to establish an algorithm to get a "good basis".
- Other gauge groups have integral representations similar to $Sp(1)$. Then, we can apply $Sp(1)$ results for classical gauge groups.